A Hilbert-Schmidt operatorial approach to study quantum mechanics and geometry of Non-commutative spaces

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Moyal plane: Hilbert-Schmidt formalism

Start with the Hiesenberg alg. in 2D Moyal plane $\mathbb{R}^2_*$

$$[\hat{x}_i, \hat{x}_j] = i\theta\epsilon_{ij} \ ; \ [\hat{x}_i, \hat{p}_j] = i\delta_{ij} \ ; \ [\hat{p}_i, \hat{p}_j] = 0 \ i, j \in \{1, 2\} \quad (1)$$

Auxiliary Hilbert space (furnishes a representation of just coordinate sub alg.)

$$\mathcal{H}_c = \text{Span}_{\mathbb{C}}\{ \, |n\rangle = \frac{(\hat{b}^\dagger)^n}{\sqrt{n!}} \, |0\rangle \, ; \hat{b} = \frac{(\hat{x}_1 + i\hat{x}_2)}{\sqrt{2\theta}}, [\hat{b}, \hat{b}^\dagger] = 1 \} \quad (2)$$

Quantum Hilbert space (furnishes a representation of the entire Heisenberg alg.)

$$\mathcal{H}_q = \{\psi(\hat{x}_1, \hat{x}_2) \mid \psi \in \mathcal{B}(\mathcal{H}_c), Tr_{\mathcal{H}_c}(\psi^\dagger \psi) < \infty\}, \quad (3)$$

with inner product

$$(\psi_1, \psi_2) := Tr_{\mathcal{H}_c}(\psi_1^\dagger \psi_2) \quad (4)$$

Moyal plane: Hilbert-Schmidt formalism

The action is given by

$$\hat{X}_i\psi = \hat{X}_i^L\psi = \hat{x}_i\psi;\ \hat{P}_i\psi = \frac{1}{\theta}\epsilon_{ij}[\hat{x}_j,\psi] = \frac{1}{\theta}(\hat{X}_i^L - \hat{X}_i^R)\psi. \quad (5)$$

Coordinate basis: Clearly $\nexists$ any common eigenstates for $\hat{x}_i$'s.
However, $\exists$ maximally localized coherent states:

$$|z\rangle = e^{-\bar{z}\hat{b}+z\hat{b}^\dagger}|0\rangle \in \mathcal{H}_c; z = \frac{1}{\sqrt{2\theta}}(x_1 + ix_2)\ ;\ \Delta x_1\Delta x_2 = \frac{\theta}{2} \quad (6)$$

Introduce the Voros basis : $|z) := |z\rangle\langle z| = \rho_z \in \mathcal{H}_q; \hat{B}|z) = z|z)$,

where ρ_z can be viewed as density matrix over $\mathcal{H}_c$.
Coordinate representation in this coherent state basis for $|\psi) \in \mathcal{H}_q$

$$(z|\psi) = \mathrm{Tr}_{\mathcal{H}_c}(|z\rangle\langle z|\psi) = \langle z|\psi|z\rangle \quad (7)$$

$\mathcal{H}_q$ as an Algebra

$\mathcal{H}_q$ is equipped with the structure of an algebra

$$m : \mathcal{H}_q \otimes \mathcal{H}_q \to \mathcal{H}_q;\ \ |\psi_1) \otimes |\psi_2) \mapsto |\psi_1\psi_2) \tag{8}$$

Coordinate representation of composite operator

$$(z|\psi_1\psi_2) = (z|\psi_1) *_v (z|\psi_2);\ \ *_v = e^{\overleftarrow{\partial_{\bar{z}}}\overrightarrow{\partial_z}} \text{is the Voros star product} \tag{9}$$

Momentum eigenstates: With Orthogonality and completeness relations

$$|\vec{p}) = \sqrt{\frac{\theta}{2\pi}} e^{i\vec{p}\cdot\hat{\vec{x}}};\ \ \hat{P}_i|\vec{p}) = p_i|\vec{p});\ \ {}_V(\vec{x}|\vec{p}) = \frac{1}{2\pi} e^{-\frac{\theta}{4}\vec{p}^2} e^{i\vec{p}\cdot\vec{x}}$$

$$(\vec{p'}|\vec{p}) = \delta^2(\vec{p'} - \vec{p}); \int d^2p|\vec{p})(\vec{p}| = \int d^2x|\vec{x})_V *_V {}_V(\vec{x}| = \mathbb{1}_{\mathcal{H}_q}$$

QM with space-time noncommutativity

$$\boxed{\text{QM in } 1+1 \text{ D} \;:\; [\hat{t},\hat{x}] = \theta;\;\; \theta > 0.}$$

Recall: Abstract Schrödinger equation ($\theta = 0$)

$$(\hat{P}_t + H)|\psi\rangle = 0,\;\; \text{where } \hat{P}_t + H = 0 \text{ (FCC)} \tag{10}$$

Take overlap with $|t,x\rangle$: $(\hat{t}|t,x\rangle = t|t,x\rangle$ and $\hat{x}|t,x\rangle = x|t,x\rangle)$

$$\langle t,x|\hat{P}_t|\psi\rangle = -i\frac{\partial}{\partial t}\psi(t,x); \langle t,x|\hat{P}_x|\psi\rangle = -i\frac{\partial}{\partial x}\psi(t,x), \tag{11}$$

with $\psi(t,x) := \langle t,x|\psi\rangle$ (formally).

$$(\psi|\phi)_t = \int \psi^*(t,x)\phi(t,x)dx \;;\;\; \psi(t,x) \in L^2(\mathbb{R}^1)$$

Continued...

