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Nonassociative differential geometry and gravity with non-geometric fluxes

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Why nonassociative gravity?

Geometry of fluxes

Example: electron in a strong magnetic field B. In this regime, due to the minimal coupling with the background gauge field, the dynamics takes place in a reduced phase space. It coincides with the electron coordinates thus the electron coordinates become noncommutative: $[x, y] = \frac{i\hbar}{B}$.

Similarly open strings endpoints test a noncommutative space (brane) in the presence of a nonvanishing constant B-field flux.

Yang-Mills theory captures the low energy effective D-brane action. In the presence of a B-field this suggested a description in terms of Noncommutative Yang-Mills theory: YM-theory on the *nongeometric background* $[x, y] = i\theta$.

The study of Yang-Mills (and Born-Infeld) theories in these noncommutative spaces has proven very fruitful.

-it provides an exact low energy D-brane effective action (in a given $\alpha' \to 0$ sector of string theory where closed strings decouple).

-it allows to realize string theory T-duality symmetry within the low energy physics of Noncommutative (Super) Yang-Mills theories [Connes, Douglas, Schwartz 1997].

Key example: NC-torus

NC plane $[x^i, x^j] = \theta^{ij} \quad \Rightarrow \quad$ NC torus coordinates $U^i = e^{ix^i}$.

NCSYM: $U(n)$, θ^{ij}, G_{ij}, g_{SYM}, M first Chern number $\frac{1}{2\pi}\int TrF$.

NCSYM′: $U(n'), \theta'^{ij}$, G'_{ij}, g'_{SYM}, M'.

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NCSYM$'$: $U(n'), \theta'^{ij}$, G'_{ij}, g'_{SYM}, M'.

Let $\Lambda = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} \in SO(d,d,Z)\,,$ then

$$\theta' = (\mathcal{A}\theta + \mathcal{B})(\mathcal{C}\theta + \mathcal{D})^{-1},$$

$$G'^{ij} = (\mathcal{C}\theta + \mathcal{D})^i_{\ k}(\mathcal{C}\theta + \mathcal{D})^j_{\ l}\, G^{kl},$$

$$g'^2_{SYM} = \sqrt{|\,\mathrm{det}(\mathcal{C}\theta + \mathcal{D})|}\; g^2_{SYM},$$

$$\begin{pmatrix} n' \\ M' \end{pmatrix} = S(\Lambda)\begin{pmatrix} n \\ M \end{pmatrix} \qquad \text{spinor representation } S(\Lambda)$$

The rank and the bundle topology of the NCSYM theory determine the D0 and D2 brane charges in IIA string theory. Noncommutativity θ^{ij} captures the presence of nontrivial NS B field. The action of these $SO(d,d,Z)$ rotations matches the T-duality transformation on metric, string coupling constant, D-brane chages, NS field of type IIA string theory.

Closed Strings

T-dualities for closed strings on $\mathbb{T}^d$ in the presence of a constant 3-form H-flux suggest the following chain of fluxes:

$$H_{abc} \xrightarrow{T_a} f^a{}_{bc} \xrightarrow{T_b} Q^{ab}{}_c \xrightarrow{T_c} R^{abc}$$

- H and f-fluxes on Riemannian manifolds,
- Q-flux on T-fold (global description involves T-duality transformations as transition functions between local trivializations of tangent bundle);
-R-flux *non geometric background.* Expected global description in terms of noncommutative and nonassociative structures

[Lüst, Blumenhagen, et al; Mylonas, Schupp, Szabo]

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In R-flux background closed strings are expected to probe the phase space

$$[x^\mu, x^\nu] = \tfrac{i\ell_s^3}{3\hbar} R^{\mu\nu\rho} p_\rho \ , \qquad [x^\mu, p_\nu] = i\hbar\, \delta^\mu{}_\nu \qquad \text{and} \qquad [p_\mu, p_\nu] = 0 \ ,$$

with spacetime nonassociativity:

$$[x^\mu, x^\nu, x^\rho] := [x^\mu, [x^\nu, x^\rho]] + [x^\nu, [x^\rho, x^\mu]] + [x^\rho, [x^\mu, x^\nu]] = \ell_s^3\, R^{\mu\nu\rho} \ .$$

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Low energy effective action for closed strings includes the gravitational action.

