# Unitarity problems in higher derivative field theories and Quantum Gravity

**Manuel Asorey**

Universidad de Zaragoza

Departamento de Física Teórica
Universidad Zaragoza
1542

BALFEST80

DIAS, Dublin, January 2018

# BALFEST80

# Einstein Approach to Gravitation

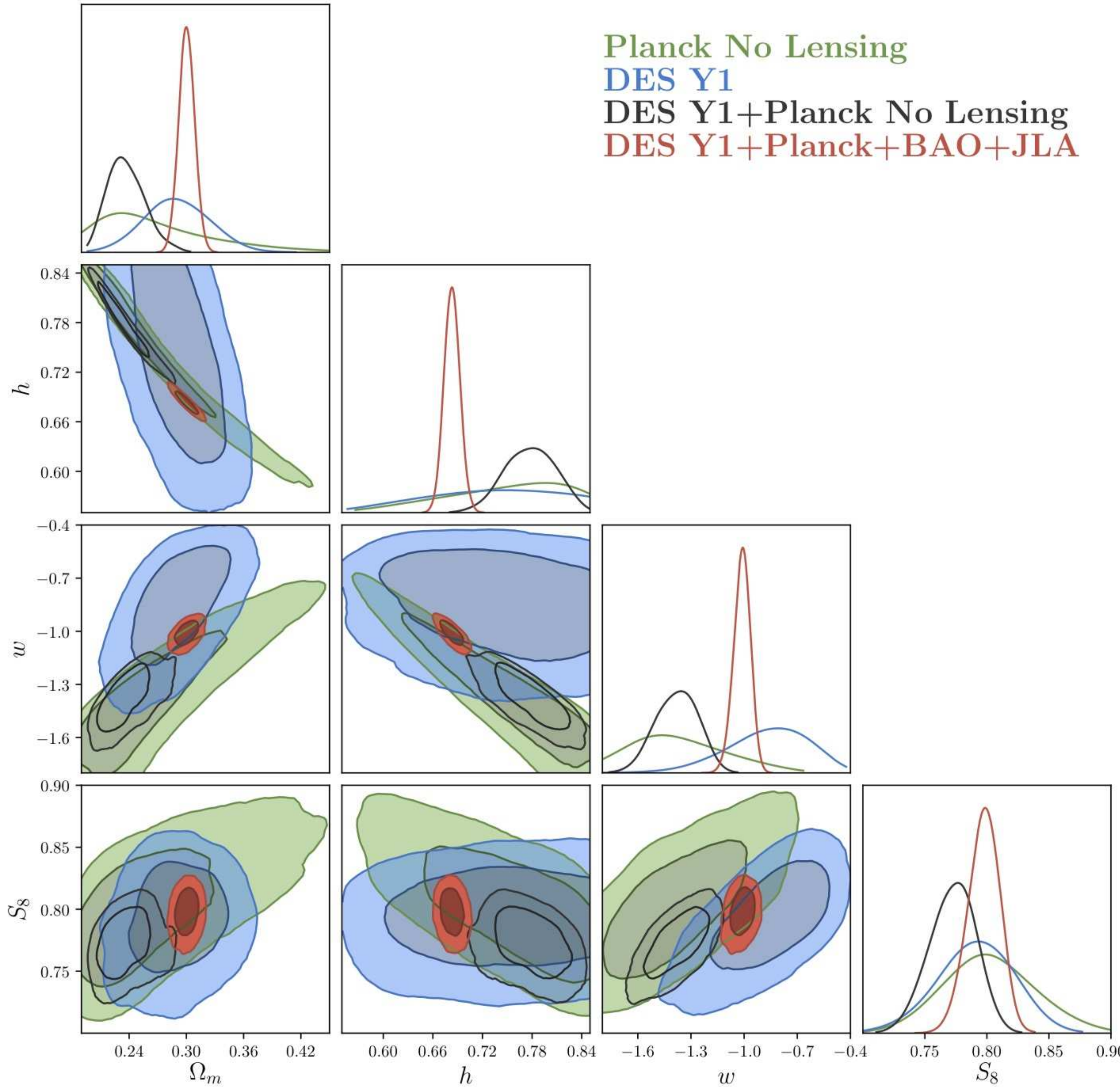
- General Relativity:

$$S_E(g) = \varkappa \int \sqrt{g} R$$

- General Relativity with Cosmological Constant:

$$S_{E_\lambda}(g) = \varkappa \int \sqrt{g} R + \lambda \int \sqrt{g}$$

Good agreement with LCDM
and Cosmological Data



Planck No Lensing
DES Y1
DES Y1+Planck No Lensing
DES Y1+Planck+BAO+JLA
$\Omega_m$
$h$
$w$
$S_8$

# Einstein Approach to Gravitation

- General Relativity:

$$S_{E_\lambda}(g) = \varkappa \int \sqrt{g} R + \lambda \int \sqrt{g}$$

Good agreement with LCDM
and Cosmological Data

$$w = -1.00 \begin{cases} +0.04 \\ -0.05 \end{cases}$$

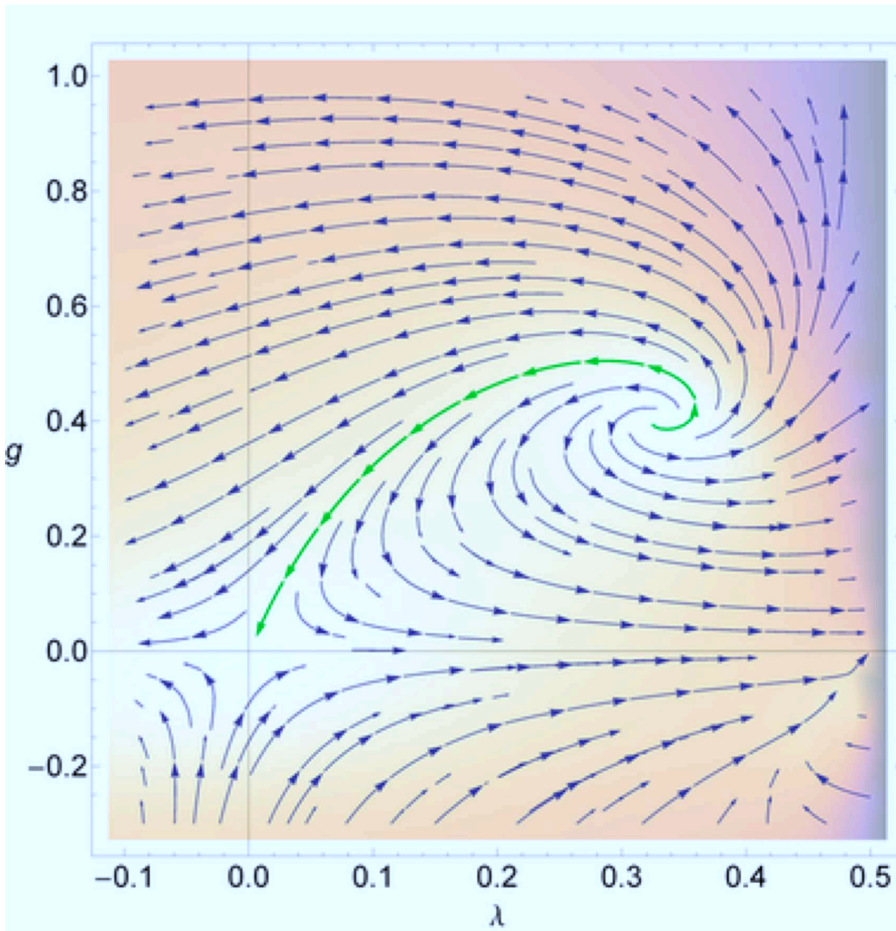
# Why to look beyond GR?

