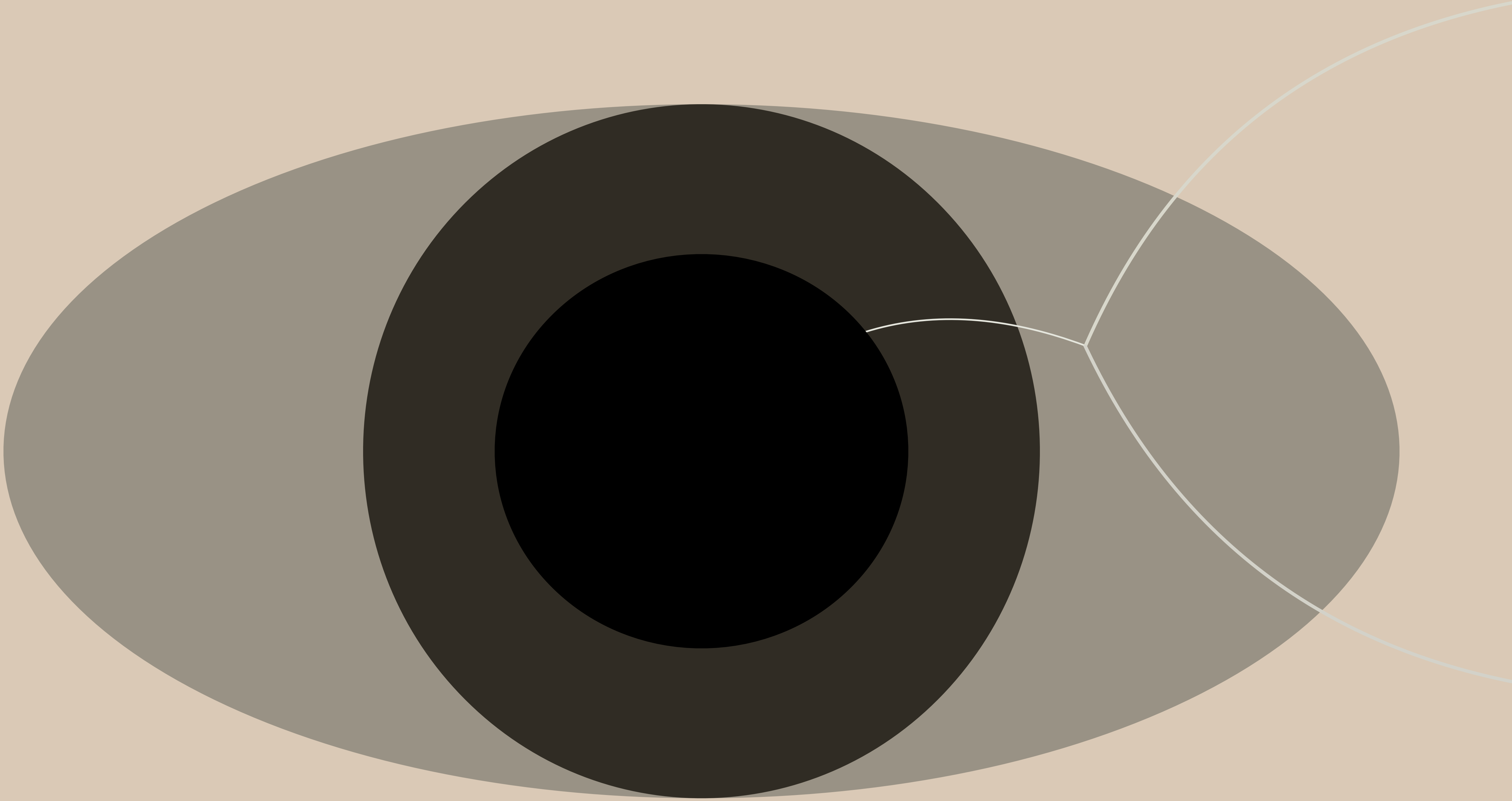

Isomonodromy, Riemann-Hilbert and Black Holes

BalFEST80 - DIAS

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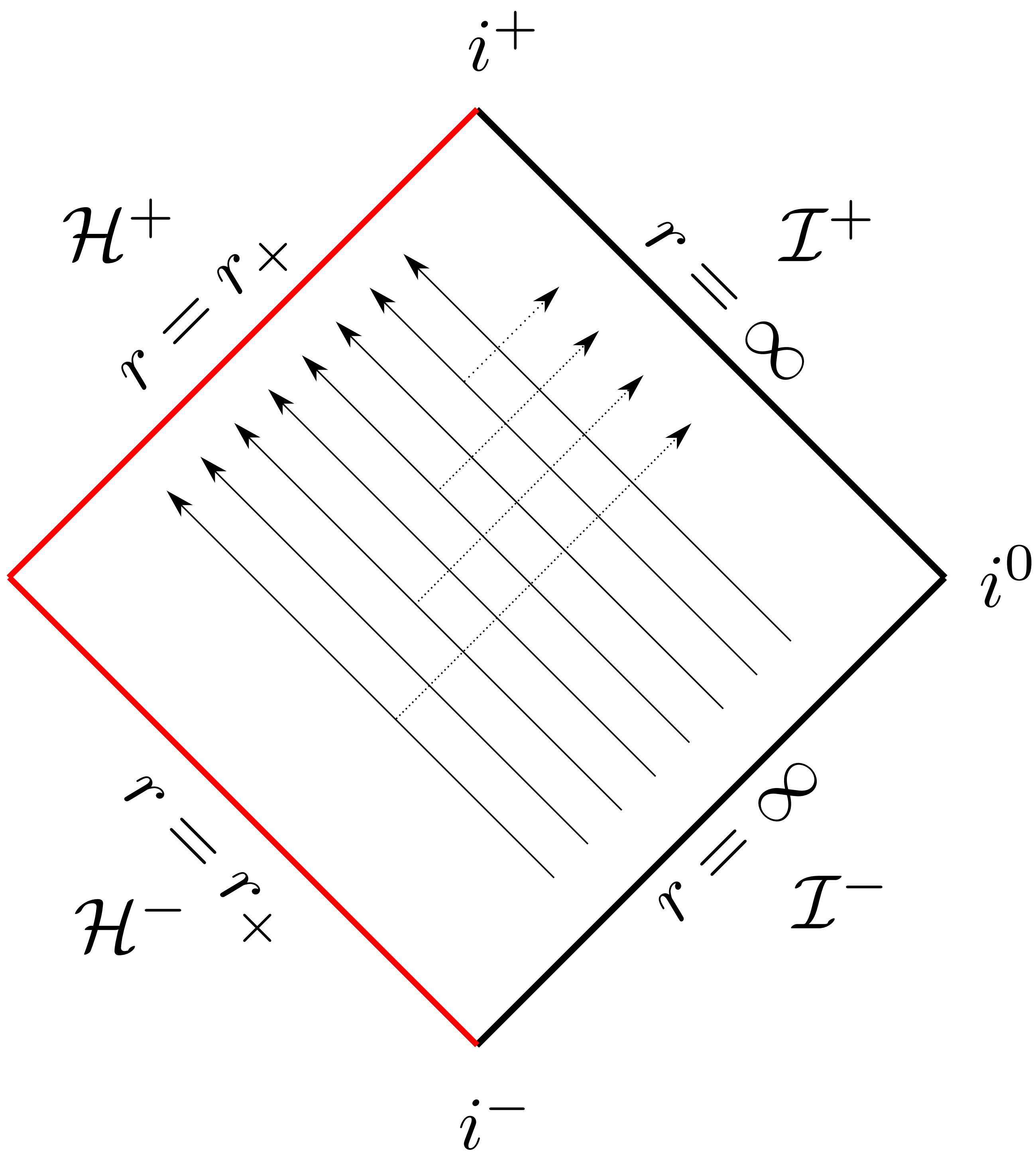
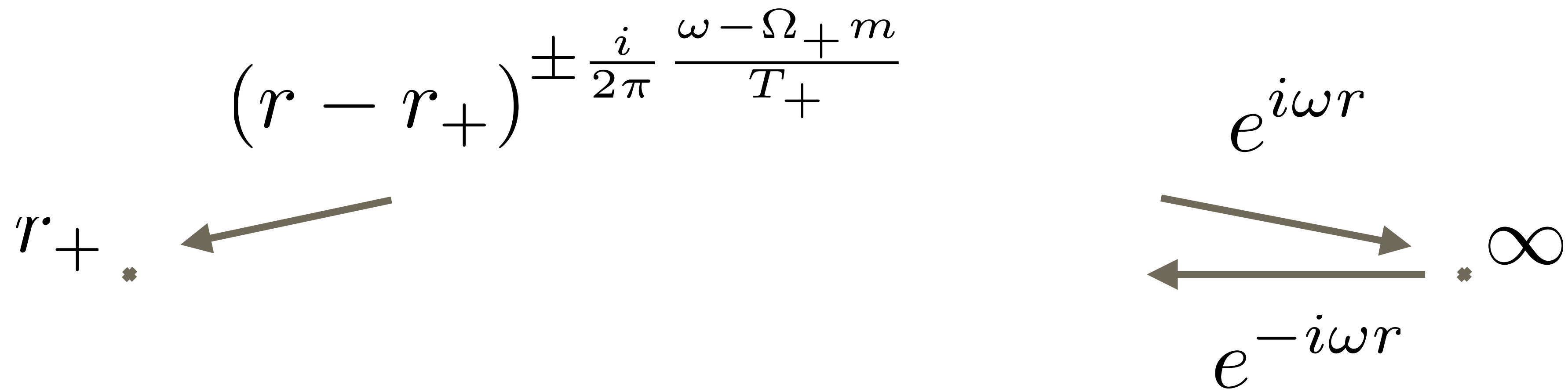
black hole scattering



$$(\nabla_a \nabla^a + \mu^2)\Psi = 0$$

For all type D (or the analogue in higher dimensions): perturbations can be written as solutions of (possibly confluent limits of) Fuchsian equations (of 2nd order)

