BOUNDARIES WITHOUT BOUNDARIES

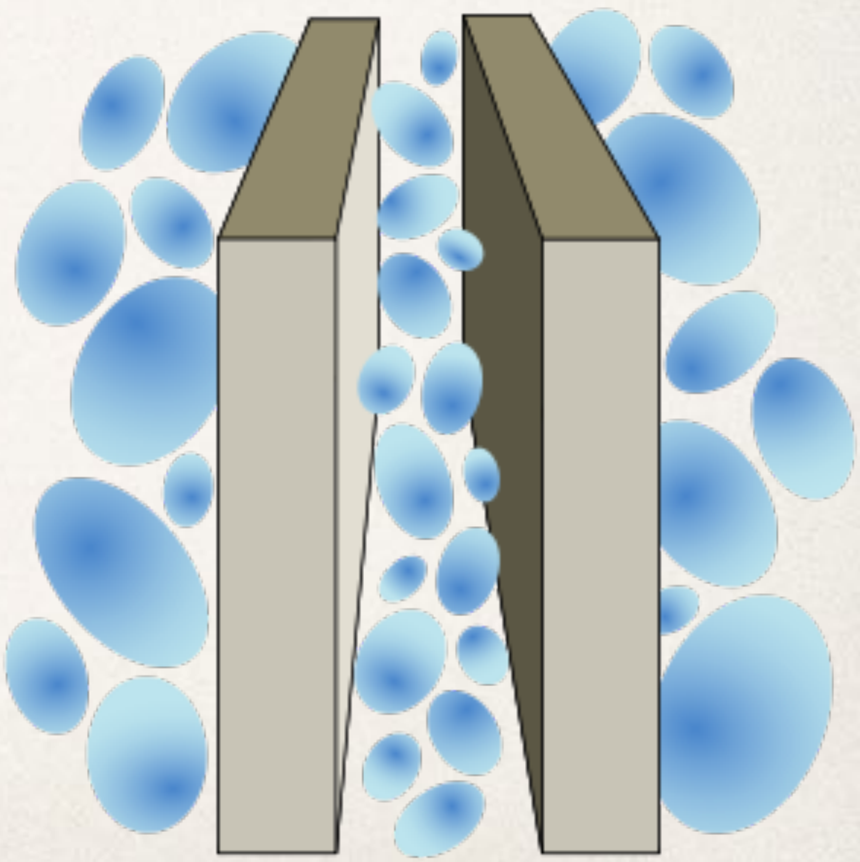
Paolo Facchi
Università di Bari & INFN, Italy

Work with:
G. Garnero, G. Marmo, J. Samuel, S. Sinha
Bari, Napoli, Bangalore

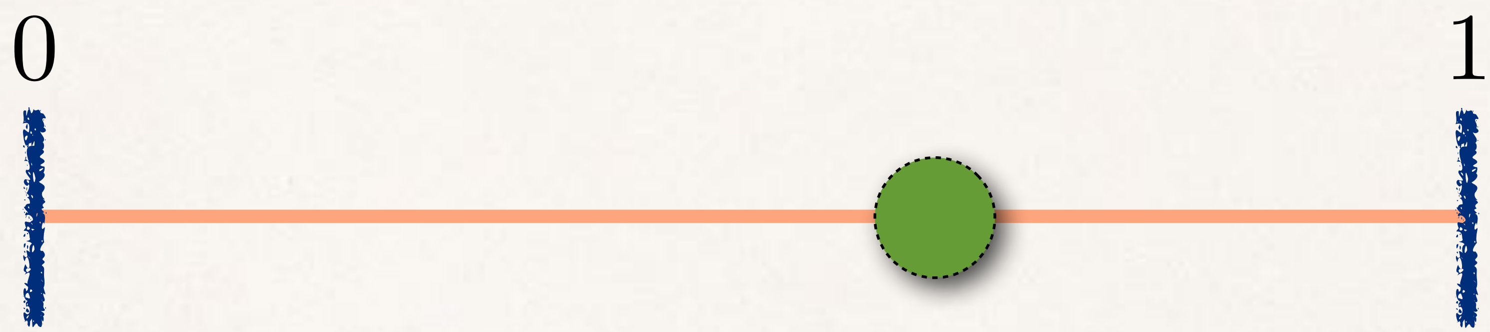
Fields, Particles and Information Geometry
BALFEST80
DIAS, Dublin, 24 January 2018

Outline

- Quantum boundary conditions
- Boundaries by reduction: from a circle to an interval
- A quantum particle on a circle: Laplacian
- Unitary folding
- Momentum on a half-line
- Momentum on an interval



A quantum particle on a segment



test functions
compact support

$$H = T = \frac{p^2}{2m} = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}$$

$$D(T) = \mathcal{D}(0,1)$$

What happens at the boundary?

$$D(T^*) = \{\psi \in L^2(0,1), \psi'' \in L^2(0,1)\} = H^2[0,1]$$

Quantum Boundary Conditions

$$T_U \psi = -\frac{\hbar^2}{2m} \psi'' \qquad\qquad U \in \mathrm{U}(2)$$

$$i\,(I + U)\,\varphi = (I - U)\,\dot{\varphi}$$

$$\varphi = \begin{pmatrix} \psi(0) \\ \psi(1) \end{pmatrix} \qquad\qquad \dot{\varphi} = \begin{pmatrix} \psi'(0) \\ -\psi'(1) \end{pmatrix}$$

