The Fully Ionised MHD Equations and Their Steady State Solution

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Outline

1. Star-Disk system
2. Testing the code
   - Setting up the Steady State Equations
   - Solving the Steady State Equations
3. Results

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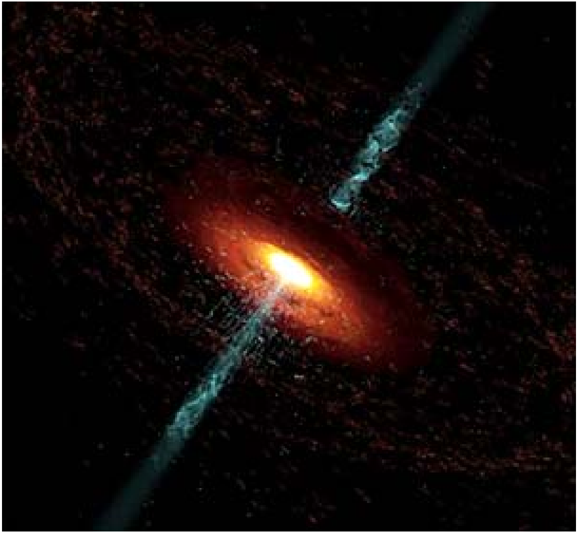
Star-Disk System



credit: FAS

Assumptions, Approximations and Consequences

Weak Ionisation: A fluid consisting of a majority neutral particles and a minority of charged particles

- collisions are dominated by neutrals
- $\rho \sim \rho_1$
- $\mathbf{v} \sim \mathbf{v}_1$
- Pressure, inertia and accelerative forces due to charged fluids are negligible

Assumptions, Approximations and Consequences

Full Ionisation: A fluid consisting entirely of charged particles

- electrons $-e$, positive ions $+Ze \Rightarrow n_e = Zn_{ion}$
- $m_e \ll m_{ion}$
- $\rho = \rho_e + \rho_{ion}$
- $\rho \mathbf{v} = \rho_e \mathbf{v}_e + \rho_{ion} \mathbf{v}_{ion}$
- Pressure, inertia and accelerative forces due to charged fluids must now be included

Testing the Code

Testing by comparison against a steady solution

- Start with multifluid magnetohydrodynamics (MHD) equations
- Assume time-independence $\frac{\partial}{\partial t} \longrightarrow 0$
- Assume one-dimensions $\frac{\partial}{\partial y}, \frac{\partial}{\partial z} \longrightarrow 0$
- Solve the steady state equations and compare

MHD multifluid equations for the charged fluids

Conservation of mass $\longrightarrow$ Continuity equation

$$\frac{\partial \rho_i}{\partial t} + \nabla \cdot (\rho_i \mathbf{v}_i) = 0$$

(ρ_i and $\mathbf{v}_i$ mass density and velocity of charged species i)

For time independence in one dimensions, this reduces to:

$$\frac{\partial}{\partial x}(\rho_i \mathbf{v}_i) = 0 \quad \Rightarrow \quad \rho_i (v_i)_x = \text{constant} \equiv Q_i$$

MHD multifluid equations for the charged fluids

Momentum equation

$$\alpha_i \rho_i \left(\mathbf{E} + \frac{1}{c} \mathbf{v}_i \times \mathbf{B} \right) = \nabla p_i - \mathbf{f}_{ij} + \rho_i \frac{D_i \mathbf{v}_i}{Dt}$$

where

$$\mathbf{f}_{ij} \equiv \rho_i \rho_j K_{ij} (\mathbf{v}_j - \mathbf{v}_i)$$

and the Lagrangian derivative is defined as

$$\frac{D_i}{Dt} \equiv \frac{\partial}{\partial t} + (\mathbf{v}_i \cdot \nabla)$$

MHD multifluid equations for the charged fluids

$$\alpha_i \rho_i \left(\mathbf{E} + \frac{1}{c} \mathbf{v}_i \times \mathbf{B} \right) = \nabla p_i - \mathbf{f}_{ij} + \rho_i \frac{D_i \mathbf{v}_i}{Dt}$$

