Entanglement witnesses from mutually unbiased bases

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In honour of A. P. Balachandran on the occasion of his 80th birthday

Outline:

1. basic intro to positive maps and entanglement witnesses
2. positive maps vs. quantum entanglement
3. Mutually Unbiased Bases (MUBs)
4. MUBs $\rightarrow$ positive maps
5. conclusions

How to play with convex sets

Positive maps

A linear map

$$\Phi \;:\; \mathcal{A}_1 \;\longrightarrow\; \mathcal{A}_2$$

is positive iff

$$a \geq 0 \implies \Phi[a] \geq 0$$

$$(a \geq 0 \iff a = xx^*)$$

It is unital iff

$$\Phi[e_1] = e_2$$

Why positive maps?

- provide generalization of $*$-homomorphisms,
- provide generalization of Jordan homomorphisms,
- unital maps define affine mappings between sets of states of $\mathbb{C}^*$-algebras.
- define universal tools for detecting quantum entanglement

The problem

How to construct and classify positive maps

The problem is hard

Related to 17th Hilbert problem

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Special classes of positive maps

- completely positive (CP) maps
- decomposable maps

CP maps $\subset$ decomposable maps $\subset$ all positive maps

Completely positive maps

$$\Phi \ : \ \mathcal{A} \ \longrightarrow \ \mathcal{B}(\mathcal{H})$$

Stinespring 1955

Φ is completely positive iff

- there exists a Hilbert space $\mathcal{K}$
- there exists $\star$-homomorhism $\pi : \ \mathcal{A} \ \longrightarrow \ \mathcal{B}(\mathcal{K})$
- there exists $V : \mathcal{K} \longrightarrow \mathcal{H}$

$$\Phi[a] = V\pi(a)V^*$$

Decomposable maps

$$\Phi \; : \; \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$$

Φ is a decomposable positive map iff

- there exists a Hilbert space $\mathcal{K}$
- there exists Jordan-homomorhism $j : \; \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{K})$
- there exists $V : \mathcal{K} \longrightarrow \mathcal{H}$

$$\Phi[a] = V j(a) V^*$$

Completely positive maps

$$\dim \mathcal{H} < \infty$$

$$\Phi \;:\; \mathcal{B}(\mathcal{H}_1) \longrightarrow \mathcal{B}(\mathcal{H}_2)$$

Stinespring, Kraus, Choi

$$\Phi(X) \;=\; \sum_\alpha V_\alpha X V_\alpha^*$$

Decomposable maps

$$\Phi = \Phi_1 + \mathrm{T} \circ \Phi_2$$

Φ_1, Φ_2 — CP maps

$d_1 \cdot d_2 \leq 6 \longrightarrow$ all positive maps are decomposable (Woronowicz)

The hard problem is the construction of non-decomposable maps

Positive maps vs. quantum entanglement

$$\mathcal{H} = \mathcal{H}_2 \otimes \mathcal{H}_1$$

Definition: (Werner)

$$\rho = \sum_{\alpha} p_\alpha \, \rho_\alpha^{(1)} \otimes \rho_\alpha^{(2)}$$ separable state = not entangled

Theorem (Horodecki):

ρ is separable iff

$$(\mathrm{id} \otimes \Phi)\rho \geq 0$$

for all positive maps $\Phi \ : \ \mathcal{B}(\mathcal{H}_2) \longrightarrow \mathcal{B}(\mathcal{H}_1)$

ρ is entangled iff there exits a positive map Φ

$$(\mathrm{id} \otimes \Phi)\rho \ngeq 0$$

Φ detects ρ

Classification of entangled states $\longleftrightarrow$ Classification of positive maps

CP property is spectral

$\{e_1, e_2, \dots\}$ –ONB in $\mathcal{H}_1$

$$E_{ij} := |e_i\rangle\langle e_j| \in M_{d_1}(\mathbb{C})$$

Choi matrix

$$\Phi \longrightarrow \widehat{\Phi} = \sum_{i,j} E_{ij} \otimes \Phi(E_{ij})$$

Φ is CP $\iff$ $\widehat{\Phi} \geq 0$

Quantum Information: Positive Partial Transpose (PPT)