M. Asorey, A. Ibort, G. Marmo, Int. J. Mod. Phys. A **20**, 1001 (2005)

Quantum Boundary Conditions

Self-adjoint extensions

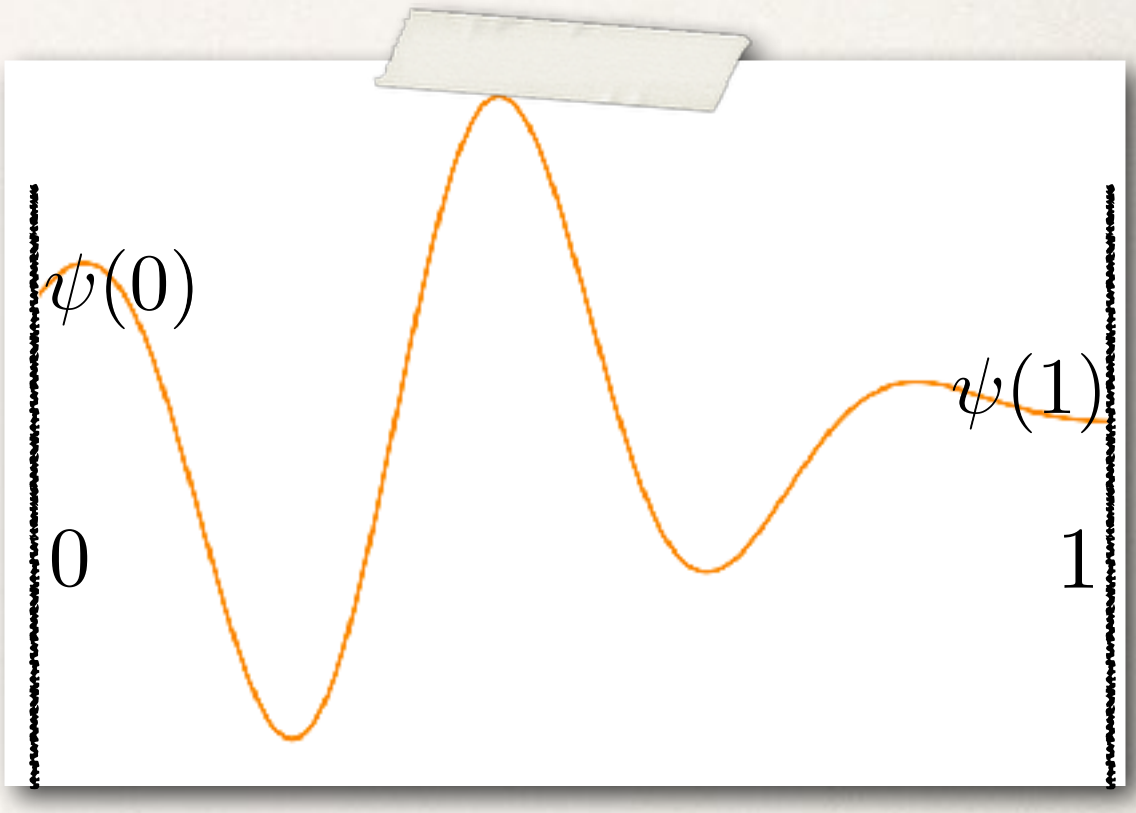
Unitary matrices at the boundary

$$i\,(I+U)\,\varphi = (I-U)\,\dot{\varphi} \qquad\qquad U \in \mathrm{U}(2)$$

M. Asorey, A. Ibort, G. Marmo, Int. J. Mod. Phys. A **20**, 1001 (2005)

Examples

$$i\,(I+U)\begin{pmatrix}\psi(0)\\ \psi(1)\end{pmatrix} = (I-U)\begin{pmatrix}\psi'(0)\\ -\psi'(1)\end{pmatrix}$$



Dirichlet

$$U = I \qquad \psi(0) = 0 = \psi(1)$$

Robin

$$U = -e^{-i\alpha} I$$

$$\alpha \in \mathbb{R} \qquad \begin{cases}\psi'(0) = -\tan\frac{\alpha}{2}\,\psi(0),\\ \psi'(1) = \tan\frac{\alpha}{2}\,\psi(1)\end{cases}$$

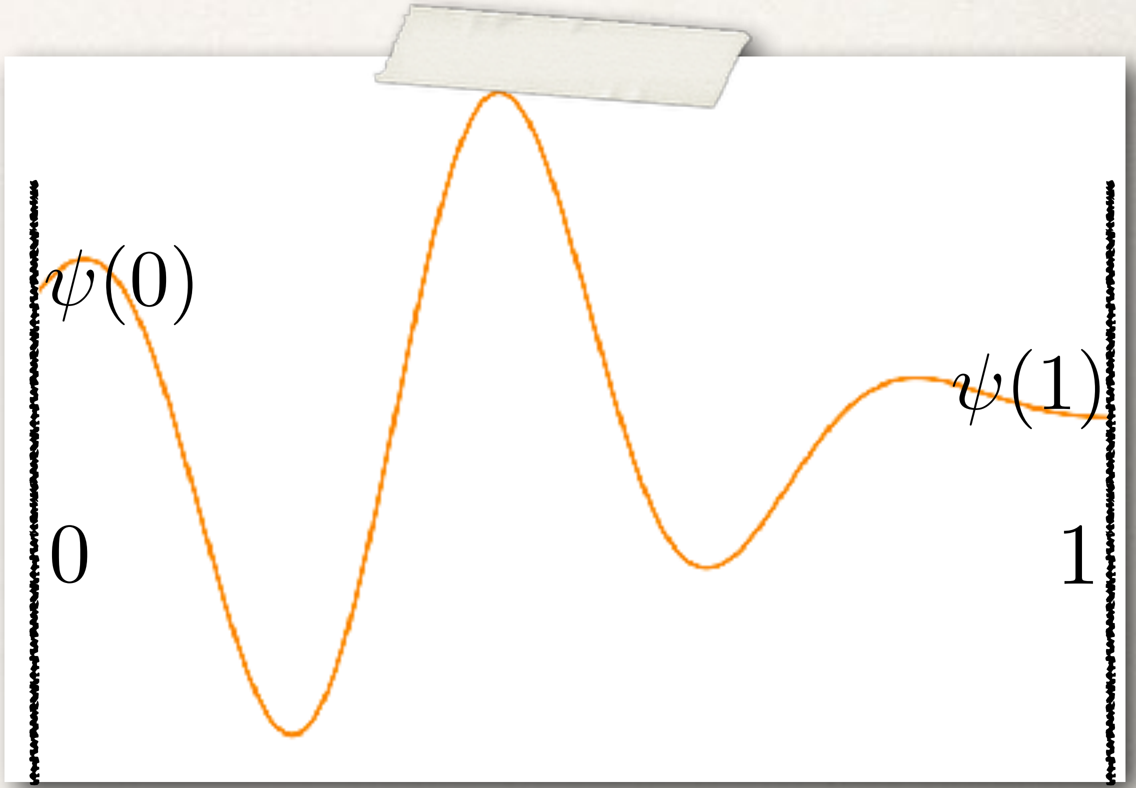
Neumann

$$U = -I$$

$$\psi'(0) = 0 = \psi'(1)$$

Examples

$$i\,(I+U)\begin{pmatrix}\psi(0)\\ \psi(1)\end{pmatrix} = (I-U)\begin{pmatrix}\psi'(0)\\ -\psi'(1)\end{pmatrix}$$