- Nonassociative Gravity as low energy action in the presence of R-flux?
- T-duality transformations act within nonassociative GR?

This motivates the study of:

• A gravity theory on the phase space $\mathcal{M}$ of coordinates $X^A = (x^\mu, p_\nu)$.

• The induced Einstein vacuum equations on the nonassociative spacetime of the coordinates x^μ.

[P.A, Dimitrijevic, Szabo, arXiv:1710.11467]

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A key observation is that the nonassociative R-flux phase space can be obtained via Drinfeld 2-cochain twist deformation of commutative phase space.

-This work is then a nontrivial nonassociative generalization of the NC differential geometry and *Riemannian geometry* constructed in [Wess Group].
-For an earlier study toward this goal see [Blumenhagen, Fuchs].
-This work is also an explicit (and self contained) example of NA differential geometry as developped for arbitrary 2-cochain twist in [Barnes, Schenkel, Szabo].

Drinfeld twist

Let Ξ be the Lie algebra of vector fields on phase space $\mathcal{M}$. Let $U\Xi$ be the universal enveloping algebra (of sums of products of vector fields). It is a Hopf algebra, i.e. a **symmetry algebra**, with, for all $u \in \Xi$,

$$\Delta(u) = u \otimes 1 + 1 \otimes u \ , \ \varepsilon(u) = 0 \ , \ S(u) = -u$$

Consider the twist [Mylonas, Schupp, Szabo]

$$\mathcal{F} = \exp\Big(-\tfrac{i\hbar}{2}\,(\partial_\mu \otimes \tilde{\partial}^\mu - \tilde{\partial}^\mu \otimes \partial_\mu) - \tfrac{i\ell_s^3}{12\hbar} R^{\mu\nu\rho}(p_\nu\partial_\rho \otimes \partial_\mu - \partial_\mu \otimes p_\nu\partial_\rho)\Big)$$

built with the vector fields $\partial_A = \Big(\frac{\partial}{\partial x^\mu} = \partial_\mu, \frac{\partial}{\partial p_\mu} = \tilde{\partial}^\mu\Big)$ and the R-flux.

The Hopf algebra $U\Xi$ is twist deformed in the quasi-Hopf algebra $U\Xi^{\mathcal{F}}$ with same product, ε and S of $U\Xi$, but new coproduct, for all $\xi \in U\Xi$,

$$\Delta_{\mathcal{F}}(\xi) = \mathcal{F}\Delta(\xi)\mathcal{F}^{-1}$$

in particular $\Delta_{\mathcal{F}}(\partial_\mu) = 1 \otimes \partial_\mu + \partial_\mu \otimes 1$ and

$$\Delta_{\mathcal{F}}(\tilde{\partial}^\mu) = 1 \otimes \tilde{\partial}^\mu + \tilde{\partial}^\mu \otimes 1 + i\kappa\, R^{\mu\nu\rho}\, \partial_\nu \otimes \partial_\rho \, , \qquad \kappa \equiv \ell_s^3/6\hbar$$

$U\Xi^{\mathcal{F}}$ is a quasi-Hopf algebra because while $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$, for $\Delta_{\mathcal{F}}$ we have

$$\Phi\, (\Delta_{\mathcal{F}} \otimes id)\, \Delta_{\mathcal{F}}(\xi) = (id \otimes \Delta_{\mathcal{F}})\, \Delta_{\mathcal{F}}(\xi)\Phi$$

where the associator Φ is given by

$$\Phi = \exp\Big(\tfrac{\ell_s^3}{6}\, R^{\mu\nu\rho}\, \partial_\mu \otimes \partial_\nu \otimes \partial_\rho\Big) =: \phi_1 \otimes \phi_2 \otimes \phi_3$$

Gravity is based on general covariance under infinitesimal diffeomorphisms, i.e. under the Lie algebra of vector fields (equivalently $U\Xi$). We construct a NC/NA Riemannian geometry covariant under the quasi Hopf algebra $U\Xi^{\mathcal{F}}$.