$X \in \mathcal{B}_+(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is PPT

$$X^\Gamma := (\mathbb{1} \otimes \mathrm{T})X \geq 0$$

Separable $\subseteq$ PPT

$d_1 \cdot d_2 \leq 6 \longrightarrow$ Separable $=$ PPT (Peres, Horodecki)

Φ decomposable $\longrightarrow$ $(\mathbb{1} \otimes \Phi)X \geq 0$ for all PPT states

The hard problem is to detect entangled PPT states

Entanglement witness

Definition:

W is block-positive $\iff \langle x \otimes y | W | x \otimes y \rangle \geq 0$

Theorem:

Φ is positive $\iff \widehat{\Phi}$ is block-positive

Definition:

W is **entanglement witness**

- W is block-positive
- W is not positive

ρ is entangled iff there exits an entanglement witness W

$$\mathrm{Tr}(W\rho) < 0$$

W detects ρ

Geometrical picture

Duality

V — real vector space

$$A \subset V \;\longrightarrow\; A^\circ := \left\{ \, y \in V^* \mid y(x) \geq 0 \;,\; x \in A \, \right\} \subset V^*$$

For an arbitary A its dual A° is a convex cone in V^*

$$\begin{aligned} A \subset B \quad &\Longrightarrow \quad B^\circ \subset A^\circ \\ (A \cap B)^\circ \quad &= \quad \mathrm{conv}(A^\circ \cup B^\circ) \\ (A \cup B)^\circ \quad &= \quad A^\circ \cap B^\circ \end{aligned}$$

Duality

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Duality – quantum states

$\mathcal{B}_{\rm sa}$:= self-adjoint elements in $\mathcal{B}(\mathcal{H}\otimes\mathcal{H})$

$\mathcal{B}_{\rm sa}$ is a real Hilbert space $\longrightarrow$ $\mathcal{B}_{\rm sa}^* \equiv \mathcal{B}_{\rm sa}$

$\mathcal{B}_+$ = positive elements in $\mathcal{B}_{\rm sa}$

$\mathcal{B}_{\rm PPT}$ = PPT elements in $\mathcal{B}_+$

$\mathcal{B}_{\rm sep}$ = separable elements in $\mathcal{B}_{\rm PPT}$

$$\mathcal{B}_{\rm sep} \subset \mathcal{B}_{\rm PPT} \subset \mathcal{B}_+ \subset \mathcal{B}_{\rm sa}$$

Duality – quantum states

$$\mathcal{B}_{\mathrm{sep}} \subset \mathcal{B}_{\mathrm{PPT}} \subset \mathcal{B}_{+} \subset \mathcal{B}_{\mathrm{sa}}$$

$$\mathcal{B}^{\circ}_{\mathrm{sep}} \supset \mathcal{B}^{\circ}_{\mathrm{PPT}} \supset \mathcal{B}^{\circ}_{+} \supset \mathcal{B}_{\mathrm{sa}}$$

$\mathcal{B}^{\circ}_{+} = \mathcal{B}_{+}$ (self-dual set)

$\mathcal{B}^{\circ}_{\mathrm{sep}} =$ block-positive operators

$\mathcal{B}^{\circ}_{\mathrm{sep}} =$ block-positive decomposable operators

$$\begin{aligned} \text{entanglement witnesses} &= \mathcal{B}^{\circ}_{\mathrm{sep}} - \mathcal{B}_{+} \\ \text{non-decomposable entanglement witnesses} &= \mathcal{B}^{\circ}_{\mathrm{PPT}} - \mathcal{B}_{+} \end{aligned}$$

Optimal witness

$$W \longrightarrow \mathcal{D}_W = \{X \geq 0 \,|\, \mathrm{Tr}(XW) < 0\}$$

Definition:

$$W_1 \text{ is finer than } W_2 \iff \mathcal{D}_{W_2} \subset \mathcal{D}_{W_1}$$

$$W \text{ is optimal} \iff \text{there is no witness finer than } W$$

Theorem:

W is optimal iff $W - A$ is no longer block-positive

where A is an arbitrary $A \in \mathcal{B}_+$

nd-optimal witness

$$W \longrightarrow \mathcal{D}_W^{\mathrm{PPT}} = \{X \in \mathcal{B}_{\mathrm{PPT}} \mid \mathrm{Tr}(XW) < 0\}$$