Likewise, for $\theta \neq 0$, the Schrödinger equation is given by

$$(\hat{P}_t + H)|\psi) = 0, \text{ with } \hat{P}_t|\psi) = \frac{1}{\theta}[\hat{x}, \psi(\hat{t}, \hat{x})], \hat{P}_x|\psi) = -\frac{1}{\theta}[\hat{t}, \psi(\hat{t}, \hat{x})]$$

$\hat{T}\psi = \hat{t}\psi, \hat{X}\psi = \hat{x}\psi$. Initially, $\mathcal{H}_q$ is the space of HS operators

$$\mathcal{H}_q = \left\{ \psi(\hat{t}, \hat{x}) = \psi(\hat{b}, \hat{b}^\dagger);\ \hat{b} = \frac{\hat{t} + i\hat{x}}{\sqrt{2\theta}} \right\};\ (\psi|\phi) = \mathrm{Tr}_{\mathcal{H}_c}(\psi^\dagger \phi)$$

With $\hat{H} = \frac{\hat{P}_x^2}{2m} + V(\hat{T}, \hat{X})$, the abstract Schrödinger equation is

$$\frac{1}{2m\theta}[\hat{t}, [\hat{t}, \psi]] + [\hat{x}, \psi] + V(\hat{t}, \hat{x})\psi = 0 \tag{12}$$

Continuity equation : $[\hat{x}, \rho] + [\hat{t}, j] = 0$

$$\rho = \psi^\dagger \psi, j = \frac{1}{2m\theta}(\psi^\dagger[\hat{t}, \psi] - [\hat{t}, \psi^\dagger]\psi) \tag{13}$$

Effective commutative theory

Define analogue of position basis in Heisenberg picture as

$$|t,x)_V = \frac{1}{\sqrt{2\pi\theta}}|z); \ \int \frac{d^2 z}{\pi}|z) * (z| = \int dtdx|t,x) * (t,x| = 1$$

so that $\psi(x,t) := (t,x|\psi) = \frac{1}{\sqrt{2\pi\theta}}\langle z|\psi|z\rangle$ and

$$\rho(t,x) = \psi^*(t,x) *_V \psi(t,x) = \Sigma_{n=0}^{\infty}\frac{1}{n!}|\partial_z^n \psi(z,\bar{z})|^2 \geq 0$$

$\Rightarrow$ can be interpreted as probability density. Inner product

$$(\psi|\phi) = \int dtdx\psi^*(t,x) *_V \phi(t,x); (t',x'|t,x) = \delta^2_{\sqrt{\theta}}(\vec{x'} - \vec{x}),$$

where $\delta^2_{\sqrt{\theta}}(\vec{x'} - \vec{x}) = \delta_{\sqrt{\theta}}(t'-t)\delta_{\sqrt{\theta}}(x'-x)$ with $\vec{x} := (t,x)$.

Note that $\delta_\sigma(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{x^2}{2\sigma^2}}$; $\int dx\delta_\sigma(x) = 1$.

Continued...

These $\delta^2_{\sqrt{\theta}}(\vec{x'} - \vec{x})$ indeed satisfy

$$\int dt' dx' \delta^2_{\sqrt{\theta}}(\vec{x'} - \vec{x}) *' f(\vec{x'}) = f(\vec{x}) \tag{14}$$

"Induced" inner product at constant time slice:

Energy-momentum eigenstate

$$|p, E) = \sqrt{\frac{\theta}{2\pi}} e^{-i(E\hat{t} - p\hat{x})}; \; (t, x|p, E) = \frac{1}{2\pi} e^{-\frac{\theta}{4}(E^2+p^2)} e^{-i(Et-px)}$$

Introduce $\mathcal{P}_E := \int dp|p, E)(p, E|$; $\mathcal{P}_E \mathcal{P}_{E'} = \mathcal{P}_E \delta(E' - E)$, along with $|\psi)_E := \mathcal{P}_E|\psi)$ so that

$$\psi_E(t, x) = (t, x|\psi)_E = \frac{1}{\sqrt{2\pi}} \int dp e^{-i(Et-px)} e^{-\frac{\theta}{4}(E^2+p^2)} \psi_E(p)$$

Note: $|t + \tau, x) = e^{i\hat{H}\tau}|t, x)$, τ should not be identified with coordinate time i.e. an eigen-value of $\hat{t}$ (Doplicher, Bal)

Continued...

One checks that

$$\int dtdx\psi_E^*(t,x) * \phi_E(t,x) = \int dtdx\psi_E^*(x) * \phi_E(x)$$
$$= \int dEdp\psi_E^*(p)\phi_E(p) \to \infty$$

Hence define, $(\psi|\phi)_t := \int_t dx\psi^*(t,x) * \phi(t,x) \ \forall \ \psi, \phi \in \mathcal{H}_q(E)$

<u>Coordinate representation of $\hat{T}, \hat{X}, \hat{P}_t, \hat{P}_x$:</u>

$$(t,x|\hat{X}\psi(\hat{t},\hat{x})) = \hat{X}_\theta(t,x|\psi(\hat{t},\hat{x})) = \hat{X}_\theta\psi(t,x), \qquad (15)$$

where $\hat{X}_\theta = x + \frac{\theta}{2}(\partial_x - i\partial_t)$. Likewise,

$$\hat{T}_\theta = t + \frac{\theta}{2}(\partial_t + i\partial_x); \ \hat{P}_t = -i\partial_t \text{ and } \hat{P}_x = -i\partial_x$$

Schrödinger equation

Schrödinger equation (time dependent)

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2m}\frac{\partial^2\psi}{\partial x^2} + V(t,x)*\psi(t,x) = -\frac{1}{2m}\frac{\partial^2\psi}{\partial x^2} + V(\hat{T}_\theta, \hat{X}_\theta)\psi(t,x)$$

Time independent Schrödinger equation

$$-\frac{1}{2m}\frac{\partial^2\psi_E(t,x)}{\partial x^2} + V(x) * \psi_E(t,x) = E\psi_E(t,x) \qquad (16)$$

Applications

① Harmonic oscillator: $V(\hat{x}) = \frac{1}{2} m\omega^2 \hat{x}^2$.

$$\text{Schrödinger equation}: [\hat{x}, \hat{\psi}] = -\frac{1}{2m\theta}[\hat{t}, [\hat{t}, \hat{\psi}]] - \frac{\theta}{2} m\omega^2 \hat{x}^2 \hat{\psi} \quad (17)$$

Ansatz: $\hat{\psi}(\hat{t}, \hat{x}) = \int dEdp e^{E\hat{t} - p\hat{x}} \widetilde{\psi}(E, p)$

- $E_n = \left(n + \frac{1}{2}\right)\omega$, No change!
- Deformed $\psi(t, x)$ and $\rho(t, x)$. For example, in the ground state

$$\widetilde{\psi}_0(t, x) = e^{-iE_0 t} e^{-\frac{1}{2\sigma_\theta^2}\left(x - \frac{\theta E_0}{2}\right)^2};\ \sigma_\theta^2 = \frac{\theta}{2} + \frac{1}{m\omega}\ \text{(un-normalized)} \quad (18)$$

$$\rho(x) = \delta_{\widetilde{\sigma}_\theta}(x - \theta E_0);\ \widetilde{\sigma}_\theta^2 = \frac{1}{2}\sigma_\theta^2 \left(1 + \frac{\theta}{2\sigma_\theta^2}\right),\ \text{(in normalized and ‘t’ ind.)}$$

Continued...