Key point:

Consider commutative algebras that carry a representation of $U\Xi$; deform these algebras so that they carry a representation of $U\Xi^{\mathcal{F}}$.

Example: Algebra of functions $\longrightarrow$ $A_\star$ $\equiv$ NC/NA algebra of functions.
Notation: $\mathcal{F} = \bar{\mathsf{f}}^{\,\alpha} \otimes \bar{\mathsf{f}}_{\,\alpha}$

$$f \star g = \bar{\mathsf{f}}^{\,\alpha}(f) \cdot \bar{\mathsf{f}}_{\,\alpha}(g)$$

NC controlled by the $\mathcal{R}$-matrix $\mathcal{R} = \mathcal{F}^{-2}$,

$$f \star g := \bar{R}^{\,\alpha}(g) \star \bar{R}_{\,\alpha}(f) =: {}^{\alpha}g \star {}_{\alpha}f \ .$$

NA controlled by the associator Φ,

$$(f \star g) \star h = {}^{\phi_1}f \star ({}^{\phi_2}g \star {}^{\phi_3}h) \ .$$

$U\Xi^{\mathcal{F}}$-**covariance** of $\star$-product: let $\triangle_{\mathcal{F}}(\xi) = \xi_{(1)} \otimes \xi_{(2)}$ then

$${}^{\xi}(f \star g) = {}^{\xi_{(1)}}f \star {}^{\xi_{(2)}}g$$

This $\star$-product on coordinates functions gives the R-flux algebra:

$$[x^\mu, x^\nu]_\star = 2i\kappa\, R^{\mu\nu\rho}\, p_\rho \ , \qquad [x^\mu, p_\nu]_\star = i\hbar\, \delta^\mu{}_\nu \qquad \text{and} \qquad [p_\mu, p_\nu]_\star = 0$$

Tensor algebra $\mathcal{T}_\star$

$$\tau \otimes_\star \tau' = \bar{\mathsf{f}}^{\alpha}(\tau) \otimes \bar{\mathsf{f}}_{\alpha}(\tau'),$$

Exterior algebra $\Omega^\bullet_\star$

$$\theta \wedge_\star \theta' = \bar{\mathsf{f}}^{\alpha}(\theta) \wedge \bar{\mathsf{f}}_{\alpha}(\theta').$$

In particular we have the $A_\star$-bimodules of forms $\Omega_\star$ and of vector fields $\Xi_\star$.

$$\begin{aligned} f \star \mathrm{d}\tilde{x}_\mu &= f \cdot \mathrm{d}\tilde{x}_\mu = \mathrm{d}\tilde{x}_\mu \star f \ , \\ f \star \mathrm{d}x^\mu &= f \cdot \mathrm{d}x^\mu - \tfrac{i\kappa}{2} R^{\mu\nu\rho} (\partial_\nu f) \cdot \mathrm{d}\tilde{x}_\rho = \mathrm{d}x^\mu \star f - \mathrm{d}\tilde{x}_\rho \star i\kappa R^{\mu\nu\rho} (\partial_\nu f) \end{aligned}$$

$$\begin{aligned} f \star \partial_\mu &= f \cdot \partial_\mu = \partial_\mu \star f \ , \\ f \star \tilde{\partial}^\mu &= f \cdot \tilde{\partial}^\mu + \tfrac{i\kappa}{2} R^{\mu\nu\rho} (\partial_\nu f) \cdot \partial_\rho = \tilde{\partial}^\mu \star f + \partial_\rho \star i\kappa R^{\mu\nu\rho} (\partial_\nu f) \end{aligned}$$