Ingredients: bases of “plane waves” (Jost functions):



Normalization:

$$j = -i\, Q(r)[R(r)^* \partial_r R(r) - R(r) \partial_r R(r)^*].$$

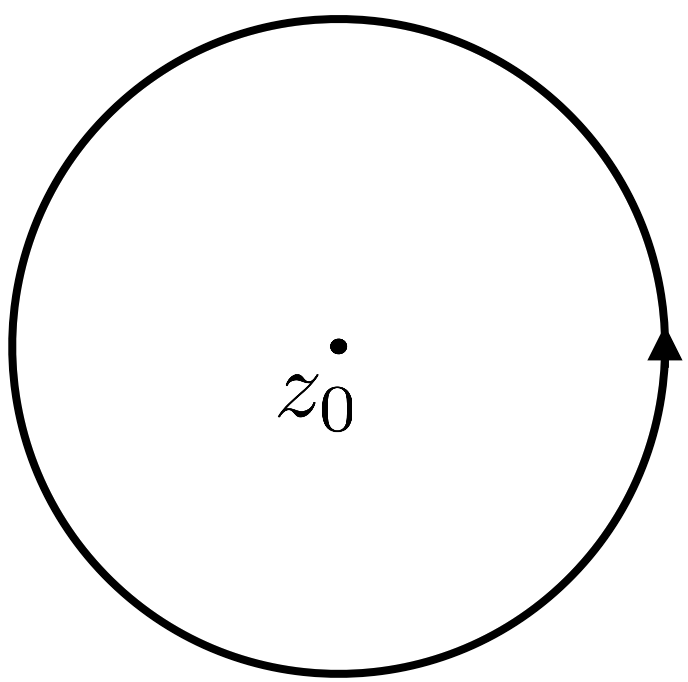
“Boundary conditions”:

$$R_+^- = \frac{1}{\mathcal{T}} R_\infty^+ + \frac{\mathcal{R}}{\mathcal{T}} R_\infty^-,$$

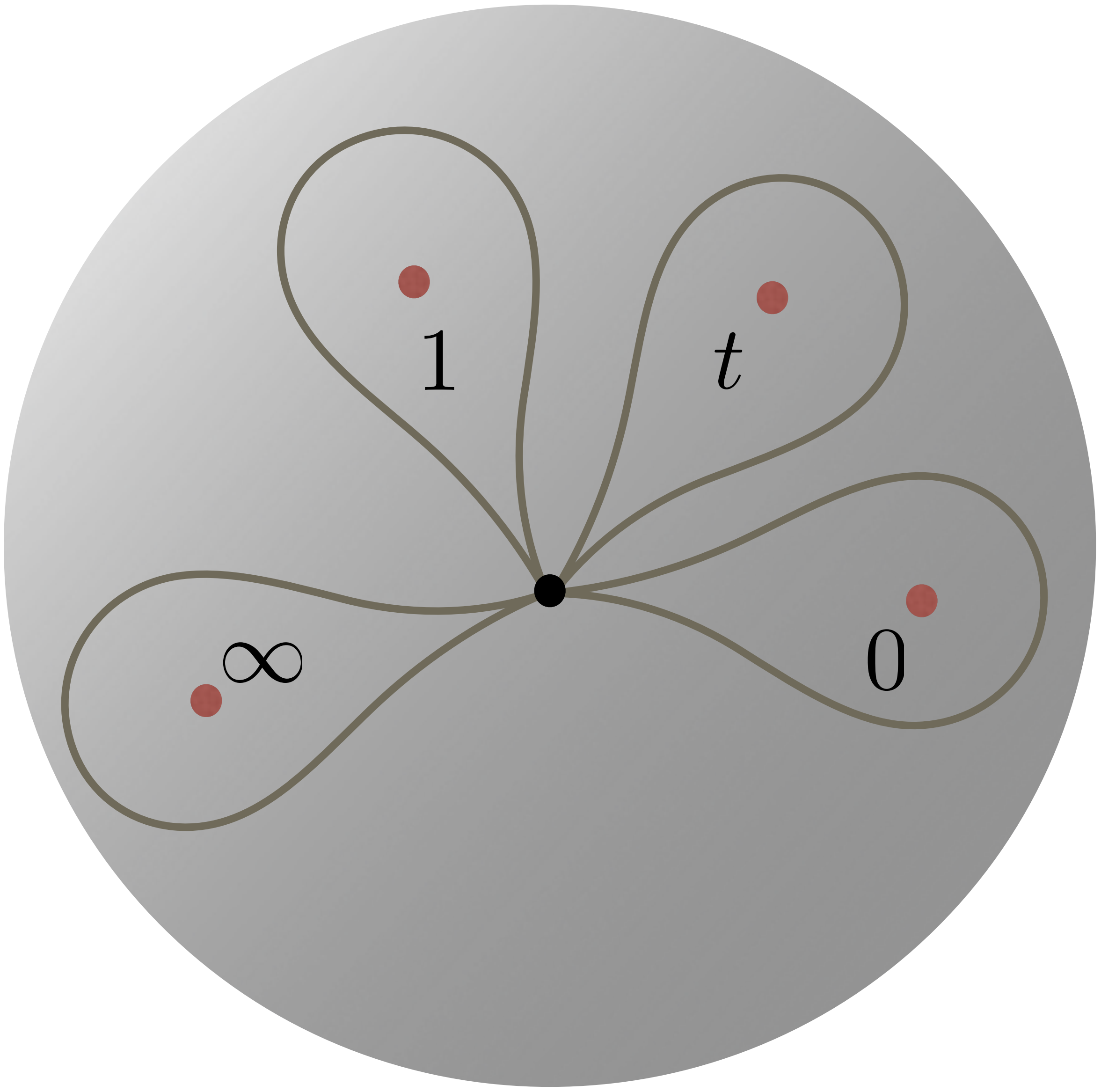
Scattering problem = connection problem for EDO's:

$$R^1_C(r) = E^{11}_{+C} R^1_+(r) + E^{12}_{+C} R^2_+(r)$$
$$R^2_C(r) = E^{21}_{+C} R^1_+(r) + E^{22}_{+C} R^2_+(r)$$

Connection coefficients for EDO’s: tied to monodromy



$$M_0 = \begin{pmatrix} e^{2\pi i \alpha_+} & 0 \\ 0 & e^{2\pi i \alpha_-} \end{pmatrix}$$



$$M_\infty M_0 M_t M_1 = \mathbb{1}$$

$$M_{+C} = E_{+C}^{-1} \begin{pmatrix} e^{2\pi i \alpha_C^1} & 0 \\ 0 & e^{2\pi i \alpha_C^2} \end{pmatrix} E_{+C}$$

two important facts

- Monodromy problem solves scattering:

$$|\mathcal{T}|^2 = \left| \frac{2 \sin 2\pi\theta_i \sin 2\pi\theta_j}{\cos 2\pi(\theta_i - \theta_j) - \cos \pi\sigma_{ij}} \right| \qquad \mathrm{Tr}\, M_i M_j = 2 \cos 2\pi\sigma_{ij}$$

- If one given solution has prescribed (Frobenius) behavior at two different points, then the associated monodromies commute.

$y(z)$ “regular” at z_0 and z_1 $\longleftrightarrow$ $M_0 M_1 = M_1 M_0$

composite monodromy is specified (up to integer!)

scattering problem is mapped to calculation of composite monodromy in terms of parameters in ODE.

angular eigenvalues are obtained by calculating parameters in angular ODE in terms of the composite monodromy

quasi-normal modes are obtained by calculating the angular separation constant and radial parameters in terms of composite monodromy

$$y'' + \left(\frac{1-\theta_0}{z} + \frac{1-\theta_1}{z-1} + \frac{1-\theta_{t_0}}{z-t_0} \right) y' + \left(\frac{q_1 q_2}{z(z-1)} - \frac{t_0(t_0-1)K_0}{z(z-1)(z-t_0)} \right) y = 0$$

$$\frac{d}{dz}\Phi(z) = A(z)\Phi(z) \qquad A(z) = \frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_t}{z-t}$$

$$\Phi(z) = \begin{pmatrix} y^{(1)}(z) & y^{(2)}(z) \\ w^{(1)}(z) & w^{(2)}(z) \end{pmatrix}$$

$$A(z) = \left(\frac{d}{dz}\Phi(z) \right) \Phi(z)^{-1}$$

“holomorphic flat connection"

$$F(z) = dA + A \wedge A = 0$$

$$\partial_z^2 y - (\mathrm{Tr}A + \partial_z \log A_{12})\partial_z y + (\det A - \partial_z A_{11} + A_{11}\partial_z \log A_{12})y = 0,$$

$$y'' + p(z,t)y' + q(z,t)y = 0,$$

$$p(z,t) = \frac{1-\theta_0}{z} + \frac{1-\theta_1}{z-1} + \frac{1-\theta_t}{z-t} - \frac{1}{z-\lambda},$$

$$q(z,t) = \frac{\kappa_1(\kappa_2+1)}{z(z-1)} - \frac{t(t-1)K}{z(z-1)(z-t)} + \frac{\lambda(\lambda-1)\mu}{z(z-1)(z-\lambda)},$$