Write

$$\nabla p_e \equiv \frac{\partial}{\partial x} p_e \hat{\imath} + \frac{\partial}{\partial y} p_e \hat{\jmath} + \frac{\partial}{\partial z} p_e \hat{k} \approx \frac{\partial}{\partial x} p_e \hat{\imath}$$

Examine electron momentum equation, using m_e negligibly small, in y- and z-directions only:

$$E_y = -\frac{1}{c}((v_e)_z B_x - (v_e)_x B_z) - \frac{\rho_{ion} K_{e,ion}}{\alpha_e}((v_{ion})_y - (v_e)_y)$$

$$E_z = -\frac{1}{c}((v_e)_x B_y - (v_e)_y B_x) - \frac{\rho_{ion} K_{e,ion}}{\alpha_e}((v_{ion})_z - (v_e)_z)$$

MHD multifluid equations for the overall fluid

Conservation of mass $\longrightarrow$ Continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

For time independence in one dimensions, this reduces to:

$$\frac{\partial}{\partial x}(\rho \mathbf{v}) = 0 \quad \Rightarrow \quad \rho(v)_x = \text{constant} \equiv Q$$

MHD multifluid equations for the overall fluid

Momentum equation for charged particles:

$$\alpha_i \rho_i \left(\mathbf{E} + \frac{1}{c} \mathbf{v}_i \times \mathbf{B} \right) = \nabla p_i - \mathbf{f}_{ij} + \rho_i \frac{D_i \mathbf{v}_i}{Dt}$$

Use charge neutrality and current density

$$\sum_i^N \alpha_i \rho_i = 0 \qquad \mathbf{J} \equiv \sum_i^N \alpha_i \rho_i \mathbf{v}_i$$

and

$$\mathbf{f}_{e,ion} = -\mathbf{f}_{ion,e}$$

and sum over i to get:

$$\frac{1}{c} \mathbf{J} \times \mathbf{B} = \nabla p + \sum_i^N \rho_i \frac{D_i \mathbf{v}_i}{Dt}$$

Lagrangian Derivative

$$\sum_{i=1}^{N=2} \rho_i \frac{D_i}{Dt}\mathbf{v}_i = \sum_{i=1}^{N=2} \rho_i \frac{\partial}{\partial t}\mathbf{v}_i + \sum_{i=1}^{N=2} \rho_i (\mathbf{v}_i \cdot \nabla)\mathbf{v}_i$$

after a LOT of algebra

$$= \rho \frac{D}{Dt}\mathbf{v} - \frac{1}{\alpha_e \alpha_{ion}} (\mathbf{J} \cdot \nabla) \left(\frac{\mathbf{J}}{\rho}\right)$$

MHD multifluid equations for the overall fluid

Momentum equation for overall fluid

$$\frac{\partial \rho \mathbf{v}}{\partial t} = \frac{1}{c}\mathbf{J} \times \mathbf{B} - \nabla \cdot (\rho \mathbf{v}\mathbf{v}) - \nabla p + \frac{1}{\alpha_e \alpha_{ion}}(\mathbf{J} \cdot \nabla)\left(\frac{\mathbf{J}}{\rho}\right)$$

Reduce this equation to time-independence and one-dimensions.

MHD multifluid equations for the overall fluid

Examine the term $(\mathbf{J} \cdot \nabla) \left(\frac{\mathbf{J}}{\rho} \right)$

Use

$$\mathbf{J} \equiv \frac{4\pi}{c} \nabla \times \mathbf{B} = \frac{4\pi}{c} \frac{\partial}{\partial x} (0, -B_z, B_y)$$

and so

$$(\mathbf{J} \cdot \nabla) \frac{\mathbf{J}}{\rho} \approx \left(\frac{\partial}{\partial x} (0, -B_z, B_y) \cdot \begin{pmatrix} \frac{\partial}{\partial x} \\ 0 \\ 0 \end{pmatrix} \right) \begin{pmatrix} \frac{J_x}{\rho} \\ \frac{J_x}{\rho} \\ \frac{J_x}{\rho} \end{pmatrix} = 0$$