Dirichlet+Robin

$$U = \begin{pmatrix} 1 & 0 \\ 0 & -\mathrm{e}^{-\mathrm{i}\alpha} \end{pmatrix} \qquad \begin{cases} \psi(0) = 0 \\ \psi'(1) = \tan\frac{\alpha}{2}\,\psi(1) \end{cases}$$

$$\alpha \in \mathbb{R}$$

Periodic

$$U = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

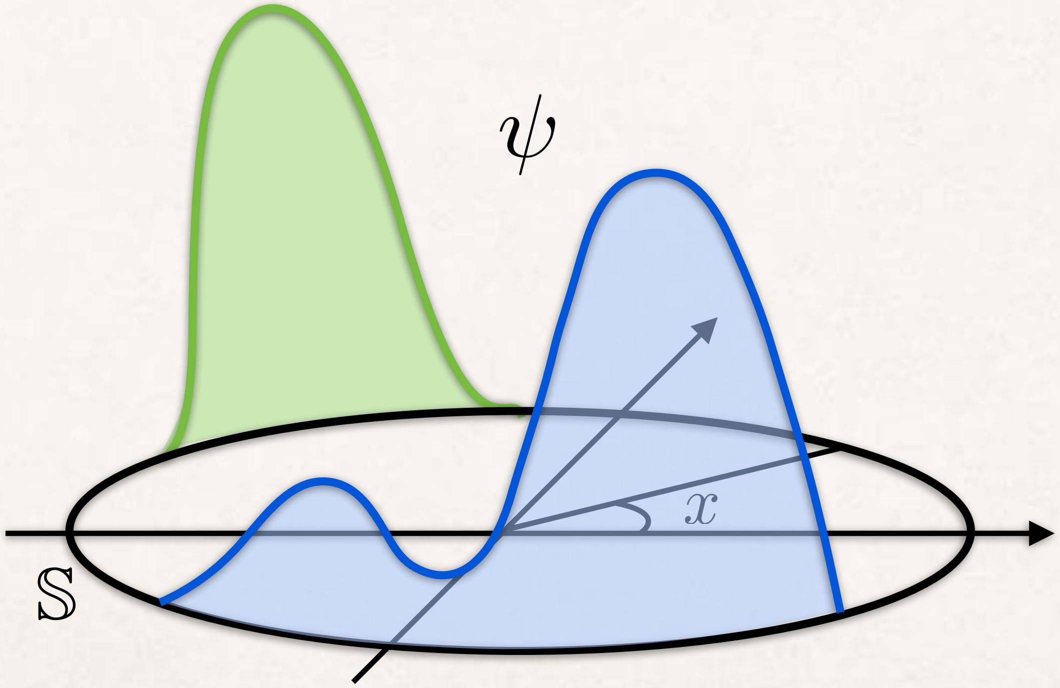
$$\begin{cases} \psi(0) = \psi(1) \\ \psi'(0) = \psi'(1) \end{cases}$$

Antiperiodic

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{cases} \psi(0) = -\psi(1) \\ \psi'(0) = -\psi'(1) \end{cases}$$

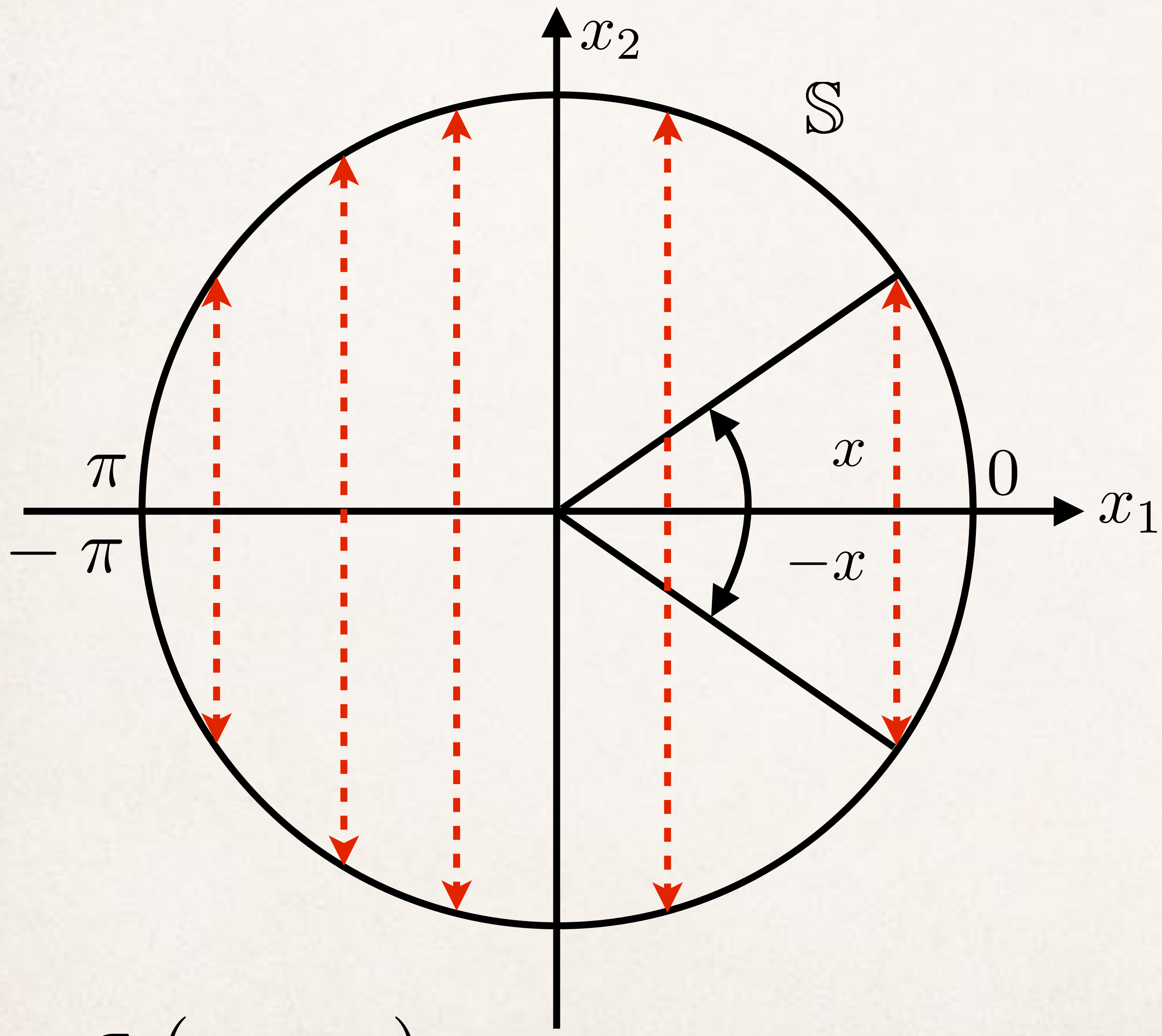
Boundaries without boundaries

Generate quantum boundary conditions starting from a particle on a manifold without boundaries?