Can embed system into flat holomorphic connection

$$A_z(z,t) = A(z), \qquad A_t(z,t) = -\frac{A_t}{z-t}$$

change parameters with t : isomonodromy deformations!

$$\frac{\partial A_0}{\partial t} = \frac{1}{t}[A_t, A_0], \qquad \frac{\partial A_1}{\partial t} = \frac{1}{t-1}[A_t, A_1],$$
$$\frac{\partial A_t}{\partial t} = \frac{1}{t}[A_0, A_t] + \frac{1}{t-1}[A_1, A_t].$$

making use of obvious conserved quantities:

$$A_i = \begin{pmatrix} p_i + \theta_i & -q_i p_i \\ \frac{1}{q_i}(p_i + \theta_i) & -p_i \end{pmatrix}, \quad i = 0, 1, t,$$

Claim: can find parameters so that:

$$\lambda(t_0) = t_0, \qquad \mu(t_0) = K - K_0$$

1508.04046

Schlesinger equations: Painlevé VI transcendent:

$$\ddot{\lambda} = \frac{1}{2}\left(\frac{1}{\lambda} + \frac{1}{\lambda-1} + \frac{1}{\lambda-t}\right)\dot{\lambda}^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{\lambda-t}\right)\dot{\lambda}$$
$$+ \frac{\lambda(\lambda-1)(\lambda-t)}{2t^2(1-t)^2}\left(\theta_\infty^2 - \theta_0^2\frac{t}{\lambda^2} + \theta_1^2\frac{t-1}{(\lambda-1)^2} + \left(1-\theta_t^2\right)\frac{t(t-1)}{(\lambda-t)^2}\right)$$

Actually Hamiltonian system:

$$\frac{d\lambda}{dt} = \{K, \lambda\}, \qquad \frac{d\mu}{dt} = \{K, \mu\}$$

$$K(\lambda, \mu, t) = \frac{\lambda(\lambda-1)(\lambda-t)}{t(t-1)}\left[\mu^2 - \left(\frac{\theta_0}{\lambda} + \frac{\theta_1}{\lambda-1} + \frac{\theta_t - 1}{\lambda-t}\right)\mu + \frac{\kappa_1(\kappa_2+1)}{\lambda(\lambda-1)}\right]$$

“Effective potential”: tau function

$$\frac{d}{dt}\log\tau(t,\{\theta_i\}) = \frac{1}{t}\mathrm{Tr}(A_0A_t) + \frac{1}{t-1}\mathrm{Tr}(A_1A_t)$$

Initial value problem for tau:

$$\frac{d}{dt}\log\tau(t,\{\theta_i\}^-)\Big|_{t=t_0} = \frac{\theta_0(\theta_t-1)}{t_0} + \frac{\theta_1(\theta_t-1)}{t_0-1} + K_0$$

$$\frac{d^2}{dt^2}\log\tau(t,\{\theta_i\}^-)\Big|_{t=t_0} = -\frac{\theta_0(\theta_t-1)}{t_0^2} - \frac{\theta_1(\theta_t-1)}{(t_0-1)^2} + \frac{\kappa_1(\theta_t-1)}{t_0(t_0-1)} - \frac{2t_0-1}{t_0(t_0-1)}K_0$$

Formally can be inverted to give monodromy data in terms of ODE parameters

$$\{\theta_i\} = \{\theta_0, \theta_1, \theta_t, \theta_\infty, \sigma_{0t}, \sigma_{1t}, \sigma_{01}\}$$

constants of Schlesinger motion

Determined from Fricke-Jimbo

Two-dimensional symplectic system!

By definition, Schlesinger motion on monodromy coordinates is trivial

tau-function is then the generating function of Hamilton-Jacobi transformation

$$\frac{d}{dt} \log \tau(t, \{\theta_i\}^-) \Big|_{t=t_0} = \frac{\theta_0(\theta_t - 1)}{t_0} + \frac{\theta_1(\theta_t - 1)}{t_0 - 1} + K_0$$

Second condition can be understood from the "Toda" equation

$$\frac{d}{dt}t(t-1)\frac{d}{dt}\log\tau^{-} + \alpha(\alpha+\kappa_{\pm}) + \beta(\beta+\kappa_{\mp}) = C\frac{\tau\tau^{--}}{(\tau^{-})^2}$$

Further analysis yield

$$\tau(t,\{\theta_i\}) = 0$$

Malgrange and Palmer interpreted the tau-function as the determinant of a Cauchy-Riemann operator

$$\tau(t,\{\theta_i\}) = \det(\partial_{\bar{z}} - A(z))$$

So, no surprises…

Gauss decomposition of fundamental solution

$$\Phi(z) = e^{\chi_L \sigma^-} e^{-\frac{1}{2}\phi_c \sigma^3} e^{\chi_R \sigma^+} = \begin{pmatrix} 1 & 0 \\ \chi_L & 1 \end{pmatrix} \begin{pmatrix} e^{-\frac{1}{2}\phi_c} & 0 \\ 0 & e^{\frac{1}{2}\phi_c} \end{pmatrix} \begin{pmatrix} 1 & \chi_R \\ 0 & 1 \end{pmatrix},$$

$$\mathcal{A}_z = [\partial_z \Phi(z)]\Phi^{-1}(z) \qquad \mathcal{A}_{12} = \mu$$ “oper” condition