MHD multifluid equations for the overall fluid

Finally

$$\frac{\partial \rho \mathbf{v}}{\partial t} = \frac{1}{c}\mathbf{J} \times \mathbf{B} - \nabla \cdot (\rho \mathbf{v}\mathbf{v}) - \nabla p + \frac{1}{\alpha_e \alpha_{ion}} (\mathbf{J} \cdot \nabla) \left(\frac{\mathbf{J}}{\rho} \right)$$

using the $p = a^2 \rho$, becomes:

$$0 = \frac{\partial}{\partial x} \left(\begin{array}{c} -\frac{1}{2}|B|^2 \\ B_x B_y \\ B_x B_z \end{array} \right) - \frac{\partial}{\partial x} \rho (v)_x \left(\begin{array}{c} (v)_x \\ (v)_y \\ (v)_z \end{array} \right) - \frac{\partial}{\partial x} \left(\begin{array}{c} a^2 \rho \\ 0 \\ 0 \end{array} \right)$$

or

$$\begin{aligned} C_x &= Q(v)_x + a^2 \rho + \frac{1}{2}|B|^2 \\ C_y &= Q(v)_y - B_x B_y \\ C_z &= Q(v)_z - B_x B_z \end{aligned}$$

Solving our steady state equations

To solve the steady state equations:

- Use the magnetic field **B** at each point in space to solve for our 12 variables

ρ, v_x, v_y, v_z, ρ_e, $(v_e)_x$, $(v_e)_y$, $(v_e)_z$, ρ_{ion}, $(v_{ion})_x$, $(v_{ion})_y$, $(v_{ion})_z$

- Then evolve the magnetic field through space using the induction equation

The first two variables to be solved are:

$$(v)_y = \frac{C_y + B_x B_y}{Q}, \text{ and } (v)_z = \frac{C_z + B_x B_z}{Q}$$

Solving our steady state equations

$$\rho = \frac{C_x - \frac{1}{2}|B|^2 \pm \sqrt{(C_x - \frac{1}{2}|B|^2)^2 - 4Q^2 a^2}}{2a^2}$$

and so

$$(v)_x = \frac{Q}{\rho}$$

Having found these we can get

$$\rho_e = \frac{\alpha_{ion}}{\alpha_{ion} - \alpha_e}\rho, \text{ and } \rho_{ion} = \frac{\alpha_e}{\alpha_e - \alpha_{ion}}\rho$$

and

$$(v_e)_x = \frac{Q_e}{\rho_e}, \text{ and } (v_{ion})_x = \frac{Q_{ion}}{\rho_{ion}}$$

Solving our steady state equations

8 variables have been calculated
4 remain

Use $\rho\mathbf{v} = \sum_i \rho_i \mathbf{v}_i$ to get

$$\begin{aligned}
\rho(v)_y &= \rho_e(v_e)_y + \rho_{ion}(v_{ion})_y \\
\rho(v)_z &= \rho_e(v_e)_z + \rho_{ion}(v_{ion})_z
\end{aligned}$$

and recall

$$\begin{aligned}
E_y &= -\frac{1}{c}((v_e)_z B_x - (v_e)_x B_z) - \frac{\rho_{ion} K_{e,ion}}{\alpha_e}((v_{ion})_y - (v_e)_y) \\
E_z &= -\frac{1}{c}((v_e)_x B_y - (v_e)_y B_x) - \frac{\rho_{ion} K_{e,ion}}{\alpha_e}((v_{ion})_z - (v_e)_z)
\end{aligned}$$

Induction equation

The 12 variables have now been solved.

Next, evolve the magnetic field using the induction equation.

For a fully ionised plasma, the Generalised Ohm's law is

$$\mathbf{E} = -\frac{1}{c}\mathbf{v} \times \mathbf{B} + \frac{B}{\alpha_e \rho_e \beta_e}\mathbf{J} - \frac{1}{c}\frac{1}{\alpha_e \rho_e}(\mathbf{J} \times \mathbf{B}) - \frac{1}{eN}\nabla p$$

and integrating gives us the fully ionised Induction Equation

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{M} = \nabla \times \left(\frac{c^2}{4\pi}\mathbf{R}(\nabla \times \mathbf{B}) - \frac{1}{eN}\nabla p \right)$$

Pressure term

Examine the term $\nabla \times \frac{1}{eN} \nabla p$.

Recall $N \equiv n_e + n_{ion}$ and $n_e = Zn_{ion}$ by charge neutrality.