P.F., G. Garnero, G. Marmo, J. Samuel, S. Sinha, arXiv:1711.03029 [quant-ph]

Reduction by symmetry



$x \in (-\pi, \pi)$

$$\Pi : \mathbb{S} \to \mathbb{S},$$

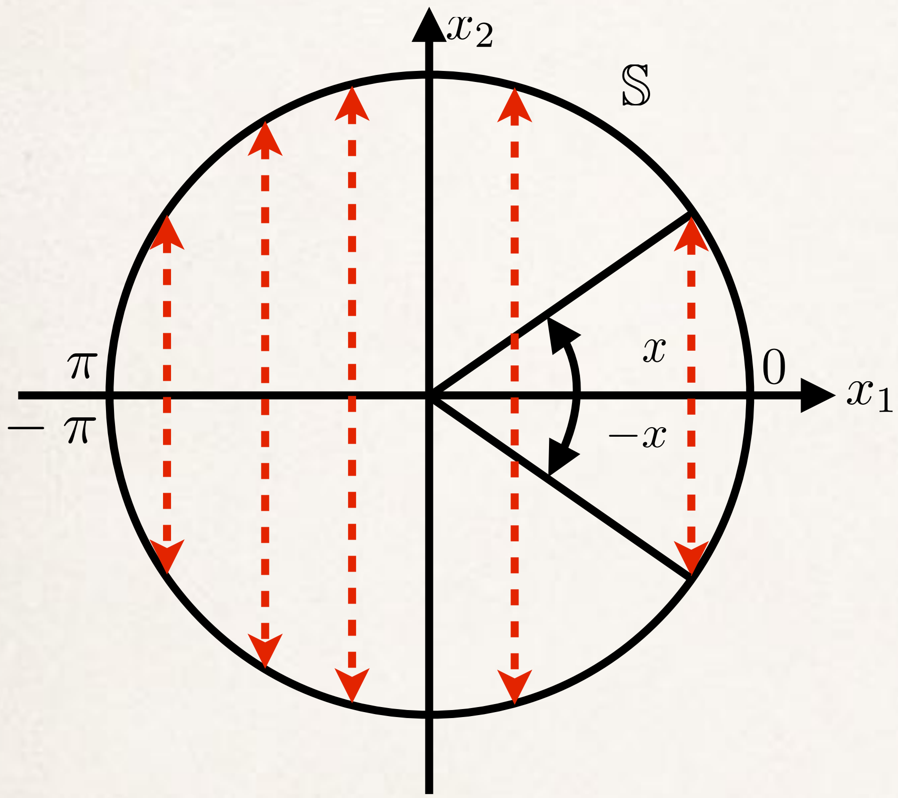
$$\Pi(x_1, x_2) = (x_1, -x_2)$$

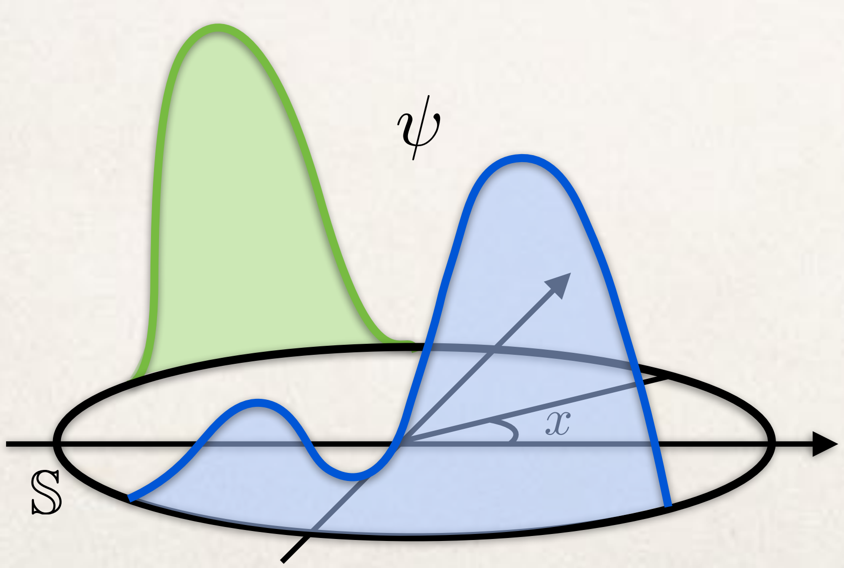
$$\Pi(x) = -x$$

$$\mathbb{S}_+ = \mathbb{S}/\Pi = [0, \pi]$$

Manifold with boundary

QBC by reduction





$$\Pi(x) = -x$$

$$P : L^2(\mathbb{S}) \to L^2(\mathbb{S})$$

$$(P\psi)(x) = \psi(\Pi(x)) = \psi(-x)$$

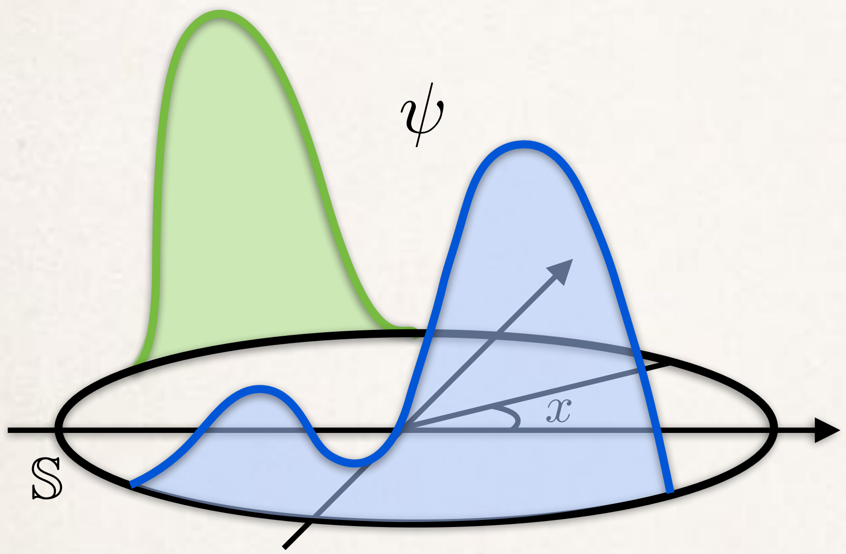
parity operator

$$L^2(\mathbb{S}) = \mathcal{H}_+ \oplus \mathcal{H}_-$$

$$\mathcal{H}_\pm = \{\psi \in L^2(-\pi,\pi) \,|\, \psi(-x) = \pm\psi(x)\}$$

even/odd functions

QBC by reduction



Free particle Hamiltonian

$$H = \frac{p^2}{2m} = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}$$

domain of self-adjointness

$$\mathrm{H}^2(\mathbb{S}) = \{\psi \in \mathrm{H}^2[-\pi, \pi] : \psi(-\pi) = \psi(\pi)\,,\ \psi'(-\pi) = \psi'(\pi)\}$$