Flat condition is equivalent to Liouville equation

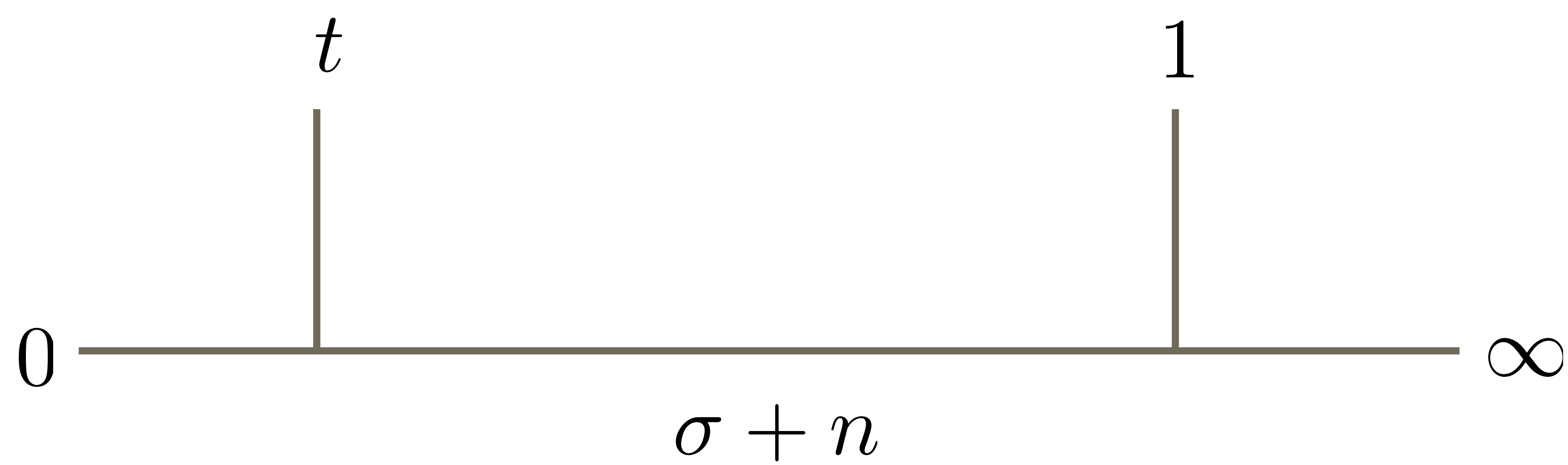
$$\partial_z \partial_{\bar{z}} \phi_c = \mu^2 e^{\phi_c}$$

General “classical" solution

$$\phi_c(z, z^*) = \log\left(-\frac{2}{\mu^2} \frac{\partial_z \zeta \partial_{z^*} \zeta^*}{(\zeta(z) - \zeta^*(z^*))^2} \right)$$

solutions are in one-to-one correspondence with profiles of Liouville tau function = conformal block of Virasoro primaries

$$\tau(t,\{\theta_i\}) = \langle \Phi_\infty(\infty)\Phi_1(1)\mathcal{P}_\Delta \Phi_t(t)\Phi_0(0)\rangle$$



Zamolodchikov 89: coefficients can be calculated recursively

AGT (AFLT) 09-10: function can be computed from Nekrasov partition function of N=2 instantons with matter (Duistermaat-Heckman)

GIL 12-13: tau function for c=1 (!) satisfies Painlevé VI

$$\tau(t) = \sum_{n\in\mathbb{Z}} C(\vec{\theta}, \sigma_{0t} + n) s^n t^{(\sigma+2n)^2/4-(\theta_0-\theta_t)^2/4} \mathcal{B}(\vec{\theta}, \sigma_{0t} + n; t)$$

$$C(\vec{\theta}, \sigma) = \frac{\prod_{\epsilon,\epsilon'=\pm} G(1 + \frac{1}{2}(\theta_t + \epsilon\theta_0 + \epsilon'\sigma))G(1 + \frac{1}{2}(\theta_1 + \epsilon\theta_\infty + \epsilon'\sigma))}{\prod_{\epsilon=\pm} G(1 + \epsilon\sigma)}$$

$$\mathcal{B}(\vec{\theta}, \sigma; t) = (1-t)^{\theta_t\theta_1/2} \sum_{\lambda,\mu\in\lambda} \mathcal{B}_{\lambda,\mu}(\vec{\theta}, \sigma) t^{|\lambda|+|\mu|}$$

$$\mathcal{B}_{\lambda,\mu}(\vec{\theta}, \sigma) = \prod_{(i,j)\in\lambda} \frac{((\theta_t + \sigma + 2(i-j))^2 - \theta_0^2)((\theta_1 + \sigma + 2(i-j))^2 - \theta_\infty^2)}{16 h_\lambda^2(i,j)(\lambda'_j + \mu_i - i - j + 1 + \sigma)^2} \times$$

$$\prod_{(i,j)\in\mu} \frac{((\theta_t - \sigma + 2(i-j))^2 - \theta_0^2)((\theta_1 - \sigma + 2(i-j))^2 - \theta_\infty^2)}{16 h_\mu^2(i,j)(\lambda_i + \mu'_j - i - j + 1 - \sigma)^2}$$