UV divergences and theories BRG

# Why to look beyond GR?

- GR theory is not PT-renormalizable: UV problem
- Modified gravity al large distances: IR problem
- GR could be nonPT-renormalizable:
  $\exists$ non-trivial renormalization group IR fixed point

# Quantum Gravity



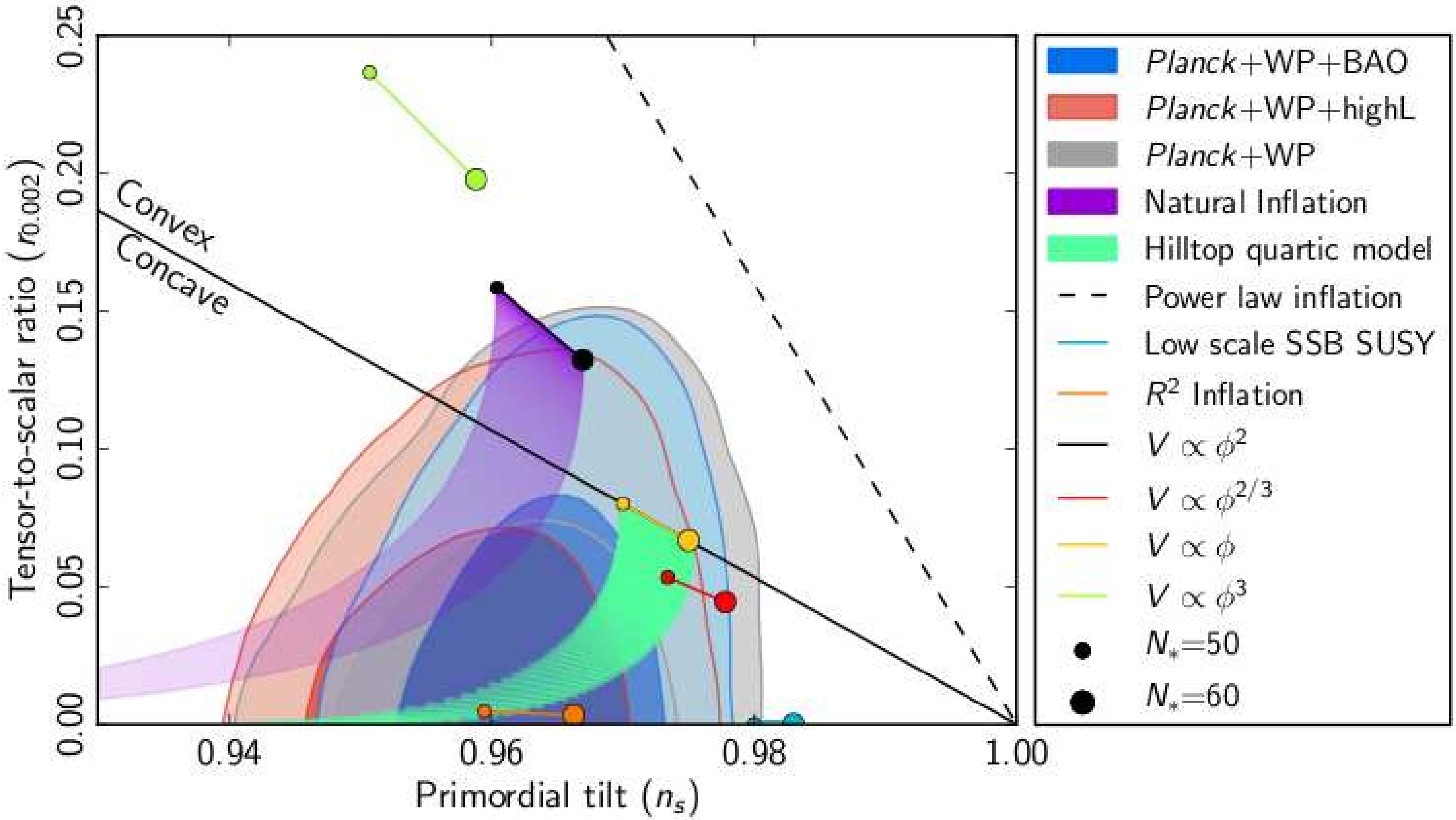

Exact Renormalization Group

# Why to look at beyond GR?

- GR theory is not renormalizable: UV problem
- Modified gravity at large distances: IR problem
- Although GR could be nonPT-renormalizable:
  ∃ non-trivial renormalization group IR fixed point
- Inflation data support BGR models: Starobinsky

$$S_{ES}(g) = \varkappa \int \sqrt{g} R + \lambda \int \sqrt{g} + \mu \int \sqrt{g} R^2$$