Pairing between forms and vectors $\langle\ ,\ \rangle : \Omega \times \Xi \to C^\infty(\mathcal{M})$ is deformed to

$$\langle\ ,\ \rangle_\star := \langle\ ,\ \rangle \circ \mathcal{F}^{-1} \ : \Omega_\star \times \Xi_\star \to A_\star\ , \tag{1}$$

explicitly

$$\langle\, \omega\, ,\, u\, \rangle_\star = \Big\langle\, \bar{\mathsf{f}}^{\,\alpha}(\omega)\, ,\, \bar{\mathsf{f}}_{\,\alpha}(u)\, \Big\rangle\ . \tag{2}$$

pai

Covariance under $U\Xi^\mathcal{F}$: $\ {}^\xi\langle\, \omega\, ,\, u\, \rangle_\star = \langle\, {}^{\xi_{(1)}}\omega\, ,\, {}^{\xi_{(2)}}u\, \rangle_\star\ .$

The pairing generalizes to

$$\langle\ ,\ \rangle_\star : \Omega_\star \wedge_\star \Omega_\star \times \Xi \wedge_\star \Xi \to A_\star\ .$$

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The pairing generalizes to

$$\langle\ ,\ \rangle_\star : \Omega_\star \wedge_\star \Omega_\star \times \Xi \wedge_\star \Xi \to A_\star\ .$$

Notice that $\langle\, \omega\, ,\quad \rangle_\star$ is a right $A_\star$-linear map $\Xi_\star \to A_\star$. More in general we consder right $A_\star$-linear maps betweeen $A_\star$-bimodules

$$L : V_\star \to W_\star$$

(e.g. $V_\star, W_\star$ star-tensor products of $\Omega_\star$ and $\Xi_\star$).

Action of $U\Xi^{\mathcal{F}}$ on $L : V_\star \to W_\star$ is the **adjoint** action:

$${}^{\xi}L(v) := ({}^{\xi}L)(v) = {}^{\xi_{(1)}}(L({}^{S(\xi_{(2)})}v))$$

so that we have **covariance**: ${}^{\xi}(L(v)) = {}^{\xi_{(1)}}L({}^{\xi_{(2)}}v)$.

Composition of two right $A_\star$-linear maps is defined by

$$L_1 \bullet L_2 := {}^{\phi_1}L_1 \circ {}^{\phi_2}L_2 \circ {}^{\phi_3}$$

so that we have **covariance**: ${}^{\xi}(L_1 \bullet L_2) = {}^{\xi_{(1)}}L_1 \bullet {}^{\xi_{(2)}}L_2$
moreover $L_1 \bullet L_2$ is again right $A_\star$-linear.

Summary: we defined $\star,\ \otimes_\star,\ \wedge_\star, \langle\ ,\ \rangle_\star, \bullet$. They are invariant under $U\Xi^{\mathcal{F}}$.

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Summary: we defined $\star,\ \otimes_\star,\ \wedge_\star, \langle\ ,\ \rangle_\star, \bullet$. They are invariant under $U\Xi^{\mathcal{F}}$.

More differential geometry:

Connection

A connection is a linear map

$$\begin{aligned} \nabla^\star\ :\ \Xi_\star &\longrightarrow \Xi_\star \otimes_\star \Omega_\star \\ u &\longmapsto \nabla^\star u = u^i \otimes_\star \omega_i\ , \end{aligned}$$

where $u^i \otimes_\star \omega_i \in \Xi_\star \otimes_\star \Omega_\star$, which satisfies the right Leibniz rule

$$\nabla^\star(u \star f) = \left({}^{\bar{\phi}_1}\nabla^\star({}^{\bar{\phi}_2}u)\right) \star {}^{\bar{\phi}_3}f + u \otimes_\star \mathrm{d}f$$

Coefficients of the connection along a vector field ∂_A: $\nabla^\star_{\partial_A}\partial_B = \partial_C \star \Gamma^C_{BA}$.