Definition:

$$W_1 \text{ is nd-finer than } W_2 \iff \mathcal{D}_{W_2}^{\mathrm{PPT}} \subset \mathcal{D}_{W_1}^{\mathrm{PPT}}$$

W is nd-optimal $\iff$ no non-decomposable witness finer than W

Theorem:

W is nd-optimal iff $W - D$ is no longer block-positive

where $D \in \mathcal{B}_{\mathrm{PPT}}$ is arbitrary

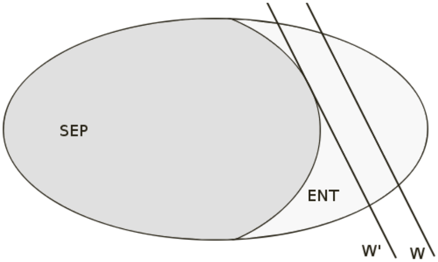
optimal vs. nd-optimal witness

Theorem:

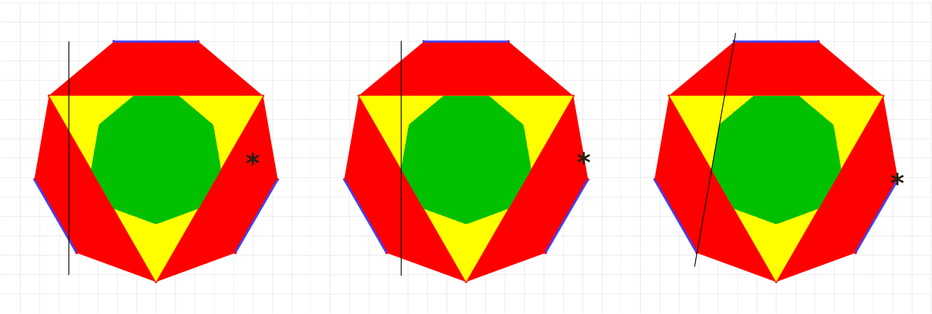
W is nd-optimal if and only if

both W and $W^\Gamma := (\mathrm{id} \otimes \mathrm{T})W$ are optimal

Banach separation theorem



W' is finer than W



Spanning property

$$\langle x_k \otimes y_k | W | x_k \otimes y_k \rangle = 0$$

Definition:

- W has a spanning property if $x_k \otimes y_k$ span $\mathcal{H}_1 \otimes \mathcal{H}_2$
- W has a bi-spanning property if also $x_k \otimes y_k^*$ span $\mathcal{H}_1 \otimes \mathcal{H}_2$

Theorem:

W has a spanning property $\Rightarrow$ W is optimal

W has a bi-spanning property $\Rightarrow$ W is nd-optimal

Exposed witnesses

$$W \longrightarrow P_W = \{\, x \otimes y \,|\, \langle x \otimes y | W | x \otimes y \rangle = 0 \,\}$$

W is exposed

If for any block-positive operator W' such that

$$\langle x \otimes y | W' | x \otimes y \rangle = 0 \ \text{ for } \ x \otimes y \in P_W$$

one has $W' = aW$, with $a > 0$

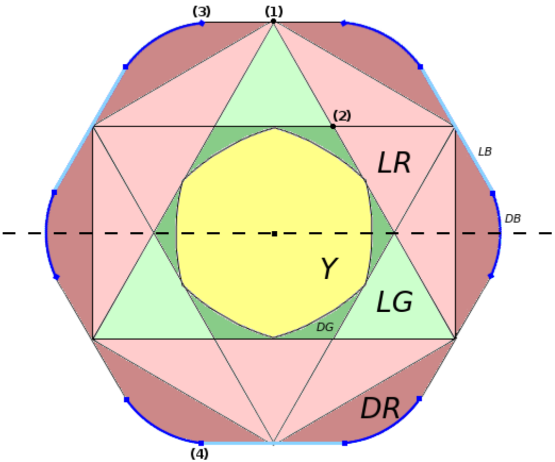
Non-decomposable witness

$$\text{exposed} \longrightarrow \text{extremal} \longrightarrow \text{nd-optimal}$$

$$\text{exposed} \longrightarrow \text{bi-spanning} \longrightarrow \text{nd-optimal}$$

Theorem [Straszewicz]

For a compact convex set exposed elements are dense in the set of extremal elements.