Then $N \propto \rho$. Also, $p \propto \rho$.

Then

$$\nabla \times \left(\frac{1}{eN} \nabla p \right) \propto \nabla \times \left(\frac{1}{\rho} \nabla \rho \right) \propto \nabla \times \nabla (\ln \rho) = 0$$

Induction equation

So the induction equation for a fully ionised plasma

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{M} = \nabla \times \left(\frac{c^2}{4\pi} \mathbf{R} (\nabla \times \mathbf{B}) - \frac{1}{eN} \nabla p \right)$$

becomes, in a time-independent and one-dimensional system,

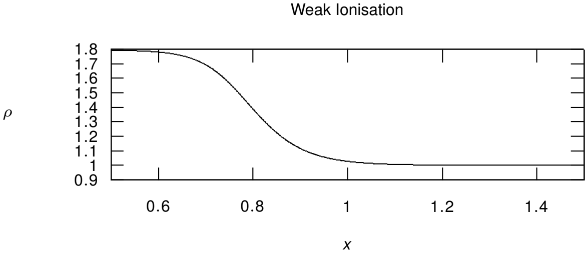
$$\frac{\partial}{\partial x} \mathbf{M} = \frac{\partial}{\partial x} \left(\frac{c^2}{4\pi} \mathbf{R} \frac{\partial}{\partial x} \mathbf{B} \right)$$

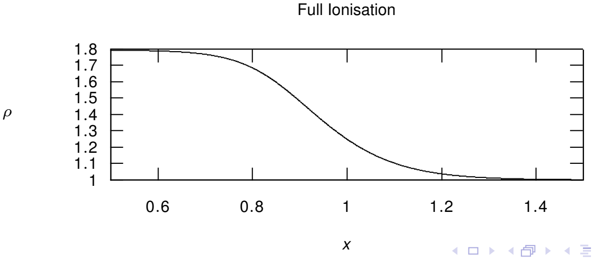
which gives us

$$\mathbf{M} - \mathbf{M}_R = \frac{c^2}{4\pi} \mathbf{R} \frac{\partial}{\partial x} \mathbf{B}$$

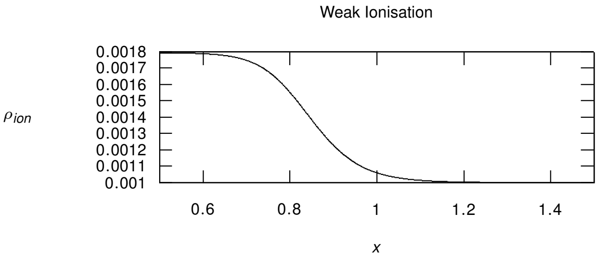
to complete the solution of the steady state equations.

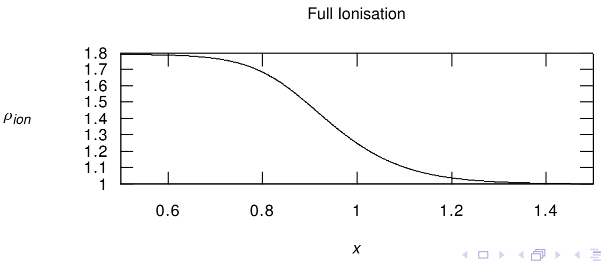
Results


Weak Ionisation
ρ
1.8
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1
0.9
0.6
0.8
1
1.2
1.4
x

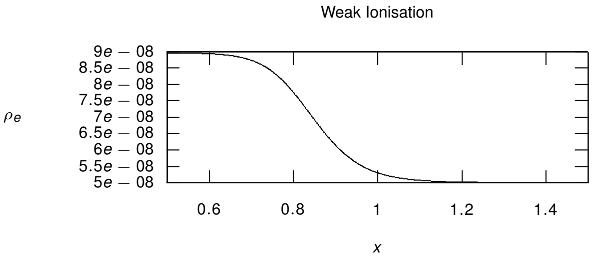

Full Ionisation
ρ
1.8
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1
0.6
0.8
1
1.2
1.4
x