second Sobolev space

$$H\,P = P\,H$$ commutes with parity

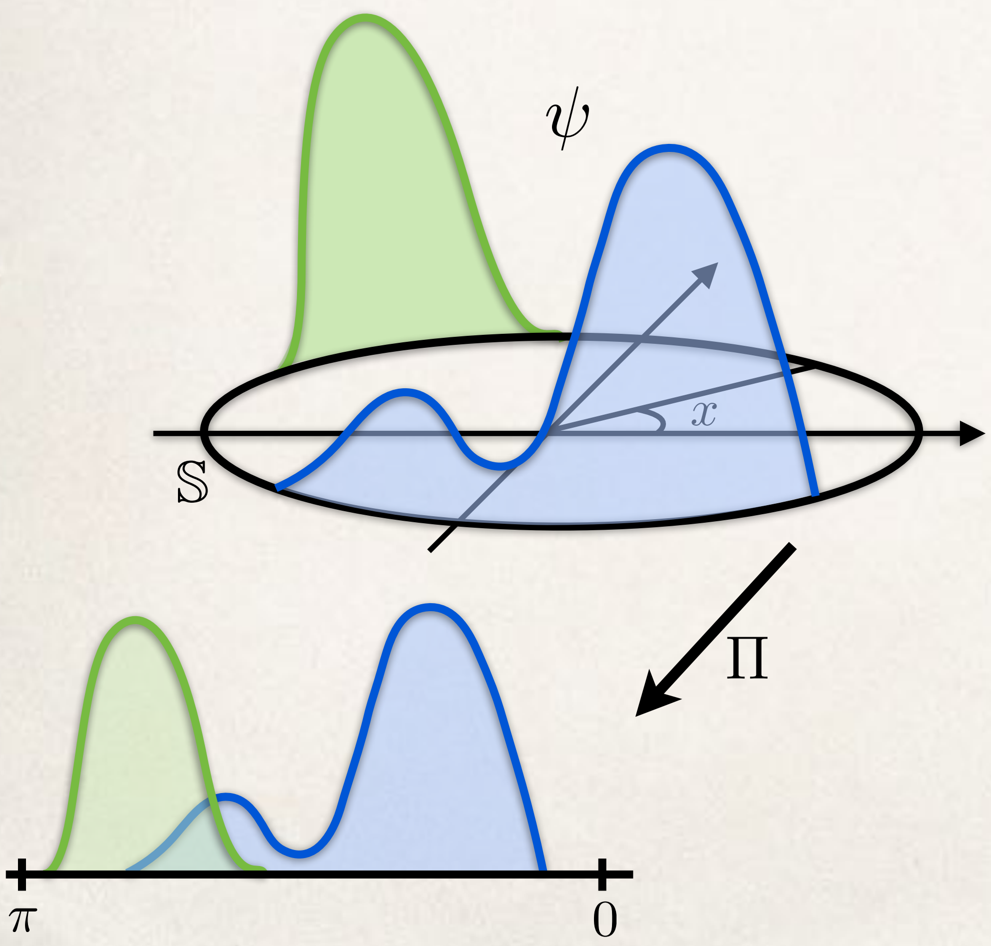
$$L^2(\mathbb{S}) = \mathcal{H}_+ \oplus \mathcal{H}_-$$ invariant subspaces

QBC by reduction

$$\mathrm{H}^2(\mathbb{S}) = \{\psi \in \mathrm{H}^2[-\pi,\pi] : \psi(-\pi) = \psi(\pi)\,,\ \psi'(-\pi) = \psi'(\pi)\}$$ circle

restriction to even functions:

$$D(H|_{\mathcal{H}_+}) = \mathrm{H}^2(\mathbb{S}) \cap \mathcal{H}_+ = \{\psi \in \mathrm{H}^2[-\pi,\pi] \cap \mathcal{H}_+ : \psi'(-\pi) = 0 = \psi'(\pi)\}$$



$$U_+ : \mathcal{H}_+ \to L^2(0,\pi)$$ interval

$$\phi(x) = (U_+\psi)(x) = \sqrt{2}\,\psi(x) \qquad x \in [0,\pi]$$

$$H_+ := U_+\, H|_{\mathcal{H}_+}\, U_+^\dagger = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}$$ on $L^2(0,\pi)$