- Shift in θE_0 is indicative of parity violation
- $\widetilde{\sigma}_\theta \to \sqrt{\theta}$ as $\omega \to \infty \Rightarrow$ squeezing below $\sim \sqrt{\theta}$ is not possible

② Expectation values and uncertainty relations:

$$\langle \mathcal{O}_\theta \rangle_t := \int dx \widetilde{\psi}_0^*(t,x) * \mathcal{O}_\theta \widetilde{\psi}_0(t,x);\ \Delta \mathcal{O}_\theta = \sqrt{\langle \mathcal{O}_\theta^2 \rangle - \langle \mathcal{O}_\theta \rangle^2}$$

- $\langle \hat{X}_\theta \rangle_t = 0; \langle \hat{T}_\theta \rangle_t = t;$
- $\Delta X_\theta = \frac{1}{\sqrt{2m\omega}}; \Delta T_\theta = \sqrt{\frac{\theta}{2}(1+\theta m\omega)};\ \Delta P_x = \sqrt{\frac{m\omega}{2}}$
- $\Delta X_\theta \Delta T_\theta = \frac{\theta}{2}\sqrt{1+\frac{1}{m\omega\theta}} \to \frac{\theta}{2}$ as $\omega \to \infty$ and $\Delta X_\theta \Delta P_x = \frac{1}{2}$.
- For coherent state ($|z)$) only $\Delta X_\theta \Delta T_\theta = \frac{1}{2}$

Continued...

③ Deformation in (i) Ehrenfest theorem:

$$\partial_t \langle \hat{X}_\theta \rangle_t = \frac{\langle \hat{P}_x \rangle_t}{m}; \;\; \partial_t \langle \hat{P}_x \rangle_t = -\left\langle \frac{\partial V(x)}{\partial x} \right\rangle_t - \frac{\theta}{2} \left\langle \frac{\partial^2 V}{\partial x^2} (\partial_x - i\partial_t) \right\rangle_t$$

Showing the presence of an additional force of noncommutative origin.

(ii) Fermi golden rule:

$$T_{i \to f} = \frac{P_{i \to f}}{T} = \frac{1}{T} \left| \int_0^t dt \left[\langle f | V(\hat{t}) | i \rangle + \frac{\theta}{2} \langle f | \frac{\partial V}{\partial t} (\partial_t + i\partial_x) | i \rangle \right] \right|^2$$

Geometry through spectral triple

1. For Riemannian manifold M:

$$(\mathcal{A}, \mathcal{H}, \mathcal{D}) = (C^\infty(M), L^2(M, S), -i\partial\!\!\!/) \tag{19}$$

Pure states are evaluation maps i.e. $\delta_{\vec{x}} : f \mapsto f(\vec{x}) \ \forall \ f \in C^\infty(M)$.
Connes distance:

$$d(\omega_1, \omega_2) = \sup_{a \in B} |\omega_1(a) - \omega_2(a)|; \ \ B = \{a \in \mathcal{A} : \|[\mathcal{D}, \pi(a)]\|_{op} \leq 1\}$$

gives the usual geodesic distance.

2. Spectral triple for Moyal plane $(\mathbb{R}^2_*)$

$\mathcal{D}_M = \sigma^1 P_1 + \sigma^2 P_2$ acts on $\Phi = \begin{pmatrix} |\phi_1) \\ |\phi_2) \end{pmatrix} \in \mathcal{H}_q \otimes \mathbb{C}^2$, by default

$$\Rightarrow [\mathcal{D}_M, \pi(a)]\Phi = \sqrt{\frac{2}{\theta}} \left[\begin{pmatrix} 0 & i\hat{b}^\dagger \\ -i\hat{b} & 0 \end{pmatrix}, \pi(a) \right] \Phi$$

Spectral triple of Moyal plane

Thus one identifies

$$\mathcal{D}_M = \sqrt{\frac{2}{\theta}} \begin{pmatrix} 0 & i\hat{b}^\dagger \\ -i\hat{b} & 0 \end{pmatrix} \xrightarrow{SO(2)} = \sqrt{\frac{2}{\theta}} \begin{pmatrix} 0 & \hat{b}^\dagger \\ \hat{b} & 0 \end{pmatrix}$$

This can also act on $\mathcal{H}_c \otimes \mathbb{C}^2$ from the left so that finally one has the spectral triple

$$\mathcal{A} = \mathcal{H}_q;\ \mathcal{H} = \mathcal{H}_c \otimes \mathbb{C}^2;\ \mathcal{D}_M = \sqrt{\frac{2}{\theta}} \begin{pmatrix} 0 & \hat{b}^\dagger \\ \hat{b} & 0 \end{pmatrix} \tag{20}$$

Pure states that we shall consider are

1. $\rho_m := |m\rangle\langle m|,\ m = 0, 1, 2...$ harmonic oscillator states
2. $\rho_z := |z\rangle\langle z| = |z),\ z \in \mathbb{C}$

Eigen-spinors of $\mathcal{D}_M$

$$|0\rangle\rangle := \begin{pmatrix} |0\rangle \\ 0 \end{pmatrix},\ |m\rangle\rangle_\pm := \frac{1}{\sqrt{2}} \begin{pmatrix} |m\rangle \\ \pm|m-1\rangle \end{pmatrix};\ m = 1, 2, 3, \ldots \tag{21}$$

Continued...