# Inflation Models



# Higher derivative theories

- Smoothing singularities
- Renormalizable theories of Gravity
- Classical Instabilities Ostrogradsky
- Quantum Ghosts: Unitarity loss

# Lovelock Theorem

$$S_{ES}(g) = \varkappa \int \sqrt{g} R + \lambda \int \sqrt{g} + \beta \int \sqrt{g} \left( R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \right)$$

$2^{nd}$-order EoM + Symmetric 2-tensor $G_{\mu\nu}$
+ Bianchi identities

# Horndeski scalar-tensor theories

$$
\begin{aligned}
S_H &= \int \sqrt{g}\,[K(\phi,X) - G_3(\phi,X)\Box\phi + G_4(\phi,X)R] \\
&+ \int \sqrt{g}\left[G_{4,X}[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi) + G_5(\phi,X)G_{\mu\nu}(\nabla^\mu\nabla^\nu\phi)\right] \\
&- \int \sqrt{g}\left[\tfrac{1}{6}G_{5,X}[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi)\right] \\
&+ \int \sqrt{g}\left[2(\nabla^\mu\nabla_\alpha\phi)(\nabla^\alpha\nabla_\beta\phi)(\nabla^\beta\nabla_\mu\phi)\right]
\end{aligned}
$$

$$X = -\tfrac{1}{2}\nabla_\mu\phi\nabla^\mu\phi$$

# Ostrogradski instabilities
# [1850]

# Ostrogradski instabilities

Theories with high time derivatives are unstable

Classical Mechanics:
A Lagrangian with higher derivatives:

$$L(x, \dot{x}, \ddot{x})$$

canonical variables

$$q_1 = x \qquad p_1 = \partial_{\dot{x}} L - \frac{d}{dt}\partial_{\ddot{x}} L$$
$$q_2 = \dot{x} \qquad p_2 = \partial_{\ddot{x}} L$$

# Ostrogradski instabilities

If L is regular there exist a function $\Lambda(q_1, q_2, p_2)$ such that

$$\partial_{\ddot{x}} L \Big|_{\substack{x=q_1, \dot{x}=q_2 \\ \ddot{x}=\Lambda(q_1,q_2,p_2)}} = p_2$$

$$H = p_1 q_2 + p_2 \Lambda(q_1, q_2, p_2) - L(q_1, q_2, \Lambda(q_1, q_2, p_2))$$

$H$ is unbounded below in $p_1$.

Similar arguments apply for higher time derivative Lagrangians.

Theories with high time derivatives are unstable

# Super-renormalizable high derivative theories

$$S_{SR}(g) = \varkappa \int \sqrt{g} R + \lambda \int \sqrt{g}$$

$$+ \sum_{n=1}^{N} \int \sqrt{g} \, \left( \alpha_n \, R_{\mu\nu\alpha\beta} \, \Box^n R^{\mu\nu\alpha\beta} - \beta_n \, R_{\mu\nu} \, \Box^n R^{\mu\nu} + \gamma_n \, R \, \Box^n R \right)$$