$$D(H_+) = \{\psi \in \mathrm{H}^2[0,\pi] : \psi'(0) = 0 = \psi'(\pi)\}$$

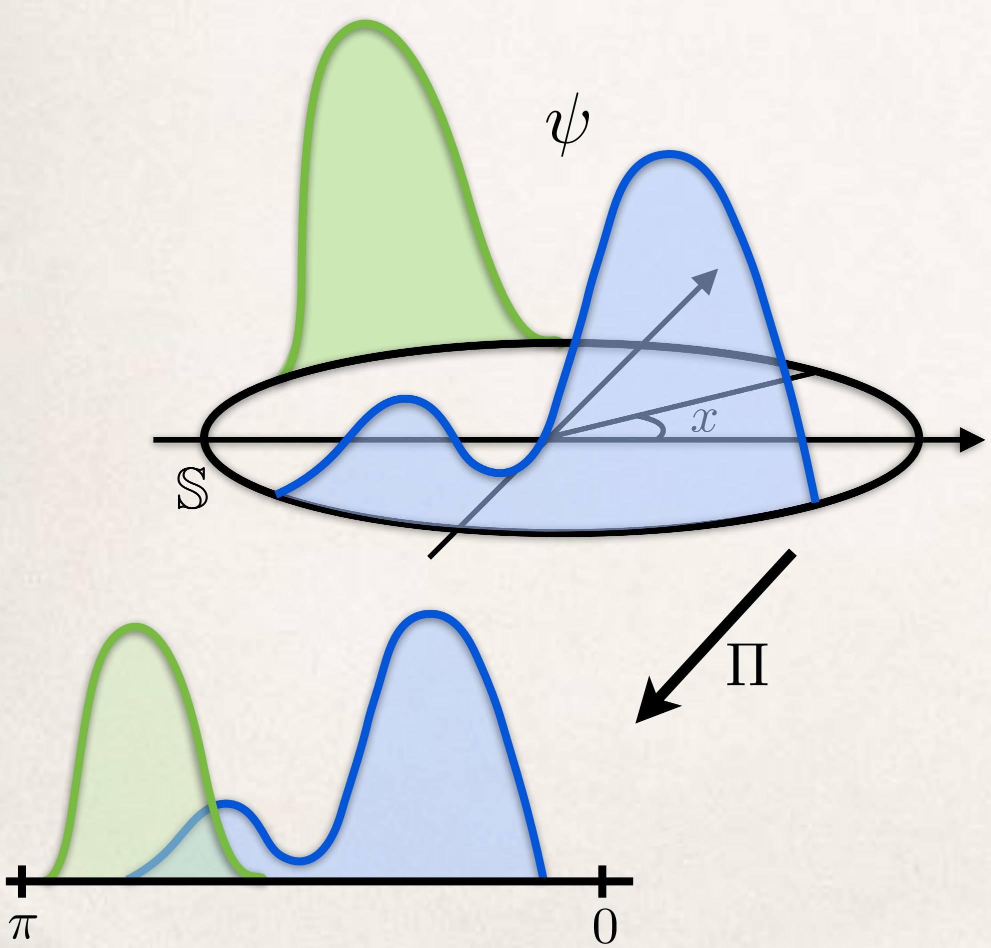
Neumann

QBC by reduction

$$\mathrm{H}^2(\mathbb{S}) = \{\psi \in \mathrm{H}^2[-\pi,\pi] : \psi(-\pi) = \psi(\pi)\,,\ \psi'(-\pi) = \psi'(\pi)\}$$ circle

restriction to odd functions:

$$D(H|_{\mathcal{H}_-}) = \mathrm{H}^2(\mathbb{S}) \cap \mathcal{H}_- = \{\psi \in \mathrm{H}^2[-\pi,\pi] \cap \mathcal{H}_- : \psi(-\pi) = 0 = \psi(\pi)\}$$



$$U_- : \mathcal{H}_- \to L^2(0,\pi)$$ interval

$$\phi(x) = (U_-\psi)(x) = \sqrt{2}\,\psi(x) \qquad x \in [0,\pi]$$

$$H_- = U_-\, H|_{\mathcal{H}_-}\, U_-^\dagger = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}$$ on $L^2(0,\pi)$