(3)
(1)
(2)
LR
LB
DB
Y
LG
DG
DR
(4)

New construction of postive maps

Using a concept of Mutually Unbiased Bases

Mutually Unbiased Bases (MUB)

Definition:

$$\{|\psi_k\rangle\}_{k=1}^{d} \ \& \ \{|\phi_k\rangle\}_{k=1}^{d} \quad \text{are MUB}$$

$$|\langle\psi_k|\phi_l\rangle|^2 = \frac{1}{d} \ ; \quad k,l = 1,\ldots,d$$

For $d = 2$ the are 3 MUBs

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} ; \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} ; \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Mutually Unbiased Bases (MUB)

$N(d)$ = maximal number of MUBs in d dimensions

Theorem:

- $3 \leq N(d) \leq d+1$
- $d = p^r \;\Rightarrow\; N(d) = d+1$
- $d = d_1 d_2 \;\Rightarrow\; N(d) \geq \min\{N(d_1), N(d_2)\}$

For $d = 6$ there is a numerical evidence that $N(6) = 3$

How to construct MUBs in $\mathbb{C}^d$

$|0\rangle, |1\rangle, \ldots, |d-1\rangle$ — computational basis

$$|\widetilde{k}\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} \omega^{-kj} |j\rangle$$

$$\omega = e^{2\pi i/d}$$

$$|j\rangle \longleftrightarrow |\widetilde{k}\rangle$$

Prime d

$$|0\rangle, |1\rangle, \ldots, |d-1\rangle$$ — computational basis

$$\mathcal{X}|j\rangle = |j+1\rangle \; ; \qquad \mathcal{Z}|j\rangle = \omega^j |j\rangle$$

$$\mathcal{X}, \mathcal{Z}, \mathcal{X}\mathcal{Z}, \mathcal{X}\mathcal{Z}^2, \ldots, \mathcal{X}\mathcal{Z}^{d-1}$$

$|\psi_j^{(k)}\rangle$ – eigenbasis of $\mathcal{X}\mathcal{Z}^k$ define $d+1$ MUBs

$$\mathcal{X}\mathcal{Z}^\ell |\psi_j^{(k)}\rangle = \omega^{j+k-\ell} |\psi_{j+k-\ell}^{(k)}\rangle$$

Weyl–Heisenberg group $\mathbb{H}_d$ (arbitrary d)

$$\mathcal{X}\mathcal{Z} = \omega \mathcal{Z}\mathcal{X}$$

$$U_{mn} := \mathcal{X}^m \mathcal{Z}^n \; ; \quad (m, n = 0, 1, \ldots, d-1)$$

$$U_{mn} U_{kl} = U_{m+k, n+l}$$

$$[U_{mn}, U_{m'n'}] = 0 \iff mn' - nm' = 0$$

$\frac{1}{\sqrt{d}} U_{mn}$ – define ONB in $M_d(\mathbb{C})$

$$\mathrm{Tr}[U_{mn} U^{\dagger}_{m'n'}] = d \delta_{mm'} \delta_{nn'}$$

Weyl–Heisenberg group $\mathbb{H}_d$ (arbitrary d)

$$U_{mn} := \mathcal{X}^m \mathcal{Z}^n \; ; \quad (m, n = 0, 1, \ldots, d-1)$$

$$U_{00} = \mathbb{I}_d$$

$$\mathbb{H}_d \setminus \mathbb{I}_d \;\longrightarrow\; \mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_L$$

- $\mathcal{C}_k \cap \mathcal{C}_\ell = \emptyset$
- $\mathcal{C}_k$ contains mutually commuting operators
- $|\mathcal{C}_k| = d - 1$

eigenbasis of $\mathcal{C}_k$ are Mutually Unbiased

$$L \leq d + 1 \;\; ; \;\; (d \text{ prime} \longrightarrow L = d + 1)$$

MUBs in Quantum Information

- quantum state tomography (discrete Wigner function)
- entropic uncertainty relations
- Quantum Key Distribution
- ...