only one loop divergences

# QFT Principles

- Poincaré invariance
- Microcausality
- Unitarity
- Renormalizability

## Implications

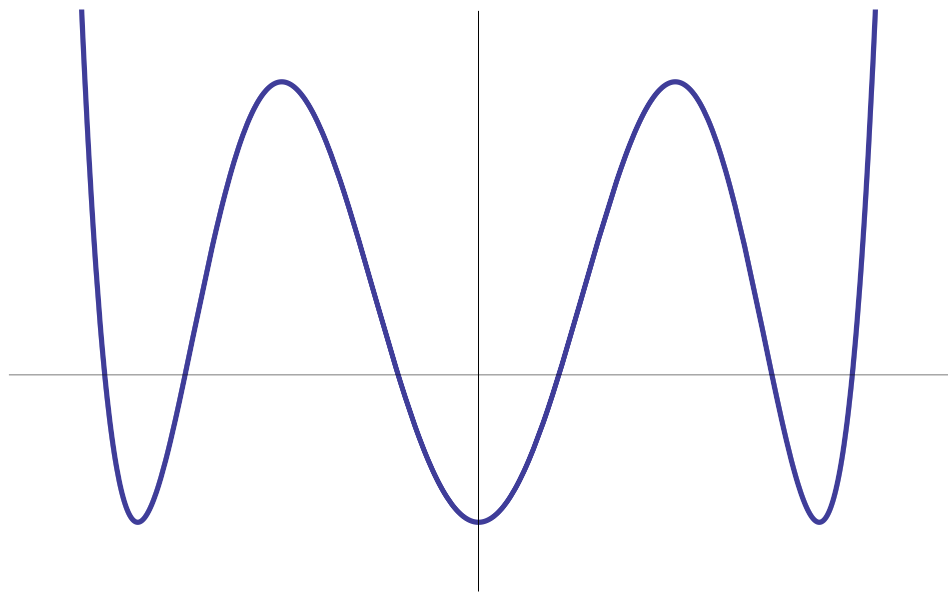
- Existence of analytic continuation to Euclidean space-time
- Osterwalder-Schräder Reflection positivity
- Existence of a Källén-Lehmann representation of the 2-point function

# Real Ghosts

$$G(k) = \left( \sum_{n=1}^{N} c_n \, k^{2n+2} \right)^{-1} = \sum_{n=1}^{N} \frac{a_n}{k^2 + m_n^2}$$