Eigen-value equation of $\mathcal{D}$:

$$\mathcal{D}\|m\rangle\rangle_\pm = \lambda^\pm_m |m\rangle\rangle_\pm ; \ \lambda^\pm_m = \pm\sqrt{\frac{2m}{\theta}}, \ \ m = 0, 1, 2, ...$$

along-with orthogonality : ${}_\pm\langle\langle m|n\rangle\rangle_\pm = \delta_{mn}; \quad {}_\pm\langle\langle m|n\rangle\rangle_\mp = 0$

as well as completeness relation

$$|0\rangle\rangle\langle\langle 0| + \sum_{m=1}^{\infty}\Big(|m\rangle\rangle_{+\,+}\langle\langle m| + |m\rangle\rangle_{-\,-}\langle\langle m|\Big) = \mathbb{1}_{\mathcal{H}_q\otimes M_2(\mathbb{C})} \quad (22)$$

We can introduce a projection operator:

$$\mathbb{P}_N = |0\rangle\rangle\langle\langle 0| + \sum_{n=1,\pm}^{N} |n\rangle\rangle_{\pm\,\pm}\langle\langle n| = \begin{pmatrix} P_N & 0 \\ 0 & P_{N-1} \end{pmatrix} \in \mathcal{H}_q \otimes M_2^d(\mathbb{C}), \quad (23)$$

where $P_N = \sum_{m=o}^{N} |m\rangle\langle m| \in \mathcal{H}_q$

Spectral triple of fuzzy sphere ($\mathbb{S}^2_*$)

Coordinate algebra : $[\hat{x}_i, \hat{x}_j] = i\lambda\epsilon_{ijk}\hat{x}_k$; $i, j, k = 1, 2, 3.$

$$\hat{\vec{x}}^2|n, n_3\rangle = r_n^2|n, n_3\rangle = \lambda n(n+1)|n, n_3\rangle;$$
$$\hat{x}_3|n, n_3\rangle = \lambda n_3|n, n_3\rangle$$

Auxiliary space for the entire $\mathbb{R}^3_*$;

$$\mathcal{H}_c = \bigoplus_n \mathcal{H}_c^{(n)} \ ; \ \ \mathcal{H}_c^{(n)} = Span\{|n, n_3\rangle \mid \text{n is fixed}, \ -n \leq n_3 \leq n\}$$

Quantum Hilbert space for the fuzzy sphere of radius r_n:

$$\mathcal{H}_q = \bigoplus_n \mathcal{H}_q^{(n)}; \ \mathcal{H}_q^{(n)} = Span\{|n, n_3\rangle\langle n, n_3'| \mid \text{n fixed}, \ -n \leq n_3, n_3' \leq n\}$$

Spectral triple is $\mathcal{A} = \mathcal{H}_q^{(n)}; \mathcal{H} = \mathcal{H}_c^{(n)} \otimes \mathbb{C}^2, \mathcal{D} = \frac{1}{r_n}\vec{J} \otimes \vec{\sigma}$

Continued...

Eigen-spinors:

$$|n, n_3\rangle\rangle_+ := f(n, n_3)\,|n, n_3\rangle \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + g(n, n_3)\,|n, n_3+1\rangle \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$|n, n_3'\rangle\rangle_- := -g(n, n_3')\,|n, n_3\rangle \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + f(n, n_3')\,|n, n_3+1\rangle \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

where $f(n, n_3) = \sqrt{\frac{n+n_3+1}{2n+1}}, \quad g(n, n_3) = \sqrt{\frac{n-n_3}{2n+1}}.$

- $\lambda^+_{n_3} = \frac{n}{r_n}\ , \quad -n-1 \leq n_3 \leq n,$ yielding (2n+2)-fold degeneracy
- $\lambda^-_{n_3'} = -\frac{(n+1)}{r_n}\ , \quad -n \leq n_3' \leq n-1,$ yielding 2n-fold degeneracy.

Perelemov coherent state

$$|z\rangle = e^{\frac{\theta}{2\lambda}(\hat{x}_- - \hat{x}_+)}|n, n\rangle \quad (\varphi = 0)$$

$|z| = \tan\left(\frac{\theta}{2}\right)$ is stereographically projected coordinate.

Connes spectral distance

$$d(\omega_1,\omega_2) = \sup_{a\in B} |\omega_1(a) - \omega_2(a)|;\ \ B = \{a \in \mathcal{A} : \|[\mathcal{D},\pi(a)]\|_{op} \leq 1\}$$

Also $\left|\omega(a) - \omega'(a)\right| = |\mathrm{Tr}((\rho_\omega - \rho_{\omega'})a)| = |(\Delta\rho, a)|; \Delta\rho \in \mathcal{H}_q = \mathcal{A}$

- We take ω, ω' to be normal states, so that they can be represented by density matrices $\omega \to \rho_\omega$
- Let $V_0 = \{a \in \mathcal{A} : \|[\mathcal{D},\pi(a)]\|_{op} = 0\}$, then $\omega(a) - \omega'(a) = 0$, $\forall\ a \in V_0$, (certain irreducibility condition)
- The optimal element a_s should attain the supremum value:

$$d(\omega,\omega') = |\omega(a_s) - \omega'(a_s)|;\ \ \|[\mathcal{D},\pi(a_s)]\|_{op} = 1$$

Towards an algorithm to compute finite distances

$$d(\rho,\rho') = N\|\Delta\rho\|_{\text{tr}}^2; \;\; N = \frac{1}{\inf\limits_{\Delta\rho_\perp} \|[\mathcal{D},\pi(\Delta\rho)] + [\mathcal{D},\Delta\rho_\perp]\|_{\text{op}}} \tag{24}$$

A lower bound is reached when $a_s \propto \Delta\rho$

$$d(\rho,\rho') \geq \frac{\|\Delta\rho\|_{\text{tr}}^2}{\|[\mathcal{D},\pi(\Delta\rho)]\|_{op}}; \text{ where } a_s = \frac{\Delta\rho}{\|[\mathcal{D},\pi(\Delta\rho)]\|_{op}} \tag{25}$$

In the following we shall be computing distances between pure states given by coherent states and the discrete states.