MUBs $\longrightarrow$ positive maps

MUBs $\longrightarrow$ positive map

$$N(d) = d + 1$$

$$\{|\psi_1^{(\alpha)}\rangle, \ldots, |\psi_d^{(\alpha)}\rangle\} \ ; \quad \alpha = 1, 2, \ldots, d+1$$

$$|\langle\psi_k^{(\alpha)}|\psi_\ell^{(\beta)}\rangle|^2 = \frac{1}{d} \ ; \quad \alpha \neq \beta$$

$$P_k^{(\alpha)} = |\psi_k^{(\alpha)}\rangle\langle\psi_k^{(\alpha)}|$$

General idea

Generalize very basic map — reduction map

$$\Phi[X] = \frac{1}{d-1}\left(\mathbb{I}\,\mathrm{Tr}X - X\right)$$

optimal but not extremal

Main result

$$\Phi[X] = \frac{1}{d-1}\left\{ 2\mathbb{I}\,\mathrm{Tr}X - \sum_{\alpha=1}^{d+1}\sum_{k,\ell=1}^{d} \mathcal{O}_{kl}^{(\alpha)}\mathrm{Tr}[XP_\ell^{(\alpha)}]P_k^{(\alpha)} \right\}$$

$\mathcal{O}_{kl}^{(\alpha)}$ — orthogonal matrices for $\alpha = 1, 2, \ldots, d+1$

$$\mathcal{O}^{(\alpha)}\mathbf{n}_* = \mathbf{n}_* \ ; \ \mathbf{n}_* = (1, 1, \ldots, 1)$$

Theorem:

Φ is a unital and trace-preserving positive map

$$\mathbb{H}_1 = \{ \ X = X^\dagger \mid \mathrm{Tr} X = 1 \ \}$$

$$S(\mathcal{H}) = \{\rho \in \mathbb{H}_1 \mid \rho \geq 0\}$$

$$\mathbf{B}_{\mathrm{in}} \subset S(\mathcal{H}) \subset \mathbf{B}_{\mathrm{out}}$$

$$X \in \mathbf{B}_{\mathrm{in}} \iff \mathrm{Tr} X^2 \leq \frac{1}{d-1}$$

$$X \in \mathbf{B}_{\mathrm{out}} \iff \mathrm{Tr} X^2 \leq 1$$

$$d = 2 \longrightarrow \mathbf{B}_{\mathrm{in}} = S(\mathcal{H}) = \mathbf{B}_{\mathrm{out}} = \text{Bloch ball}$$

Property of MUBs

$$x \in \mathcal{H} \; ; \quad \langle x|x\rangle = 1$$

$$\sum_{\alpha=1}^{d+1} \sum_{k=1}^{d} |\langle x|P_k^{(\alpha)}|x\rangle|^2 = 2$$

Quantum projective 2-design

$$P_1, \ldots, P_m \ (m \geq d^2)$$ – rank-1 projectors

For any homogeneous function of degree 2

$$f : S(\mathbb{C}^d) \longrightarrow \mathbb{C}$$

$$\frac{1}{m}\sum_{i=1}^{m} f(P_i) = \int_{S(\mathbb{C}^d)} f(|x\rangle\langle x|) d\mu_H(x)$$

$$\sum_{i=1}^{m} P_i \otimes P_i = k\Pi_{\mathrm{sym}}$$

$$x \in \mathcal{H} \ ; \quad \langle x|x\rangle = 1 \ \longrightarrow \ \sum_{i=1}^{m} |\langle x|P_i|x\rangle|^2 = k$$

$$\Phi[X] = \frac{1}{d-1}\left\{2\mathbb{I}\,\mathrm{Tr}X - \sum_{\alpha=1}^{d+1}\sum_{k,\ell=1}^{d}\mathcal{O}_{kl}^{(\alpha)}\mathrm{Tr}[XP_\ell^{(\alpha)}]P_k^{(\alpha)}\right\}$$

$$\mathbf{B}_{\text{in}} \subset S(\mathcal{H}) \subset \mathbf{B}_{\text{out}}$$

$$\Phi[\mathbf{B}_{\text{out}}] \subset \mathbf{B}_{\text{in}}$$

$$\Phi[\partial\mathbf{B}_{\text{out}}] \subset \partial\mathbf{B}_{\text{in}}$$

$$P = |x\rangle\langle x| \longrightarrow \Phi[P] \in \partial\mathbf{B}_{\text{in}}$$