$$a_n = -a_{n+1}, m_n \leq m_{n+1}$$

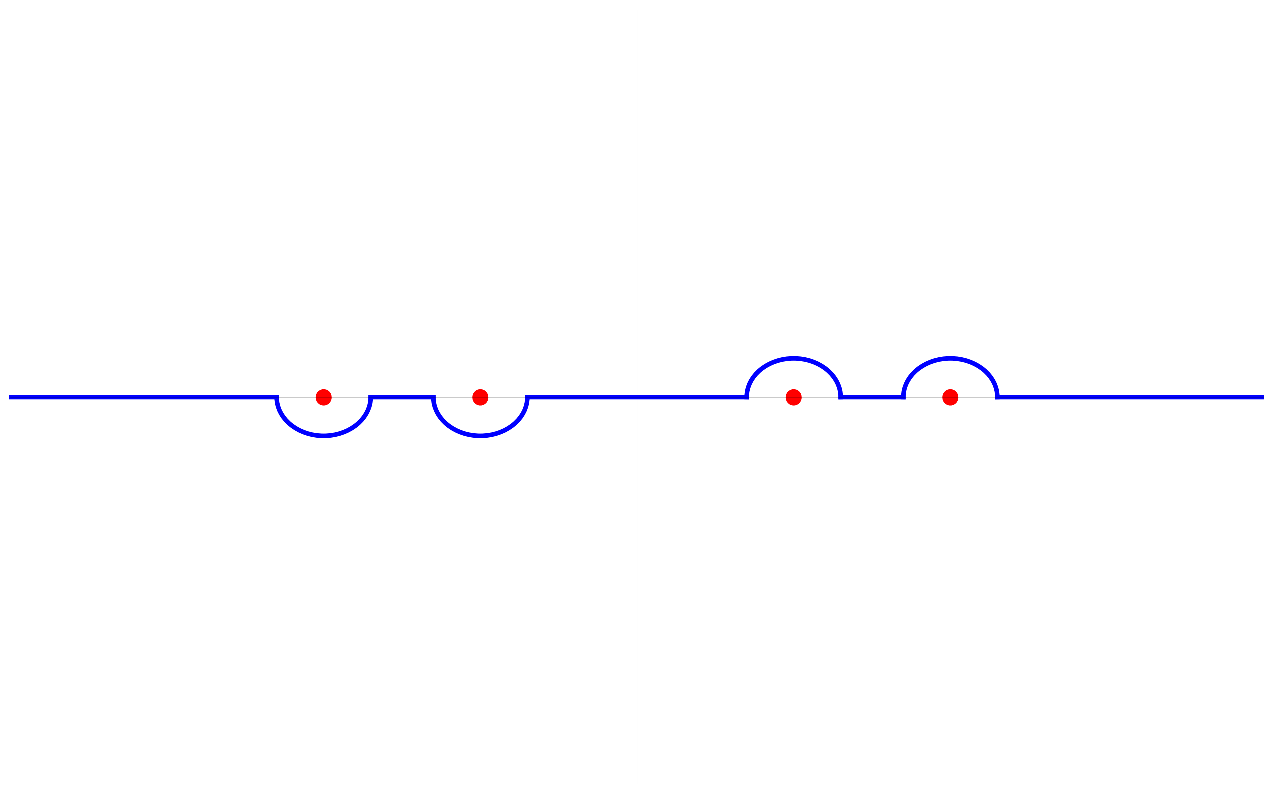
alternating signs



Negative signs corresponds to ghosts

# Real Ghosts

Analytic continuation to Euclidean time

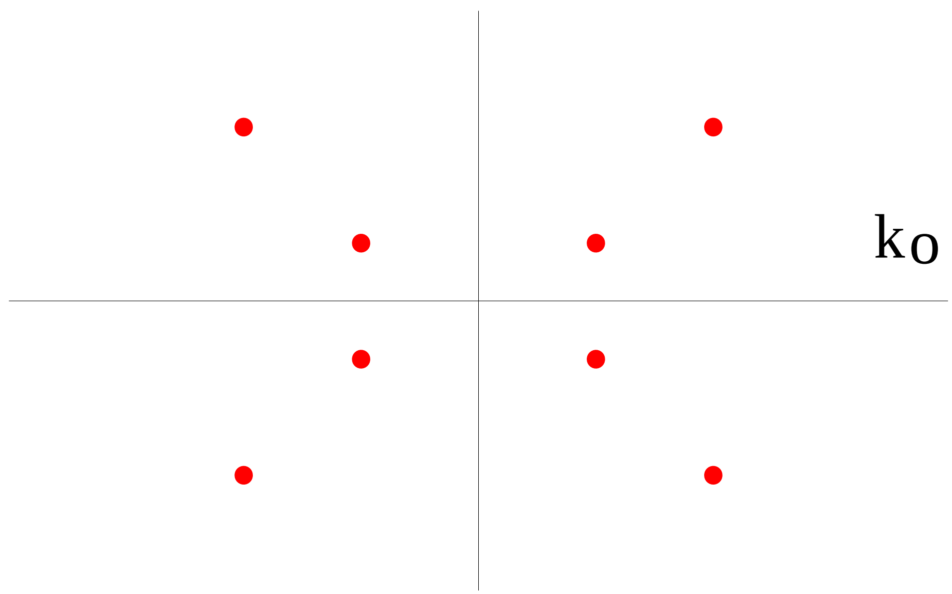


no reflection positivity

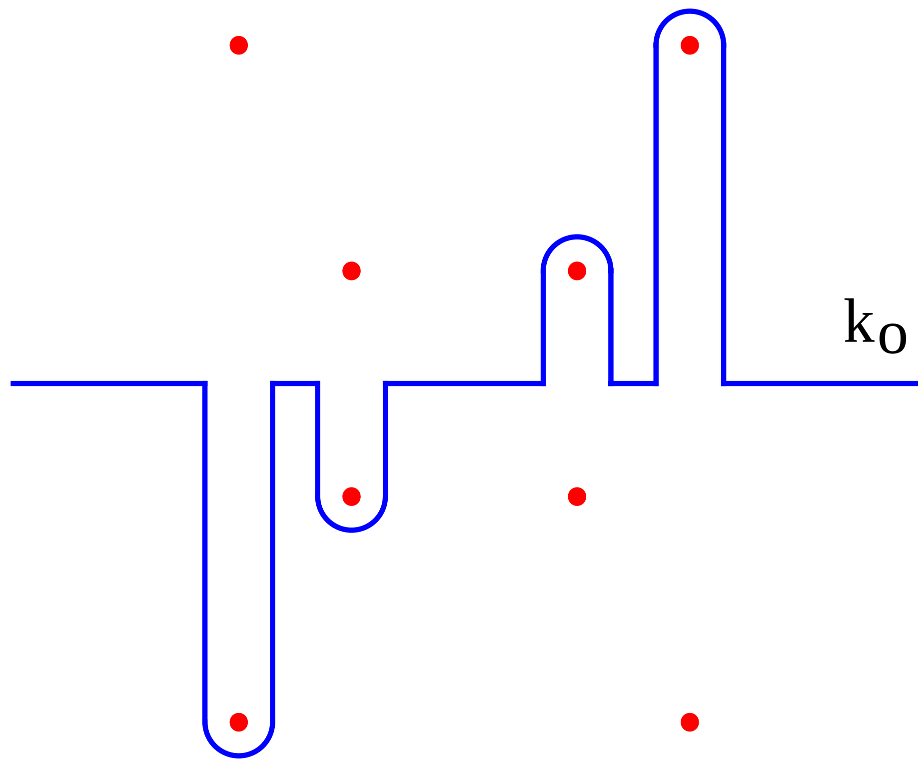
# Complex Ghosts

$$G(k) = \left( \sum_{n=1}^{N} c_n \, k^{2n+2} \right)^{-1} = \sum_{n=1}^{N} \frac{a_n}{k^2 + m_n^2} + \sum_{n=-M}^{M} \frac{b_n}{k^2 + \mu_n}$$

$$b_{-n} = b_n = b_n^*, \quad \mu_n^* = \mu_n$$



# Complex Ghosts




Violation of Poincaré invariance

# Transcendent Theories
## Ghosts Free

$$S_e(g) = \varkappa \int \sqrt{g} R + \lambda \int \sqrt{g}$$