These results achieve an combinatorial, procedural solution to the scattering problem

Scalar field in Kerr-(A)dS5

$$ds^2 = -\frac{\Delta_r}{\rho^2}\left(dt - \frac{a\sin^2\theta}{\Xi_a}d\phi - \frac{b\cos^2\theta}{\Xi_b}d\psi\right)^2 + \frac{\Delta_\theta\sin^2\theta}{\rho^2}\left(adt - \frac{(r^2+a^2)}{\Xi_a}d\phi\right)^2$$
$$+ \frac{1+r^2/l^2}{r^2\rho^2}\left(abdt - \frac{b(r^2+a^2)\sin^2\theta}{\Xi_a}d\phi - \frac{a(r^2+b^2)\cos^2\theta}{\Xi_b}d\psi\right)^2$$
$$+ \frac{\Delta_\theta\cos^2\theta}{\rho^2}\left(bdt - \frac{(r^2+b^2)}{\Xi_b}d\psi\right)^2 + \frac{\rho^2}{\Delta_r}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2,$$

$$\Delta_r = \frac{1}{r^2}(r^2+a^2)(r^2+b^2)\left(1+\frac{r^2}{l^2}\right) - 2MG,$$
$$\Delta_\theta = 1 - \frac{a^2}{l^2}\cos^2\theta - \frac{b^2}{l^2}\sin^2\theta,$$
$$\rho^2 = r^2 + a^2\cos^2\theta + b^2\sin^2\theta,$$
$$\Xi_a = 1 - \frac{a^2}{l^2}, \qquad \Xi_b = 1 - \frac{b^2}{l^2}$$

1702.01016

Radial equation

$$\frac{1}{r\Pi(r)}\frac{d}{dr}\left(r\Delta_r\frac{d\Pi(r)}{dr}\right)-\left[\lambda+\mu^2r^2+\frac{1}{r^2}(a_1a_2\omega-a_2(1-a_1^2)m_1-a_1(1-a_2^2)m_2)^2\right]$$
$$+\frac{(r^2+a_1^2)^2(r^2+a_2^2)^2}{r^4\Delta_r}\left(\omega-\frac{m_1a_1(1-a_1^2)}{r^2+a_1^2}-\frac{m_2a_2(1-a_2^2)}{r^2+a_2^2}\right)^2=0$$

Singular points = Killing horizons (complex radius)

Frobenius indices

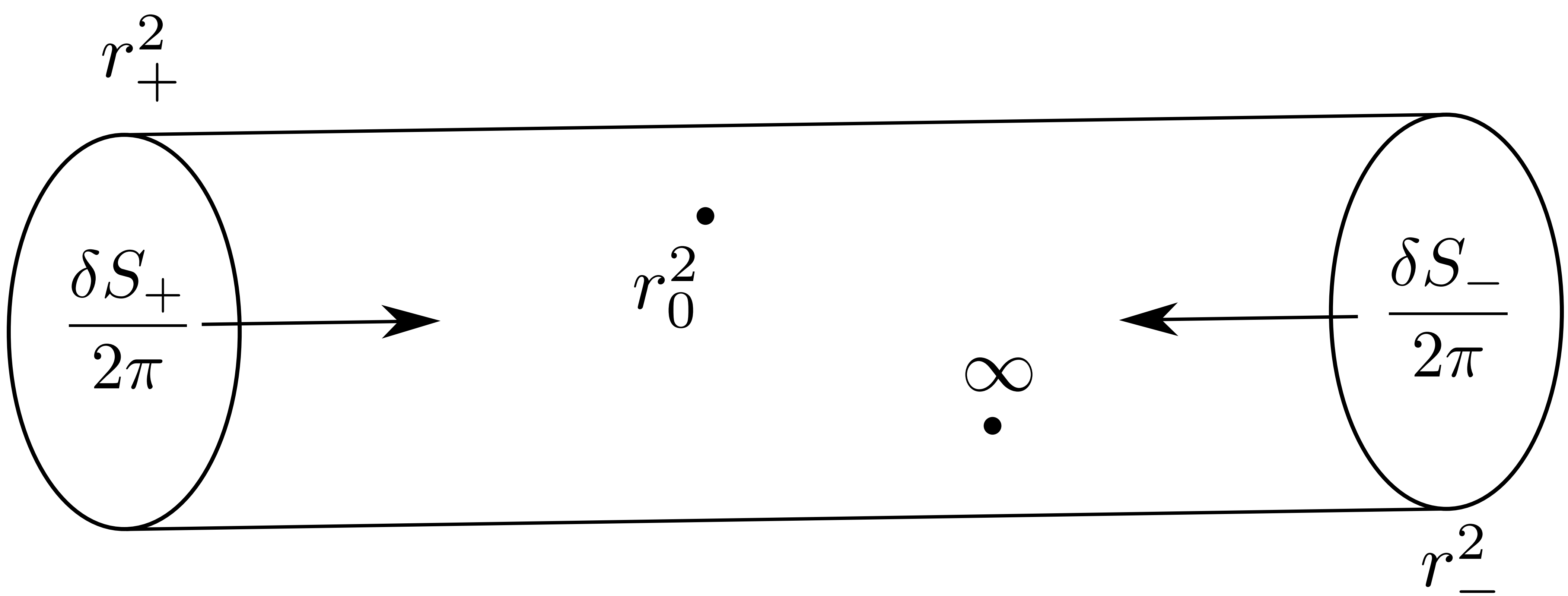
$$\theta_k=\pm\frac{i}{2\pi}\left(\frac{\omega-m_1\Omega_{k,1}-m_2\Omega_{k,2}}{T_k}\right),\quad \theta_\infty=\frac{\Delta}{2},\frac{d-\Delta}{2}=1\pm\sqrt{1+\frac{\mu^2}{4}}$$