$\nabla^\star$ uniquely lifts to a connection

$$\mathsf{d}_{\nabla^\star} \,:\, \Xi_\star \otimes_\star \Omega^\bullet_\star \longrightarrow \Xi_\star \otimes_\star \Omega^\bullet_\star \, .$$

Theorem

$$\boxed{\mathsf{T}^\star(u,v) = {}^{\phi_1}\nabla^\star_{{\phi_2}_v}{}^{\phi_3}u - {}^{\phi_1}\nabla^\star_{{\phi_2}_{\alpha}u}{}^{\phi_3\,\alpha}v + [u,v]_\star} \qquad (5) \; \boxed{\texttt{Tor}$$

Proof: True for $u = \partial_A, v = \partial_B$. Then show that $T^\star$ is well defined and right $A_\star$-linear.

Curvature

$$\mathsf{R}^\star = \mathsf{d}^\star_\nabla \bullet \mathsf{d}^\star_\nabla \; : \; \Xi_\star \;\longrightarrow\; \Xi_\star \otimes_\star \Omega^2_\star \; .$$

Is indeed right $A_\star$-linear (hence a tensor).

Define curvature as a map $R^\star : \Xi_\star \otimes_\star (\Xi_\star \wedge_\star \Xi_\star) \longrightarrow \Xi_\star$ giving on vector fields $u, v, z \in \Xi_\star$ the vector field

$$R^\star(z,u,v) = \langle\, {}^{\bar\phi_1}\mathsf{R}^{\star\,\bar\phi_2}z \, , \, {}^{\bar\phi_3}(u \wedge_\star v) \,\rangle_\star \; .$$

Theorem

$$R^\star(z,u,v) = {}^{\kappa_1\,\check\phi_1\,\phi'_1}\boldsymbol{\nabla}^\star_{\bar\rho_3\,\bar\zeta_3\,\bar\phi_3\,\phi'_{3}{}_v}\Big({}^{\bar\rho_1\,\bar\phi_1\,\kappa_2\,\check\phi_2\,\phi'_2}\boldsymbol{\nabla}^\star_{\bar\rho_2\,\bar\zeta_2\,\check\phi_3{}_u}{}^{\bar\zeta_1\,\bar\phi_2\,\kappa_3}z\Big) - {}^{\kappa_1\,\check\phi_1\,\phi'_1}\boldsymbol{\nabla}^\star_{\bar\rho_3\,\bar\zeta_3\,\bar\phi_3\,\phi'_{3}{}_{\alpha u}}\Big({}^{\bar\rho_1\,\bar\phi_1\,\kappa_2\,\check\phi_2\,\phi'_2}\boldsymbol{\nabla}^\star_{\bar\rho_2\,\bar\zeta_2\,\check\phi_3{}_{\alpha_v}}{}^{\bar\zeta_1\,\bar\phi_2\,\kappa_3}z\Big) + \boldsymbol{\nabla}^\star_{[u,v]_\star}z$$

where

$$\boldsymbol{\nabla}^\star_v u := \langle\, {}^{\bar\phi_1}\nabla^{\star\,\bar\phi_2}u \, , \, {}^{\bar\phi_3}v \,\rangle_\star$$

Proof: as for torsion.

Ricci tensor

$$\mathsf{Ric}^\star(u,v) = \langle\ \mathsf{d}x^A\ ,\ \mathsf{R}^\star(u,\partial_A,v)\ \rangle_\star\ .$$

Riemannian Geometry

Metric tensor: $\mathsf{g}^\star = g_{MN} \star \mathsf{d}x^M \otimes_\star \mathsf{d}x^N$.

Theorem

There exists one and only one torsion free and metric compatible connection.