$$+ \int \sqrt{g} \left( \alpha R_{\mu\nu\alpha\beta} \, e^{-\varepsilon \Box^n} R^{\mu\nu\alpha\beta} - \beta R_{\mu\nu} \, e^{-\varepsilon \Box^n} R^{\mu\nu} + \gamma R \, e^{-\varepsilon \Box^n} R \right)$$

- $n = 1$ has UV divergences in Minkowski space-time
- $n = 2$ leads to finite results both in Euclidean and Minkowski space-times. But ...

# Transcendent Theories
## Ghosts Free

# Reflection positivity

## Scalar field theories

$$S_4(g) = \int \phi^\dagger \left(\Box^2 - m^2\right) \phi$$

$$S_e(g) = \int \phi^\dagger \, e^{-\varepsilon \Box^n} \phi$$

$$S_{ep}(g) = \int \phi^\dagger \, e^{-\varepsilon \Box^n} \left(\Box - m^2\right) \phi$$

## Schwinger 2-p functions

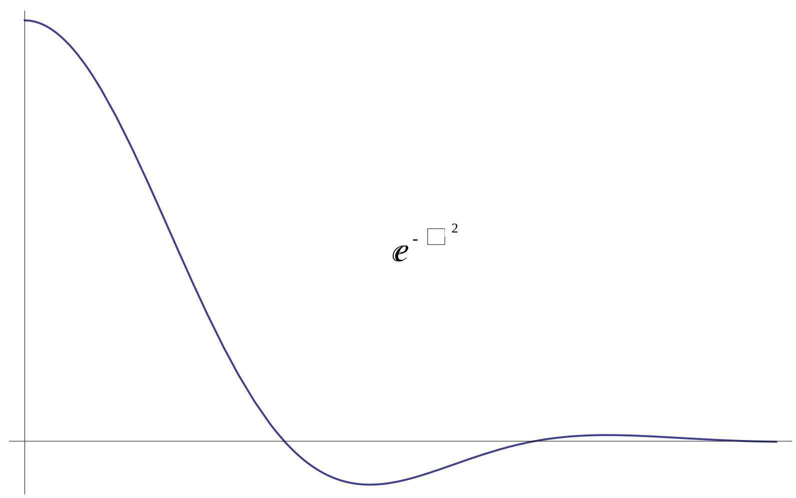
$$S_2(x, y) = \langle \phi^\dagger(x) \phi(y) \rangle$$