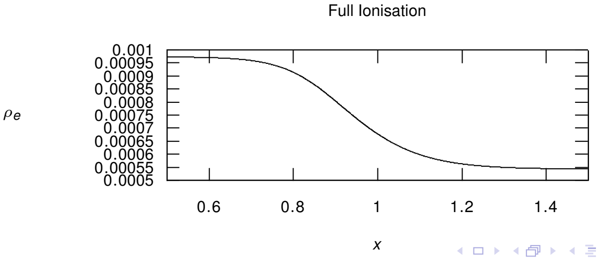
Results


Weak Ionisation
ρ_{ion}
0.0018
0.0017
0.0016
0.0015
0.0014
0.0013
0.0012
0.0011
0.001
0.6
0.8
1
1.2
1.4
x

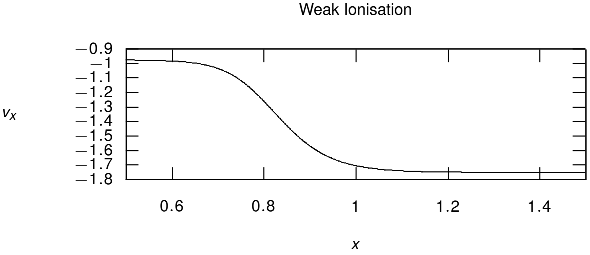

Full Ionisation
ρ_{ion}
1.8
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1
0.6
0.8
1
1.2
1.4
x

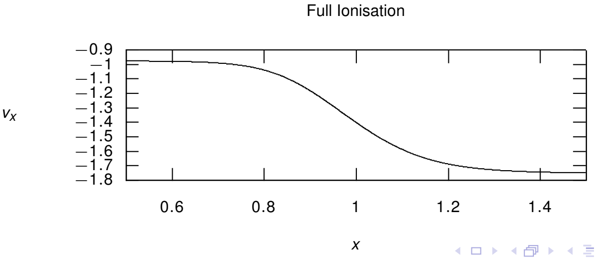
Results


Weak Ionisation
9e − 08
8.5e − 08
8e − 08
7.5e − 08
7e − 08
6.5e − 08
6e − 08
5.5e − 08
5e − 08
ρe
0.6
0.8
1
1.2
1.4
x

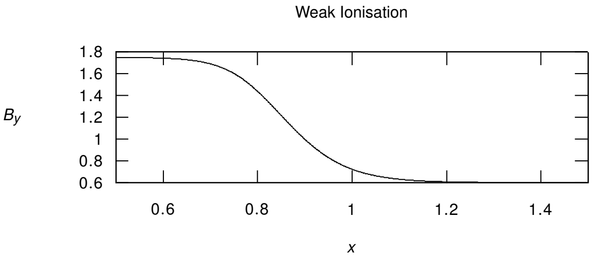

Full Ionisation
0.001
0.00095
0.0009
0.00085
0.0008
0.00075
0.0007
0.00065
0.0006
0.00055
0.0005
ρe
0.6
0.8
1
1.2
1.4
x

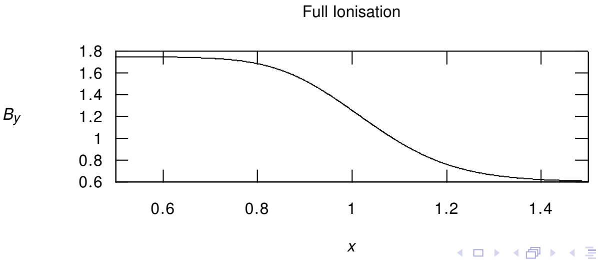
Results


Weak Ionisation
−0.9
−1
−1.1
−1.2
−1.3
−1.4
−1.5
−1.6
−1.7
−1.8
v_x
0.6
0.8
1
1.2
1.4
x


Full Ionisation
−0.9
−1
−1.1
−1.2
−1.3
−1.4
−1.5
−1.6
−1.7
−1.8
v_x
0.6
0.8
1
1.2
1.4
x

Results


Weak Ionisation
B_y
1.8
1.6
1.4
1.2
1
0.8
0.6
0.6
0.8
1
1.2
1.4
x


Full Ionisation
B_y
1.8
1.6
1.4
1.2
1
0.8
0.6
0.6
0.8
1
1.2
1.4
x

.. and ...

Thank You!