Entanglement witness

$$\Phi \longrightarrow W_\Phi = (d-1)\sum_{i,j=1}^{d} E_{ij} \otimes \Phi E_{ij} \quad \text{(Choi matrix)}$$

$$W_\Phi = 2\mathbb{I}_d \otimes \mathbb{I}_d - \sum_{\alpha=1}^{d+1} \sum_{k,\ell=1}^{d} \mathcal{O}_{k\ell}^{(\alpha)} \overline{P}_\ell^{(\alpha)} \otimes P_k^{(\alpha)},$$

W_Φ is block-positive and $W \ngeq 0$

$\mathcal{O}^{(\alpha)}$ — orthogonal matrices for $\alpha = 1, 2, \ldots, d+1$

$$\mathcal{O}^{(\alpha)}\mathbf{n}_* = \mathbf{n}_* \ ; \ \mathbf{n}_* = (1, 1, \ldots, 1)$$

Definition: stochastic matrix

$$T\mathbf{n}_* = \mathbf{n}_* \ ; \quad T_{ij} \geq 0$$

doubly stochastic if T^{t} is stochastic

Definition: pseudo-stochastic matrix

$$T\mathbf{n}_* = \mathbf{n}_* \ ; \quad T_{ij} \in \mathbb{R}$$

doubly pseudo-stochastic if T^{t} is pseudo-stochastic

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$\mathcal{O}^{(\alpha)}$ is doubly pseudo-stochastic

$\mathcal{O}^{(\alpha)}$ is doubly stochastic $\iff$ $\mathcal{O}^{(\alpha)}$ is a permutation

Special classes – permutations

$$\Phi[X] = \frac{1}{d-1}\left\{ 2\mathbb{I}\,\mathrm{Tr}X - \sum_{\alpha=1}^{d+1}\sum_{k,\ell=1}^{d} \mathcal{O}^{(\alpha)}_{kl}\,\mathrm{Tr}[X P^{(\alpha)}_\ell]\, P^{(\alpha)}_k \right\}$$

$$\mathcal{O}^{(\alpha)} = \text{permutation matrix}$$

$$\mathcal{O}^{(\alpha)} = \mathbb{I}\ ; \quad \alpha = 1, 2, \ldots, d+1$$

$$\Phi[X] = \frac{1}{d-1}(\mathbb{I}\,\mathrm{Tr}X - X) \quad \text{(reduction map)}$$

Special classes – permutations

$$\Phi[X] = \frac{1}{d-1}\left\{2\mathbb{I}\,\mathrm{Tr}X - \sum_{\alpha=1}^{d+1}\sum_{k,\ell=1}^{d}\mathcal{O}_{kl}^{(\alpha)}\mathrm{Tr}[XP_\ell^{(\alpha)}]P_k^{(\alpha)}\right\}$$

$$\mathcal{O}^{(\alpha)} = \text{permutation matrix}$$

$$\mathcal{O}^{(\alpha)} = \mathbb{I}\ ;\quad \alpha = 1,2,\ldots,d+1$$

$$\Phi[X] = \frac{1}{d-1}\left(\mathbb{I}\,\mathrm{Tr}X - X\right) \quad \text{(reduction map)}$$

Reduction map vs. quantum entanglement

$$R[X] = \frac{1}{d-1}\left(\mathbb{I}\,\mathrm{Tr}X - X\right)$$

$$\rho \in S(\mathcal{H} \otimes \mathcal{H})$$

$(\mathbb{1} \otimes R)\rho \geq 0 \implies \rho$ is not distillable

Special classes – permutations

$$\Phi[X] = \frac{1}{d-1}\left\{ 2\mathbb{I}\,\mathrm{Tr}X - \sum_{\alpha=1}^{d+1}\sum_{k,\ell=1}^{d} \mathcal{O}_{kl}^{(\alpha)} \mathrm{Tr}[XP_\ell^{(\alpha)}]P_k^{(\alpha)} \right\}$$

$$\mathcal{X}|k\rangle = |k+1\rangle \pmod d$$

$$\mathcal{O}^{(1)} = \mathcal{X} \ ; \ \mathcal{O}^{(\alpha)} = \mathbb{I} \ ; \quad \alpha = 2, \ldots, d+1$$