Distances on $\mathbb{S}^2_*$ (discrete basis)

Infinitesimal distance (In n representation): For $\rho_n := |n_3\rangle\langle n_3|$

$$d_n(\omega_{n_3+1}, \omega_{n_3}) = \sup_{a \in B} \left|\mathrm{tr}(\rho_{n_3+1} a) - \mathrm{tr}(\rho_{n_3} a)\right|$$

$$\leq \frac{\|[J_-, a]\|_{op}}{\sqrt{n(n+1) - n_3(n_3+1)}} \text{ (By Bessels Inequality)}$$

$$\leq \frac{r_n}{\sqrt{n(n+1) - n_3(n_3+1)}} \; (By \; \|[J_\pm, a]\|_{op} \leq r_n)$$

This is also the lower bound! $[\mathcal{D}, \pi(d\rho)] = \frac{1}{r_n}\left(\begin{array}{c|c} 0 & A \\ \hline -A^\dagger & 0 \end{array}\right)$

$$A = \begin{pmatrix} -\sqrt{n(n+1) - n_3(n_3-1)} & 0 & 0 \\ 0 & 2\sqrt{n(n+1) - n_3(n_3+1)} & 0 \\ 0 & 0 & -\sqrt{n(n+1) - (n_3+1)(n_3+2)} \end{pmatrix}$$

Continued...

$\Rightarrow \|[\mathcal{D}, \pi(d\rho)]\|_{op} = \frac{2}{r_n}\sqrt{n(n+1) - n_3(n_3+1)}$.
Further $\mathrm{Tr}(d\rho)^2 = 2$, which yields

$$d_n(\omega_{n_3+1}, \omega_{n_3}) = \frac{\lambda\sqrt{n(n+1)}}{\sqrt{n(n+1) - n_3(n_3+1)}}. \qquad (26)$$

<u>Finite distance ($m_3 - n_3 \geq 2$):</u>

$$\begin{aligned} d_n(\omega_{m_3}, \omega_{n_3}) &= \sup_{a\in B} \left|\mathrm{tr}(\rho_{n_3+k}a) - \mathrm{tr}(\rho_{n_3}a)\right| \quad ; \quad \text{where } k = m_3 - n_3 \\ &= \sup_{a\in B} \left|\sum_{i=1}^{k} \mathrm{tr}\left((\rho_{n_3+i} - \rho_{n_3+(i-1)}), a\right)\right| \\ &\leq \sum_{i=1}^{k} \frac{r_n}{\sqrt{n(n+1) - (n_3+i)(n_3+i-1)}}. \end{aligned}$$

Continued...

Again this upper bound is reached by

$$a_s = \sum_{p=n_3}^{m_3-1} \left(\sum_{i=1}^{m_3-p} \frac{r_n}{\sqrt{n(n+1)-(p+i)(p+i-1)}} |p\rangle\langle p| \right)$$

yielding $$d_n(\omega_{m_3}, \omega_{n_3}) = \sum_{i=1}^{k} \frac{r_n}{\sqrt{n(n+1) - (n_3+i)(n_3+i-1)}}. \tag{27}$$

Here the triangle inequality is saturated as

$$d_n(\omega_{m_3}, \omega_{n_3}) = d_n(\omega_{m_3}, \omega_{l_3}) + d_n(\omega_{l_3}, \omega_{n_3}) \quad \text{for } n_3 \leq l_3 \leq m_3$$

In particular, $d_n(\mathrm{N}, \mathrm{S}) = d_n(\rho_n, \rho_{-n}) = \sum_{k=1}^{2n} \frac{r_n}{\sqrt{k(2n+1-k)}}$.

Examples:

$d_{1/2}(\mathrm{N}, \mathrm{S}) = r_{1/2}$; $d_1(\mathrm{N}, \mathrm{S}) = \sqrt{2}\, r_1$; $d_{3/2}(\mathrm{N}, \mathrm{S}) = \left(\frac{1}{2} + \frac{2\sqrt{3}}{3}\right) r_{3/2}$.

Only in the limit $n \to \infty$ one gets $\lim_{n\to\infty} \frac{d_n(\mathrm{N},\mathrm{S})}{r_n} = \pi$

Distances on $\mathbb{S}^2_*$ (coherent state basis)

Upper bound of finite distance:
Introduce a one parameter family of pure states

$$\rho_\theta \equiv |\theta\rangle\langle\theta| = U_F(\theta)|n\rangle\langle n|U_F^\dagger(\theta) \in \mathcal{H}_n \; ; \; U_F(\theta) = e^{-i\theta J_2} \qquad (28)$$

- In terms of stereographic variable z, $\rho_z = \rho_\theta$;
- $\omega_z(a) = tr\left(\rho_z a\right);\; a^\dagger = a \in \mathcal{H}_q^{(n)}$
- Define $W(t) = \omega_{zt}(a) = tr\left(\rho_{zt} a\right)$, with $t \in [0,1]$ then

$$|\omega_z(a) - \omega_0(a)| = \left|\int_0^1 \frac{\mathrm{d}W(t)}{\mathrm{d}t} dt\right| \leq \int_0^1 \left|\frac{\mathrm{d}W(t)}{\mathrm{d}t}\right| dt \leq r_n\theta. \qquad (29)$$

The RHS is the geodesic distance of commutative sphere.
And $\nexists$ any $a \in \mathcal{A} = \mathcal{H}_q^{(n)}$ (for n-finite) saturating the upper bound.

Towards an actual computation

Ball condition in eigen-spinor basis:

$$[\mathcal{D}, \pi(a)] = \frac{1}{r_n} \left(\begin{array}{c|c} 0_{(2n+2)\times(2n+2)} & A_{(2n+2)\times 2n} \\ \hline -\ A^{\dagger}_{2n\times(2n+2)} & 0_{(2n)\times(2n)} \end{array} \right), \qquad (30)$$

where $A_{(2n+2)\times 2n} = (2n+1)_+ \langle\langle n, n_3 | \pi(a) | n, n_3' \rangle\rangle_-$ with $-n-1 \leq n_3 \leq n$ and $n-1 \leq n_3' \leq n-1$. Rectangular null matrices stem from the degeneracy of the spectrum. $\Rightarrow$

$$\|[\mathcal{D}, \pi(a)]\|^2_{\text{op}} = \|[\mathcal{D}, \pi(a)]^{\dagger}[\mathcal{D}, \pi(a)]\|_{\text{op}} = \frac{1}{r_n^2}\|AA^{\dagger}\|_{\text{op}} = \frac{1}{r_n^2}\|A^{\dagger}A\|_{\text{op}}.$$

Clearly, it is convenient to deal with $\|AA^{\dagger}\|_{\text{op}}$ as it is of lower dimension $(2n \times 2n)$.