$$D(H_-) = \{\psi \in \mathrm{H}^2[0,\pi] : \psi(0) = 0 = \psi(\pi)\}$$

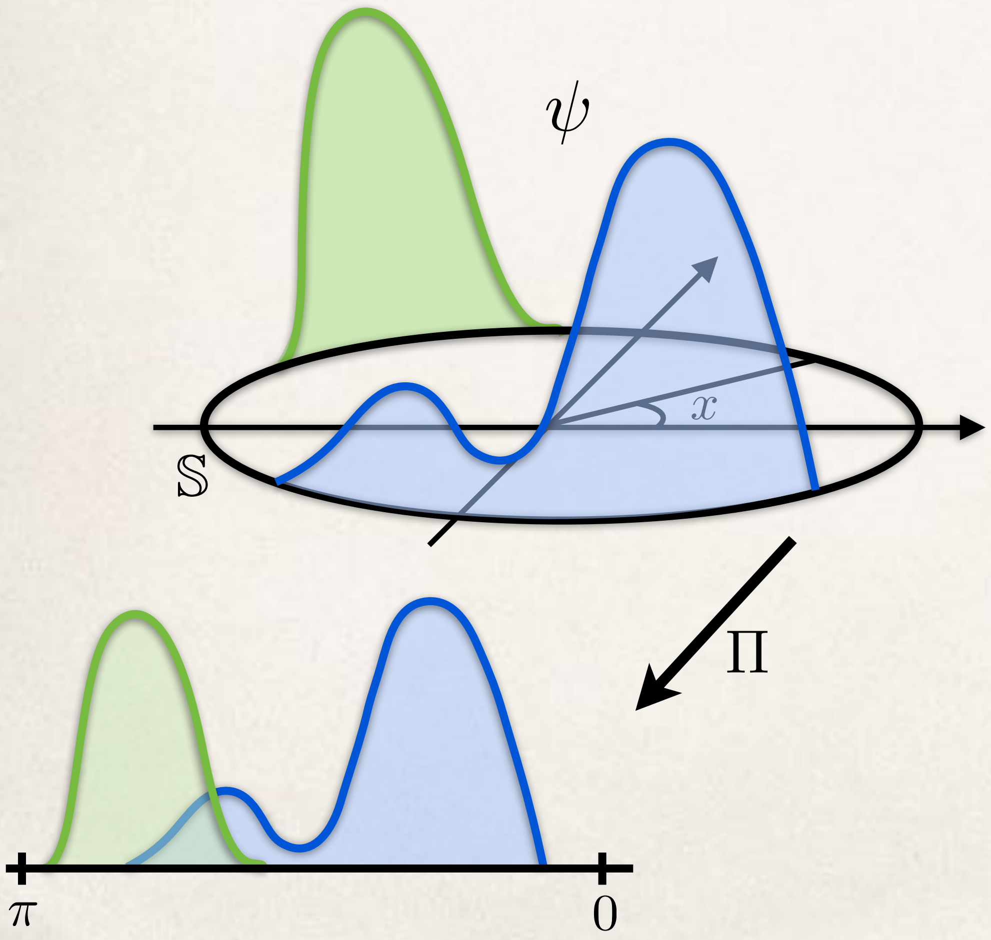
Dirichlet

QBC by reduction

$$H = \frac{p^2}{2m} = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}$$

restriction to even functions Neumann

restriction to odd functions Dirichlet



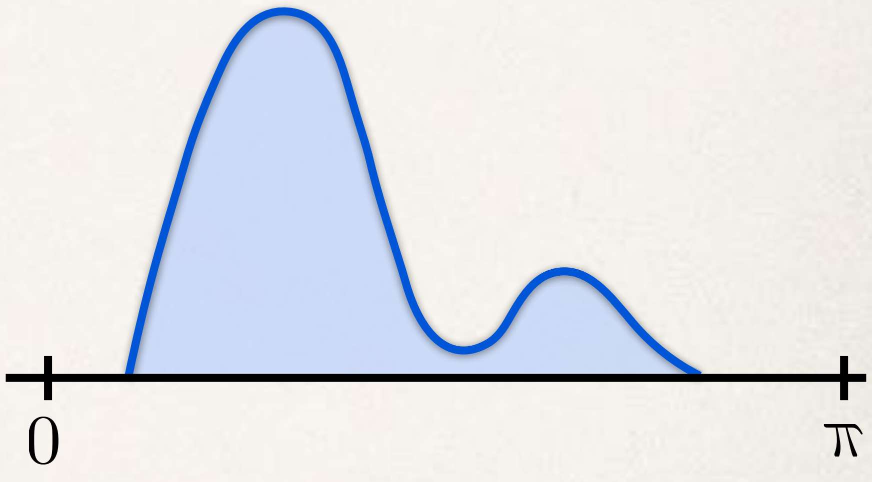
Other boundary conditions?

General QBC

spatial metric $\quad \mathrm{d}s^2 = \mathrm{d}x^2$

Hamiltonian $\quad H = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}$

Neumann bc $\quad \psi'(0) = 0 = \psi'(\pi)$



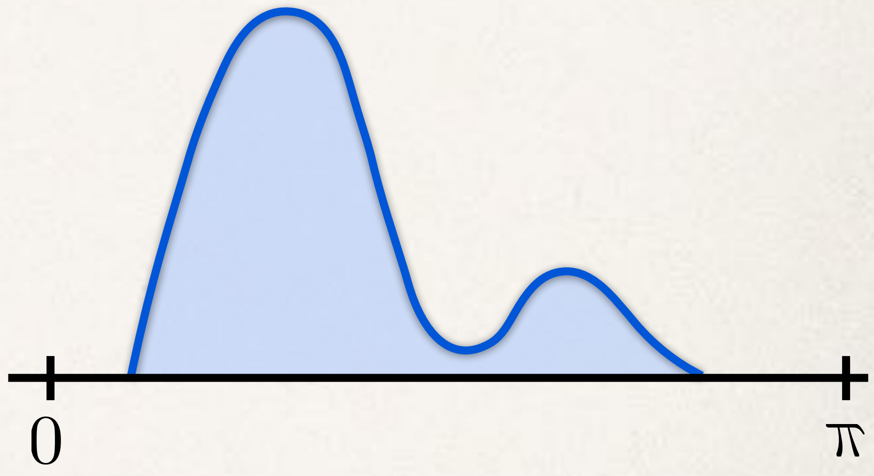
Change of coordinates on $[0, \pi]$

$$y = F(x) = \int_0^x f(t)\mathrm{d}t \qquad f \geq 0 \qquad \int_0^\pi f(t)\mathrm{d}t = \pi$$

General QBC

spatial metric

$$\mathrm{d}s^2 = \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2 \mathrm{d}y^2 = \frac{1}{[f(y)]^2}\mathrm{d}y^2$$



unitary transformation

$$\phi(y) = (U_f\,\psi)(y) = \frac{1}{\sqrt{g(y)}}\psi(F^{-1}(y)), \qquad g(y) = f(F^{-1}(y))$$

Transformed Hamiltonian

$$H_f = U_f\,H\,U_f^\dagger = -g^2\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}y^2} - gg'\frac{\hbar^2}{m}\frac{\mathrm{d}}{\mathrm{d}y} + V, \qquad V = \frac{\hbar^2}{8m}\left[(g')^2 + 2gg''\right]$$

General QBC

boundary conditions
$$\begin{cases} \phi'(0) = \nu_0\, \phi(0), \\ \phi'(\pi) = -\nu_\pi\, \phi(\pi), \end{cases}$$

Robin!

$$\nu_0 = -\frac{1}{2}\frac{f'(0)}{[f(0)]^2}, \qquad \nu_\pi = \frac{1}{2}\frac{f'(\pi)}{[f(\pi)]^2}$$

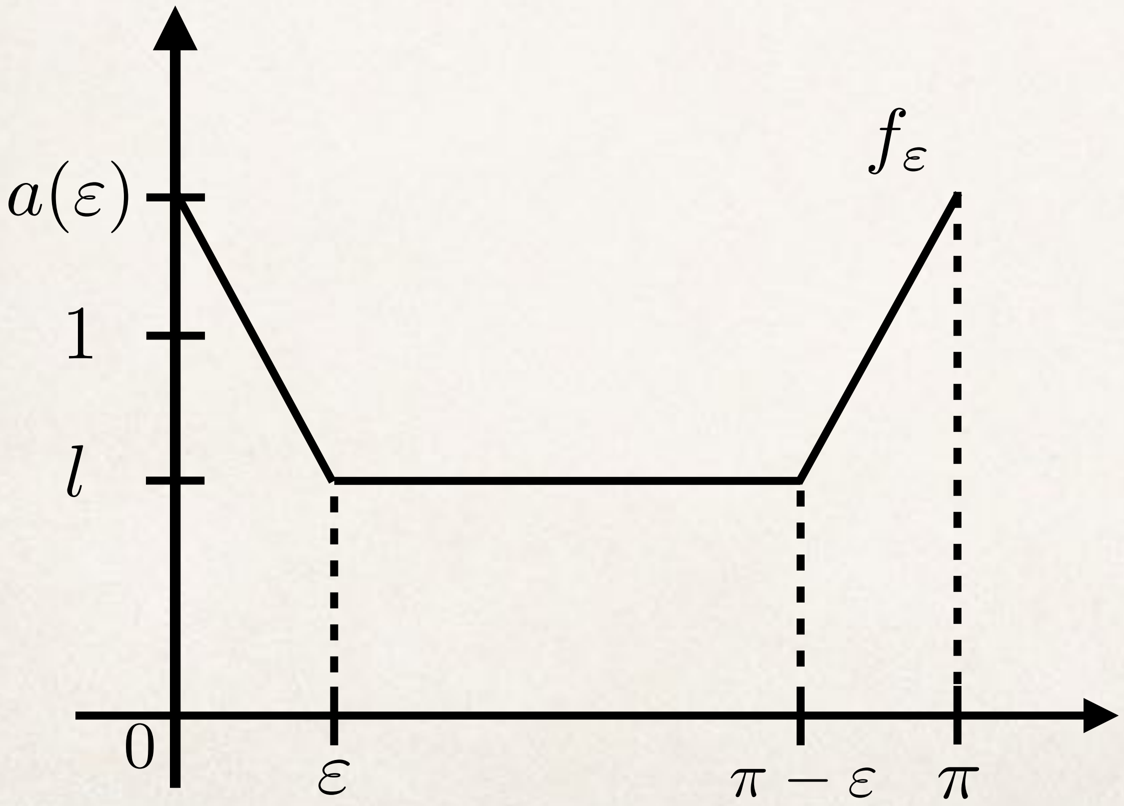
But transformed Hamiltonian ≠ kinetic energy

Change the metric only in a boundary layer!