$$\Phi X = \frac{1}{d-1}\left(2\varepsilon[X] + \sum_{i=2}^{d-1} \varepsilon[\mathcal{X}^i X \mathcal{X}^{\dagger i}] - X \right),$$

$$\varepsilon[X] = \sum_{i=1}^{d} P_i^{(1)} X P_i^{(1)}$$

A case study: $d = 3$

$$\mathcal{O}^{(\alpha)}\mathbf{n}_* = \mathbf{n}_* \ ; \quad \mathbf{n}_* = (1,1,1)$$

rotation around $\mathbf{n}_*$

$$\mathcal{O}(\varphi) = \begin{pmatrix} c_1(\varphi) & c_2(\varphi) & c_3(\varphi) \\ c_3(\varphi) & c_1(\varphi) & c_2(\varphi) \\ c_2(\varphi) & c_3(\varphi) & c_1(\varphi) \end{pmatrix} ,$$

$$\begin{aligned} c_1(\varphi) &= \frac{2}{3}\cos\varphi + \frac{1}{3} , \\ c_2(\varphi) &= \frac{2}{3}\cos\left(\varphi - \frac{2\pi}{3}\right) + \frac{1}{3} , \\ c_3(\varphi) &= \frac{2}{3}\cos\left(\varphi + \frac{2\pi}{3}\right) + \frac{1}{3} . \end{aligned}$$

$(\varphi_1, \varphi_2, \varphi_3, \varphi_4)$

$$W_\Phi = 2\mathbb{I}_d \otimes \mathbb{I}_d - \sum_{\alpha=1}^{d+1} \sum_{k,\ell=1}^{d} \mathcal{O}_{k\ell}^{(\alpha)}(\varphi_\alpha) \overline{P}_\ell^{(\alpha)} \otimes P_k^{(\alpha)},$$

$$W = \left(\begin{array}{ccc|ccc|ccc} a & \cdot & \cdot & \cdot & p^* & \cdot & \cdot & \cdot & p \\ \cdot & b & \cdot & \cdot & \cdot & q^* & q & \cdot & \cdot \\ \cdot & \cdot & c & r^* & \cdot & \cdot & \cdot & r & \cdot \\ \hline \cdot & \cdot & r & c & \cdot & \cdot & \cdot & r^* & \cdot \\ p & \cdot & \cdot & \cdot & a & \cdot & \cdot & \cdot & p^* \\ \cdot & q & \cdot & \cdot & \cdot & b & q^* & \cdot & \cdot \\ \hline \cdot & q^* & \cdot & \cdot & \cdot & q & b & \cdot & \cdot \\ \cdot & \cdot & r^* & r & \cdot & \cdot & \cdot & c & \cdot \\ p^* & \cdot & \cdot & \cdot & p & \cdot & \cdot & \cdot & a \end{array}\right)$$

$(\varphi_1, \varphi_2, \varphi_3, \varphi_4)$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^* \\ 1 & \omega^* & \omega \end{pmatrix} \begin{pmatrix} 2 \\ e^{i\varphi_1} \\ e^{-i\varphi_1} \end{pmatrix}$$

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} = -\frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^* & \omega \\ 1 & \omega & \omega^* \end{pmatrix} \begin{pmatrix} e^{i\varphi_2} \\ e^{-i\varphi_3} \\ e^{i\varphi_4} \end{pmatrix}$$

$(\varphi_1, \varphi_2 = \varphi_3 = \varphi_4 = 0)$

$$W = \left(\begin{array}{ccc|ccc|ccc} a & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & -1 \\ \cdot & b & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & c & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & c & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & a & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & b & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c & \cdot \\ -1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & a \end{array}\right)$$

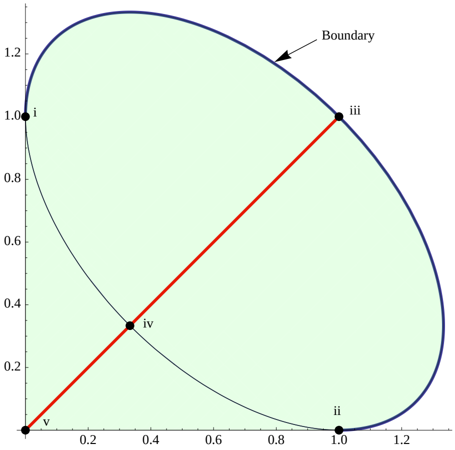
$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^* \\ 1 & \omega^* & \omega \end{pmatrix}\begin{pmatrix} 2 \\ e^{i\varphi_1} \\ e^{-i\varphi_1} \end{pmatrix}$$