Conformal group Casimir parameter ("Scaling dimension")

Entropy absorbed/gained by Killing horizon at z_k

“quantum gravity” interpretation

$$ds^2 = e^{\phi_c} dz\, dz^*$$



radial equation: inner and outer horizon correspond to hyperbolic boundaries to 2d surface (punctured cylinder)

angular equation: also conformal block, but all insertions are elliptic boundaries (punctured sphere)

separation constant: "thermal equilibrium"

$$K_0 = \frac{d}{dt} \log \tau(t, \{\theta_i\}) \Big|_{t=z_0} = \frac{d}{dt} \log \tau(t, \{\beta_i\}) \Big|_{t=u_0} = Q_0$$

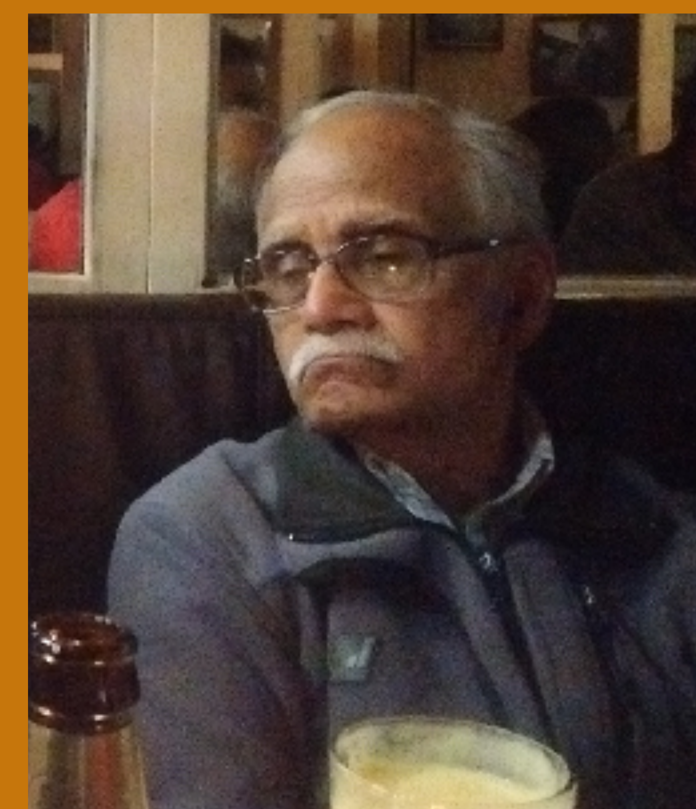
Attempt at interpretation by Guica and BCC: 1604.07383 (BTZ)

Presumably integrable structure (foliation by 2d surfaces) at play: general feature of perturbations of type D?

conclusions(?)

- procedure to compute scattering coefficients and quasi-normal modes bound to be better than matching (EP & JBA)
- construction of uniformization maps works much better than numerics (DC, RN & TA); “quadrangle functions” and beyond
- numerical algorithms for fast computation: Nekrasov expansion vs. Fredholm determinants (OL)
- applications for generic black hole perturbations (Kerr-AdS and Kerr-Newman), higher spin, etc
- application for other physical systems, ex. Rabi model (AQ, MCA)
- holographic/string interpretation (MG) ?

Thank you!

Thanks, Bal!