$n = \frac{1}{2}$ fuzzy sphere

The algebra element can be taken to be element of $su(2)$ algebra. $a = \vec{a} \cdot \vec{\sigma} \in su(2)$; $\vec{a} \in \mathbb{R}^3$. Here $A^\dagger A$ is just a number.

$$\|[\mathcal{D}, \pi(a)]\|_{op} = \frac{2}{r_{1/2}} |\vec{a}| \leq 1 \Rightarrow |\vec{a}| \leq \frac{r_{1/2}}{2}, \text{ a solid sphere}$$

Take two states $\rho_N = \rho_{\theta=0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, and

$$\rho_\theta = U(\theta) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} U^\dagger(\theta) = \frac{1}{2} \begin{pmatrix} 1 + \cos\theta & \sin\theta \\ \sin\theta & 1 - \cos\theta \end{pmatrix}$$

$$U(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \Rightarrow \Delta\rho^\dagger = \Delta\rho = \rho_\theta - \rho_0 \in su(2);$$

$\mathrm{Tr}(\Delta\rho) = 0$. Thus $\Delta\rho = \overrightarrow{\Delta\rho} \cdot \vec{\sigma}$; $\overrightarrow{\Delta\rho} = \frac{1}{2}(\sin\theta, 0, \frac{\cos\theta - 1}{2}) \in \mathbb{R}^3$

Continued...

Finally

$$d_{\frac{1}{2}}(\omega_\theta, \omega_0) = \sup_{|\vec{a}| \leq \frac{r_{1/2}}{2}} |\omega_\theta(a) - \omega_0(a)| = \sup_{|\vec{a}| \leq \frac{r_{1/2}}{2}} \left| \mathrm{Tr}_{\mathcal{H}_q^{(n)}}(\Delta\rho a) \right|$$

$= \sup_{|\vec{a}| \leq \frac{r_{1/2}}{2}} \left| 2\vec{a} \cdot \vec{\Delta\rho} \right|$ and the supremum is reached when $\vec{a} \propto \vec{\Delta\rho}$

$$d_{\frac{1}{2}}(\omega_{\theta_0}, \omega_0) = r_{\frac{1}{2}} \sqrt{(\Delta\rho)_1^2 + (\Delta\rho)_3^2} = r_{\frac{1}{2}} \sin \frac{\theta_0}{2}, \text{No role for} \Delta\rho_\perp$$

A family of $\rho_t = (1-t)\rho_0 + t\rho_\theta;\ 0 \leq t \leq 1$ of mixed states can be thought of interpolating ρ_0 and ρ_θ.
$d_{1/2}(\rho_0, \rho_t) = tr_{1/2} \sin\left(\frac{\theta}{2}\right)$ and $d_{1/2}(\rho_t, \rho_\theta) = (1-t) r_{1/2} \sin\left(\frac{\theta}{2}\right)$ satisfying

$$d(\rho_0, \rho_t) + d(\rho_t, \rho_\theta) = d(\rho_0, \rho_\theta) \tag{31}$$

$n = 1$ fuzzy sphere

Here $\|[\mathcal{D}, \pi(a)]\|_{op} = \frac{1}{r_n}\sqrt{\|A^\dagger A\|_{op}}$. Writing $M := A^\dagger A = \begin{pmatrix} M_{11} & M_{12} \\ M_{12}^* & M_{22} \end{pmatrix}$, with matrix entries

$$M_{11} = 3|a_{0,1}|^2 + 2(a_{0,0} - a_{1,1})^2 + |a_{0,-1} - 2a_{1,0}|^2 + 6|a_{1,-1}|^2,$$
$$M_{22} = 3|a_{0,-1}|^2 + 2(a_{0,0} - a_{-1,-1})^2 + |a_{1,0} - 2a_{0,-1}|^2 + 6|a_{1,-1}|^2,$$
$$M_{12} = \sqrt{2}\big[3a_{1,-1}(a_{0,1} + a_{-1,0}) + (a_{0,0} - a_{1,1})(2a_{0,-1} - a_{1,0}) + (a_{0,0} - a_{-1,-1})(2a_{1,0} - a_{0,-1})\big]$$

The two eigen-values are $E_\pm = \frac{1}{2}\left((M_{11} + M_{22}) \pm \sqrt{(M_{11} - M_{22})^2 + 4|M_{12}|^2}\right)$.
Clearly, $E_+ > E_- \ \forall\ a \in B$ which means $\inf_{a\in B}\|[\mathcal{D}, \pi(a)]\|_{\mathrm{op}} = \frac{1}{r_1}\sqrt{\min(E_+)}$; $a = \Delta\rho + \Delta\rho_\perp$

Continued...

- Writing $a_s = \Delta\rho + \Delta\rho_\perp \in su(3)$

  with $\Delta\rho = e^{i\theta\hat{J}_2}|1\rangle\langle 1|e^{-i\theta\hat{J}_2} - |1\rangle\langle 1|$.
- We write $\Delta\rho_\perp = \sum_{i=1}^{8} c_i\lambda_i$; $\lambda_i' s$ are Gell-Mann matrices.
- But orthogonality condition $(\Delta\rho, \Delta\rho_\perp) = 0$ leaves us with 7 independent parameters.