Choi-like witness

$$W = \left(\begin{array}{ccc|ccc|ccc} a & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & -1 \\ \cdot & b & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & c & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & c & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & a & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & b & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c & \cdot \\ -1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & a \end{array}\right)$$

W is block-positive if and only if (Cho-Kye-Lee 1992)

- $a, b, c \geq 0$
- $a + b + c \geq 2$
- $a \leq 1 \implies bc \geq (1-a)^2$



- $(a, b, c) = (0, 1, 1)$ — reduction map (iii)
- $(a, b, c) = (1, 1, 0)$ or $(1, 0, 1)$ — Choi maps (i) and (ii)

Conclusions

- new construction of positive maps from MUBs
- generalization of well known maps
- Problems: further analysis of
  - optimality
  - extremality
  - exposedness
  - spanning property

Happy birthday Bal!!!

Choi-like maps for d prime

$$\Phi X = \Phi_* X - \frac{1}{d-1} \sum_{\alpha=1}^{d+1} \sum_{k,\ell=1}^{d} \mathcal{O}_{kl}^{(\alpha)} \mathrm{Tr}[\widetilde{X} P_\ell^{(\alpha)}] P_k^{(\alpha)}$$

$$\mathcal{O}^{(1)} \in T^{\frac{d-1}{2}} = \text{maximal commutative subgroup (torus) of } SO(d-1)$$

$$\mathcal{O}^{(\alpha)} = \mathbb{I}_d \; ; \quad \alpha = 2, \ldots, d+1$$

$$\text{torus} \; \longrightarrow \; (\varphi_1, \ldots, \varphi_{\frac{d-1}{2}})$$

$$\lambda_0 := d-1 \; , \lambda_k = e^{i\varphi_k} = \lambda_{d-k}^* \; ; k = 1, \ldots, \frac{d-1}{2}$$

Choi-like witness for d prime $\sim\ T^{d_*}$

$$\lambda_0 := d-1\ ,\lambda_k = e^{i\varphi_k} = \lambda^*_{d-k}\ ; k = 1,\ldots,d_* := \frac{d-1}{2}$$

$$a_k = \frac{1}{d}\sum_{\ell=0}^{d-1}\omega^{k\ell}\lambda_\ell\ ; \quad \omega = e^{2\pi i/d}$$

$$W = \sum_{k,\ell=0}^{d-1} E_{k\ell}\otimes W_{k\ell}$$

$$W_{k\ell} = -E_{k\ell}\ ; \quad k\neq\ell$$

$$W_{kk} = \mathcal{X}^k W_{00}\mathcal{X}^{\dagger k}$$

$$W_{00} = \mathrm{diag}(a_0, a_1, \ldots, a_{d-1})$$

Conjecture

W has a bi-spanning property (and hence is nd-optimal) if

$$a_0 = \frac{1}{d}(d - 1 - 2[\cos\varphi_1 + \ldots + \cos\varphi_{d_*}]) \leq 1 \; ; a_k \neq 0 \; (k > 0)$$

$$\cos\varphi_1 + \ldots + \cos\varphi_{d_*} \geq -\frac{1}{2} \quad ; \quad (\varphi_1, \ldots, \varphi_{d_*}) \neq (0, \ldots, 0)$$

$$(\varphi_1, \ldots, \varphi_{d_*}) = (0, \ldots, 0) \;\longrightarrow\; \text{reduction}$$

Proof for $d = 5$

$d = 5$ vs. $d = 3$

$$\text{2D torus} \longrightarrow (\varphi_1, \varphi_2) \quad ; \quad \text{1D torus} \longrightarrow \varphi_1$$

$$\cos\varphi_1 + \cos\varphi_2 \geq -\frac{1}{2} \quad : \quad \cos\varphi_1 \geq -\frac{1}{2}$$