# Reflection positivity

$$S_{2\vartheta}(x,y) = \langle \vartheta\phi(x)\phi(y)\rangle = \langle \phi^\dagger(\vartheta x)\phi(y)\rangle$$

Unitarity implies that

$$S_{2\vartheta}(x,x) \geq 0$$

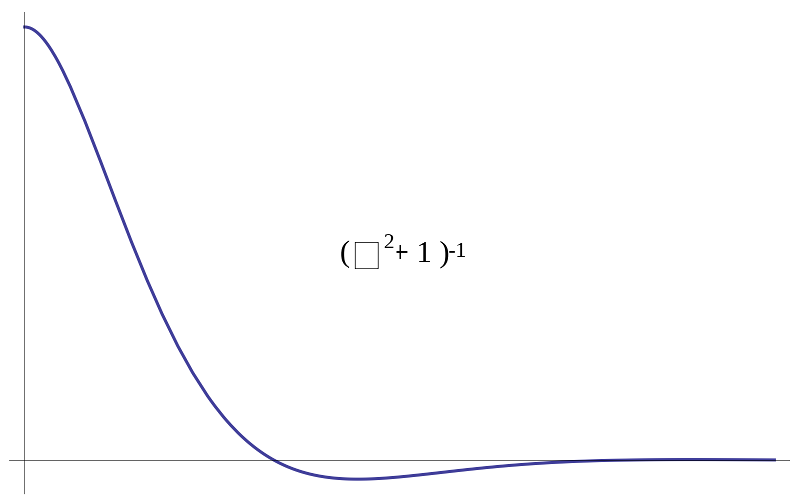


# Reflection positivity

$$S_{2\vartheta}(x,y) = \langle \vartheta\phi(x)\phi(y)\rangle = \langle \phi^\dagger(\vartheta x)\phi(y)\rangle$$

Unitarity implies that

$$S_{2\vartheta}(x,x) \geq 0$$



# Reflection positivity

$$S_{2\vartheta}(x,y) = \langle \vartheta\phi(x)\phi(y)\rangle = \langle \phi^\dagger(\vartheta x)\phi(y)\rangle$$

Unitarity implies that

$$S_{2\vartheta}(x,x) \geq 0$$

# Theorem for Exponential Kernels

$$\Delta(\tau) = S_{2\vartheta}(\mathbf{x}, \tau; \mathbf{x}, \tau) = \int \frac{d^4p}{(2\pi)^4} \frac{e^{-(p_0^2+\mathbf{p}^2)^s}}{p_0^2 + \mathbf{p}^2 + m^2}$$

Theorem:

- i) For $s \leq 1$ (packing n-sphere problem)

$$\Delta(\tau) = S_{2\vartheta}(x, x) \geq 0$$

- ii) For $s > 1$ there exist $\tau > 0$ such that

$$\Delta(\tau) = S_{2\vartheta}(x, x) < 0$$

# Reflection positivity

## Källén-Lehmann representation

$$S_2(k) = \int_0^\infty d\mu \frac{\rho(\mu)}{k^2 + \mu^2}$$

with

$$\rho(\mu) \geq 0$$

None of the transcendent kernels

$$K(x,y) = e^{-\varepsilon \Box^n}(x,y)$$

admits a Källén-Lehmann representation

[M. A., Leslaw Rachwał and Ilya Shapiro]

# CONCLUSIONS

- High time derivative theories are classically unstable
- Analytic continuation from Euclidean to Lorentzian space-time es problematic
- Ghost free theories are not necessarily unitary
- Polynomial and transcendent Euclidean theories are not reflection positive
- Källén-Lehmann representation is not possible for polynomial and transcendent theories
- Hořava-Lifshitz theories are not affected by these problems

# CONGRATULATIONS
# BAL