It is determined by $\quad \mathsf{g}_{MN} \star \mathsf{d}x^M \star \Gamma^N_{AD} = \frac{1}{2}\,(\partial_D \mathsf{g}_{AB} + \partial_A \mathsf{g}_{BD} - \partial_B \mathsf{g}_{AD}) \star \mathsf{d}x^B\ .$

Einstein vacuum equations:

$$\mathsf{Ric}^\star = 0\ .$$

We can pull back these equation on spacetime (spanned by x^μ coordinates only). Considering momentum space with the flat metric $\eta^{\mu\nu}$ we obtain Ricci tensor $\mathsf{Ric}^\circ_{\mu\nu}$ on the zero momentum leaf and spacetime Einstein equations

$$\mathsf{Ric}^\circ_{\mu\nu} = 0$$

Up to first order in R-flux (but second order in the twist deformation) we explicitly have

$$\begin{aligned}\mathsf{Ric}^\circ_{\mu\nu} = \mathsf{Ric}_{\mu\nu} + \tfrac{\ell_s^3}{12}\, R^{\alpha\beta\gamma} \Big(& \partial_\rho\big(\partial_\alpha \mathrm{g}^{\rho\sigma}\, (\partial_\beta \mathrm{g}_{\sigma\tau})\, \partial_\gamma \Gamma^{\ \tau}_{\mu\nu}\big) - \partial_\nu\big(\partial_\alpha \mathrm{g}^{\rho\sigma}\, (\partial_\beta \mathrm{g}_{\sigma\tau})\, \partial_\gamma \Gamma^{\ \tau}_{\mu\rho}\big) \\ & + \partial_\gamma \mathrm{g}_{\tau\omega}\big(\partial_\alpha (\mathrm{g}^{\sigma\tau}\, \Gamma^{\ \rho}_{\sigma\nu})\, \partial_\beta \Gamma^{\ \omega}_{\mu\rho} - \partial_\alpha(\mathrm{g}^{\sigma\tau}\, \Gamma^{\ \rho}_{\sigma\rho})\, \partial_\beta \Gamma^{\ \omega}_{\mu\nu} \\ & + (\Gamma^{\ \sigma}_{\mu\rho}\, \partial_\alpha \mathrm{g}^{\rho\tau} - \partial_\alpha \Gamma^{\ \sigma}_{\mu\rho}\, \mathrm{g}^{\rho\tau})\, \partial_\beta \Gamma^{\ \omega}_{\sigma\nu} \\ & - (\Gamma^{\ \sigma}_{\mu\nu}\, \partial_\alpha \mathrm{g}^{\rho\tau} - \partial_\alpha \Gamma^{\ \sigma}_{\mu\nu}\, \mathrm{g}^{\rho\tau})\, \partial_\beta \Gamma^{\ \omega}_{\sigma\rho}\big)\Big)\end{aligned}$$

Riemannian Geometry

Levi Civita connection determined by

$$\mathrm{d}\langle\, \mathrm{g}^\star \,,\, (\partial_A \otimes_\star \partial_B)\,\rangle_\star = \langle\, \mathrm{g}^\star \,,\, \nabla^\star(\partial_A \otimes_\star \partial_B)\,\rangle_\star$$

proceed as in classical case (use symmetry of g_{AB} and torsion free condition) to obtain:

$$\mathrm{g}_{MN} \star \mathrm{d}x^M \star \Gamma^N_{AD} = \tfrac{1}{2}\,(\partial_D \mathrm{g}_{AB} + \partial_A \mathrm{g}_{BD} - \partial_B \mathrm{g}_{AD}) \star \mathrm{d}x^B \,.$$

i.e.

$$\begin{aligned} \mathrm{d}x^M \star G_{MN} \star \Gamma^N_{AD} \;=\; & \mathrm{d}x^M \star \tfrac{1}{2}\Big(\partial_D \mathrm{g}_{AM} + \partial_A \mathrm{g}_{MD} - \partial_M \mathrm{g}_{AD} \\ & + i\kappa\, \mathcal{R}^{EF}{}_M\, (\partial_E \partial_D \mathrm{g}_{AF} + \partial_E \partial_A \mathrm{g}_{DF})\Big)\,, \end{aligned}$$

with

$$G_{MN} = \mathrm{g}_{MN} + i\kappa\, \mathcal{R}^{EF}{}_M\, \partial_E \mathrm{g}_{NF}\,,$$

Invert G_{MN} as a differential operator and obtain a unique solution for the Levi Civita connection.

This gives Einstein equations in vaccuum on $\mathcal{M}$ phase space.