- On computation, the distances for various angles of θ gives the following table obtained numerically and compared with $d_1^* := \sqrt{2}r_1 \sin\left(\frac{\theta}{2}\right))$

Data set for various distances corresponding to different angles

Angle (degree)	d_1^*/r_1	d_1/r_1
10	0.1232568334	0.1232518539
20	0.2455756079	0.2455736891
30	0.3660254038	0.3660254011
40	0.4836895253	0.4836894308
50	0.5976724775	0.5976724773
60	0.7071067812	0.7071067811
70	0.8111595753	0.8111595752
80	0.9090389553	0.9090389553
90	1	0.9999999998
100	1.0833504408	1.0833504407
110	1.1584559307	1.1584559306
120	1.2247448714	1.2247448713
130	1.2817127641	1.2817127640
140	1.3289260488	1.3289260487
150	1.3660254038	1.3660254037
160	1.3927284806	1.3927284806
170	1.4088320528	1.4088320527

Moyal plane $\mathbb{R}^2_*$

• Upper bound $d(\rho_0, \rho_z) \leq \sqrt{2\theta}|z|$; $\rho_z = |z\rangle\langle z|$ is obtained by considering a 1-parameter family of pure states $\rho_{zt} = |zt\rangle\langle zt|$; $0 \leq t \leq 1$, interpolating ρ_0 and ρ_z.

• In contrast to fuzzy sphere, this upper bound is reached by $a_s = \sqrt{\frac{\theta}{2}}\left(be^{-i\alpha} + b^\dagger e^{i\alpha}\right) \in$ Multiplier algebra.

• It's enough to show $d(\rho_0, \rho_{dz}) = \sqrt{2\theta}|dz|$ (trans. inv.)

by taking $d\rho = |dz\rangle\langle dz| - |0\rangle\langle 0| = d\bar{z}|0\rangle\langle 1| + dz|1\rangle\langle 0|$.

• $\pi(d\rho) = \begin{pmatrix} d\rho & 0 \\ 0 & d\rho \end{pmatrix}$ is a 5D matrix spanned by $|0\rangle\rangle, |1\rangle\rangle_\pm, |2\rangle\rangle_\pm$.

• Also, $\|[\mathcal{D}_M, \mathbb{P}_N \pi(a_s)]\mathbb{P}_N\|_{op} = 1$ with $N \geq 2$

Distances on doubled Moyal plane

Spectral triple:

$\mathcal{A}_T = \mathcal{H}_q \otimes M_2^d(\mathbb{C}), \mathcal{H}_T = (\mathcal{H}_c \otimes \mathbb{C}^2) \otimes \mathbb{C}^2, \mathcal{D}_T = \mathcal{D}_M \otimes \mathbb{1}_2 + \sigma_3 \otimes \mathcal{D}_2.$

Pure states: $\Omega_i^{(z)} = \rho_z \otimes \omega_i; \omega_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \omega_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$

• One can construct orthonormal eigen-spinors for $\mathcal{D}_T$ and verify Pythagoras theorem, reproducing earlier results of

$$d_t\left(\Omega_1^{(z)}, \Omega_2^{(z)}\right) = \frac{1}{|\Lambda|},\ d_l\left(\Omega_i^{(z)}, \Omega_i^{(0)}\right) = d_M(\rho_z, \rho_0) = \sqrt{2\theta}|z|$$

$$\left\{d_h(\rho_0 \otimes \omega_1, \rho_z \otimes \omega_2)\right\}^2 = \left\{d_t(\rho_0 \otimes \omega_1, \rho_0 \otimes \omega_2)\right\}^2 + \left\{d_l(\rho_0 \otimes \omega_1, \rho_z \otimes \omega_1)\right\}^2$$

Impact of Higgs field

Change the triplet $T \to \tilde{T}$ where

$$\widetilde{\mathcal{H}}_T = (\mathcal{H}_q \otimes M_2(\mathbb{C})) \otimes M_2^d(\mathbb{C}) \ni \widetilde{\Psi}; \widetilde{\mathcal{D}}_T \widetilde{\Psi} = \mathcal{D}_T \widetilde{\Psi} + \widetilde{\Psi} \mathcal{D}_T$$

so that the Dirac operator can be fluctuated. This gives rise to gauge fields, along with Higgs field $\mathcal{D}_T \to \mathcal{D}_T + H$;

$$H = c\sigma_3 \otimes a_2[\mathcal{D}_2, b_2]; \; c = ab \in \mathcal{H}_q \tag{32}$$

Under suitable condition $[c, \rho_z] = 0$ this gives rise to variation in the transverse distance.

$$d_t(\rho_z \otimes \omega_1, \rho_z \otimes \omega_2) = \frac{1}{|\Lambda(x_1, x_2)|} \tag{33}$$

Dissipation from Noncommutativity?

Consider the pair

$$\ddot{x} + \gamma\dot{x} + \omega^2 x = 0 \qquad (34)$$

$$\ddot{y} - \gamma\dot{y} + \omega^2 y = 0 \qquad (35)$$

together they define a closed system (**Bateman oscillator**) described by the Lagrangian

$$L = \dot{x}\dot{y} + \frac{\gamma}{2}(x\dot{y} - \dot{x}y) - \omega^2 xy \qquad (36)$$

where x is the D.H.O coordinate and y corresponds to its time-reversed counterpart. Under a coordinate transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad (37)$$

so that L can be written as

$$L = \frac{1}{2} g_{ij}\dot{x}_i\dot{x}_j - \frac{\gamma}{2}\epsilon_{ij}x_i\dot{x}_j - \frac{\omega^2}{2} g_{ij}x_i x_j \ ; \ g = \mathrm{diag}(1, -1) \qquad (38)$$

Lagrangian and Hamiltonian formulation of the problem

Correspondingly,

$$H = \frac{1}{2}(p_1 - \frac{\gamma x_2}{2})^2 - \frac{1}{2}(p_2 + \frac{\gamma x_1}{2})^2 + \frac{1}{2}\omega^2(x_1^2 - x_2^2) = H_1 - H_2$$

where,

$$H_1 = \frac{1}{2}(p_1 - \frac{\gamma x_2}{2})^2 + \frac{1}{2}\omega^2 x_1^2$$

and

$$H_2 = \frac{1}{2}(p_2 + \frac{\gamma x_1}{2})^2 + \frac{1}{2}\omega^2 x_2^2$$

This is not clearly bounded from below.

The problem was addressed by Vitiello, Blasone, t'hooft.

A Different Approach

By augmenting this with mass η and spring constant ϵ, s.t. the pair of equations are now coupled:

$$\ddot{x} + \gamma\dot{x} + \omega^2 x = -\epsilon y - \eta\ddot{y} \quad (39)$$

$$\ddot{y} - \gamma\dot{y} + \omega^2 y = -\epsilon x - \eta\ddot{x} \quad (40)$$

The corresponding Lagrangian and Hamiltonian written in terms of the (x_1, x_2) coordinates are given by-

$$L = \frac{(\eta+1)}{2}\dot{x}_1^2 + \frac{(\eta-1)}{2}\dot{x}_2^2 - \frac{\gamma}{2}(x_1\dot{x}_2 - x_2\dot{x}_1) - \frac{(\epsilon+\omega^2)}{2}x_1^2 - \frac{(\epsilon-\omega^2)}{2}x_2^2$$

$$H = \frac{p_1^2}{2(\eta+1)} + \frac{p_2^2}{2(\eta-1)} + \frac{\gamma}{2}\left(\frac{x_1 p_2}{\eta-1} - \frac{x_2 p_1}{\eta+1}\right) + \left(\frac{\gamma^2}{8(\eta-1)} + \frac{(\epsilon+\omega^2)}{2}\right)x_1^2 + \left(\frac{\gamma^2}{8(\eta+1)} + \frac{(\epsilon-\omega^2)}{2}\right)x_2^2$$

The positive definiteness of the Hamiltonian can now be ensured if we demand $\eta > 1$ and $\epsilon > \omega^2$.

Continued...

Carrying out a canonical transformation:

$$x_1 \longrightarrow \left(\frac{\eta+1}{\eta-1}\right)^{\frac{1}{4}} x_1 \ , \quad p_1 \longrightarrow \left(\frac{\eta-1}{\eta+1}\right)^{\frac{1}{4}} p_1$$

$$x_2 \longrightarrow \left(\frac{\eta-1}{\eta+1}\right)^{\frac{1}{4}} x_2 \ , \quad p_2 \longrightarrow \left(\frac{\eta+1}{\eta-1}\right)^{\frac{1}{4}} p_2 \quad .$$

the resulting H is

$$H = \frac{p_1^2}{2\mu} + \frac{p_2^2}{2\mu} + \frac{\gamma}{2\mu}(x_1 p_2 - x_2 p_1) + \frac{1}{2}\mu\omega_1^2 x_1^2 + \frac{1}{2}\mu\omega_2^2 x_2^2 \quad (41)$$

where $\mu = \sqrt{(\eta+1)(\eta-1)}$ can be regarded as the new mass parameter and the frequencies are given by -

$\omega_1^2 = \frac{\gamma^2}{4(\eta^2-1)} + \frac{\epsilon+\omega^2}{\eta+1} \ , \quad \omega_2^2 = \frac{\gamma^2}{4(\eta^2-1)} + \frac{\epsilon-\omega^2}{\eta-1} \quad .$

Putting the system in the ambient Moyal plane

We now place the system in Moyal plane where we have,

$$[\hat{x}_1, \hat{x}_2] = [\hat{y}, \hat{x}] = i\theta \tag{42}$$

Carrying out the path integral in the Voros basis, the action S:

$$S = \int_{t_0}^{t_f} dt \, \Bigg[\frac{\theta}{2} \left\{ \dot{\bar{z}}(t) + \frac{i\gamma}{2\mu}\bar{z}(t) \right\} \left(\frac{1}{2\mu} + \frac{i\theta}{2\hbar}\partial_t \right)^{-1} \left\{ \dot{z}(t) - \frac{i\gamma}{2\mu} z(t) \right\}$$
$$- \frac{\mu\theta}{2} \left(\omega_1^2 + \omega_2^2\right) \bar{z}(t) z(t) - \frac{\mu\theta}{4} (\omega_1^2 - \omega_2^2) \left(z^2(t) + \bar{z}^2(t) \right) \Bigg] \tag{43}$$

resulting in the pair of equations of motion–

$$\ddot{x}_1 + \left\{ \frac{\gamma}{\mu} - \frac{\mu\theta}{\hbar}\omega_2^2 \right\} \dot{x}_2 + \left\{ \omega_1^2 - \frac{\gamma^2}{4\mu^2} \right\} x_1 = 0 \quad . \tag{44}$$

$$\ddot{x}_2 - \left\{ \frac{\gamma}{\mu} - \frac{\mu\theta}{\hbar}\omega_1^2 \right\} \dot{x}_1 + \left\{ \omega_2^2 - \frac{\gamma^2}{4\mu^2} \right\} x_2 = 0 \quad . \tag{45}$$

Restoring the Bateman form

- Finally, on re-writing the equations (44) and (45) in terms of the original coordinates x, y, and then taking the limit $\eta = 0 = \epsilon$, we get-

$$\ddot{x} + (\gamma + \frac{\theta\omega^2}{\hbar} - \frac{\theta\gamma^2}{4\hbar})\dot{x} + \omega^2 x = 0 \quad . \tag{46}$$

and,

$$\ddot{y} - (\gamma + \frac{\theta\omega^2}{\hbar} - \frac{\theta\gamma^2}{4\hbar})\dot{y} + \omega^2 y = 0 \quad . \tag{47}$$

Comments

- The result indicates $\gamma_R = (\gamma + \frac{\theta\omega^2}{\hbar} - \frac{\theta\gamma^2}{4\hbar})$ is non-zero even if $\gamma = 0$ to begin with (a purely noncommutative effect). Therefore, noncommutativity can lead to dissipation!
- On the other hand, a solution of θ can be found so that $\gamma_R = 0$.
- Does this suggest a duality? Commutative dissipative system can be mapped to a NC non-dissipative system.

The presented works are mainly based on the following papers:

- Partha Nandi, Sayan K. Pal, Aritra N. Bose, B.C., *Ann. Phys.* 386, 305 (2017).
- *"Revisiting Connes' finite spectral distance on noncommutative spaces: Moyal plane and fuzzy sphere"*, Y. Chaoba Devi, A. Patil, A.N. Bose, K. Kumar, F.G. Scholtz, B.C., *arXiv: 1608.05270*.
- K.Kumar, B.C., *Phys. Rev. D* 97, 086019 (2018).
- *"Connecting dissipation and noncommutativity: A Bateman system case study"*, S.K. Pal, P. Nandi, B.C., *arXiv: 1803.03334*.

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Thank You!