From Neumann to Robin

$$H_f = U_f\, H\, U_f^\dagger = -g^2 \frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}y^2} - gg'\frac{\hbar^2}{m}\frac{\mathrm{d}}{\mathrm{d}y} + V,$$

$$V = \frac{\hbar^2}{8m}\left[(g')^2 + 2gg''\right] \qquad \nu_0 = -\frac{1}{2}\frac{f'(0)}{[f(0)]^2}, \qquad \nu_\pi = \frac{1}{2}\frac{f'(\pi)}{[f(\pi)]^2}$$



sequence of functions

$$f_\varepsilon(x) \to k \qquad x \in (0,\pi)$$

$$H_{f_\varepsilon} \to -\frac{\hbar^2}{2m}k^2\frac{\mathrm{d}^2}{\mathrm{d}y^2} = -\frac{\hbar^2}{2M}\frac{\mathrm{d}^2}{\mathrm{d}y^2}$$

$$\lim_{\varepsilon\to 0}\nu_0 = \mu_0 > 0,$$

$$\lim_{\varepsilon\to 0}\nu_\pi = \mu_\pi > 0.$$

Symmetry?

Crucial ingredient for reduction:

$H\,P = P\,H$ Hamiltonian commutes with parity

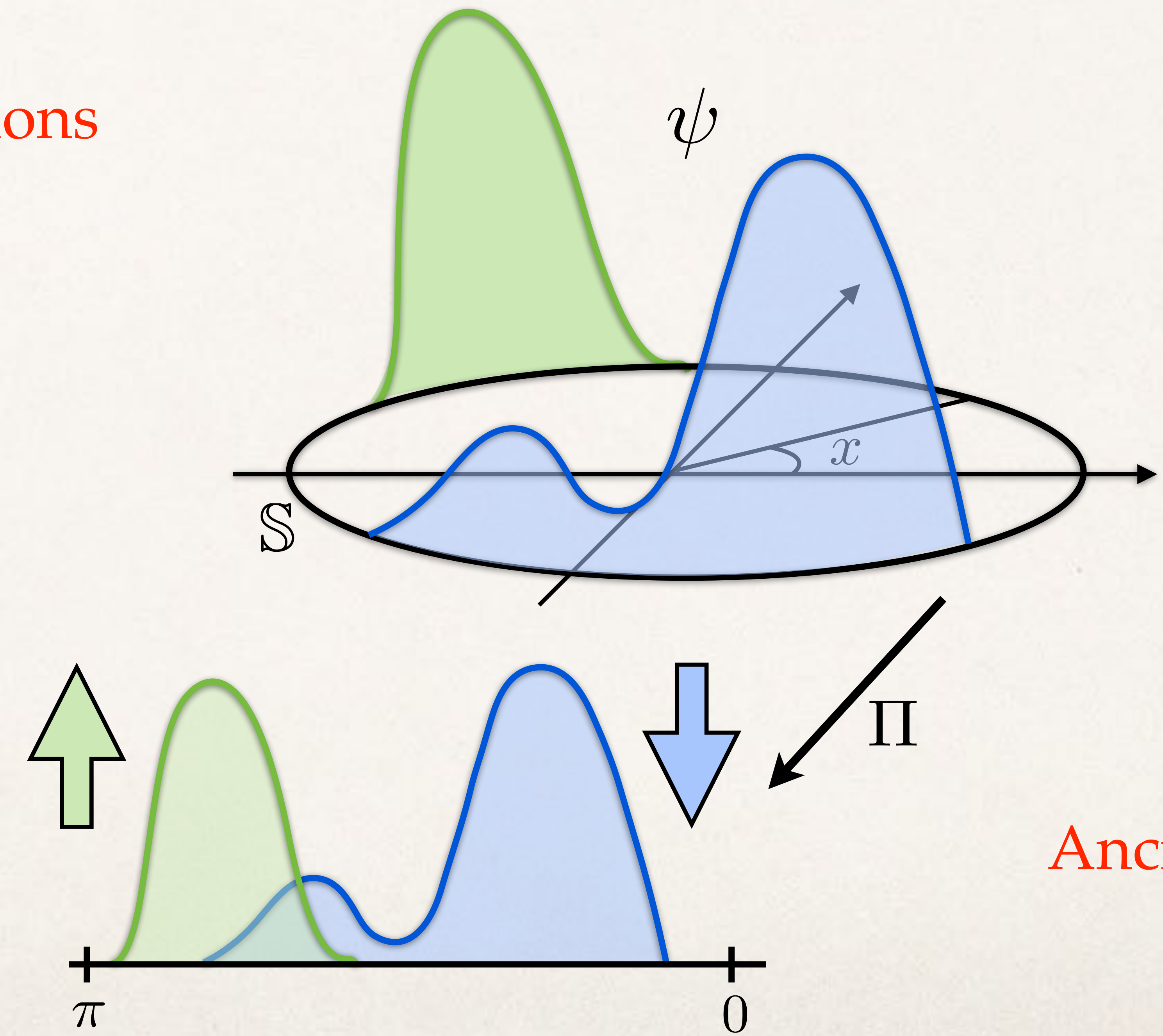
$L^2(\mathbb{S}) = \mathcal{H}_+ \oplus \mathcal{H}_-$ invariant subspaces

What if $[H, P] \neq 0$ and there are no invariant subspaces?

Replace projections by unitaries: folding

Folding

Replace projections
by unitaries



Ancillary spin

Momentum on the half-line

$$p = -i\hbar \frac{\mathrm{d}}{\mathrm{d}x} \qquad \text{on} \ \ L^2(\mathbb{R}_+)$$

it admits no self-adjoint extensions!

Momentum on the half-line

Consider the momentum on the full line

$$p = -i\hbar \frac{\mathrm{d}}{\mathrm{d}x}$$

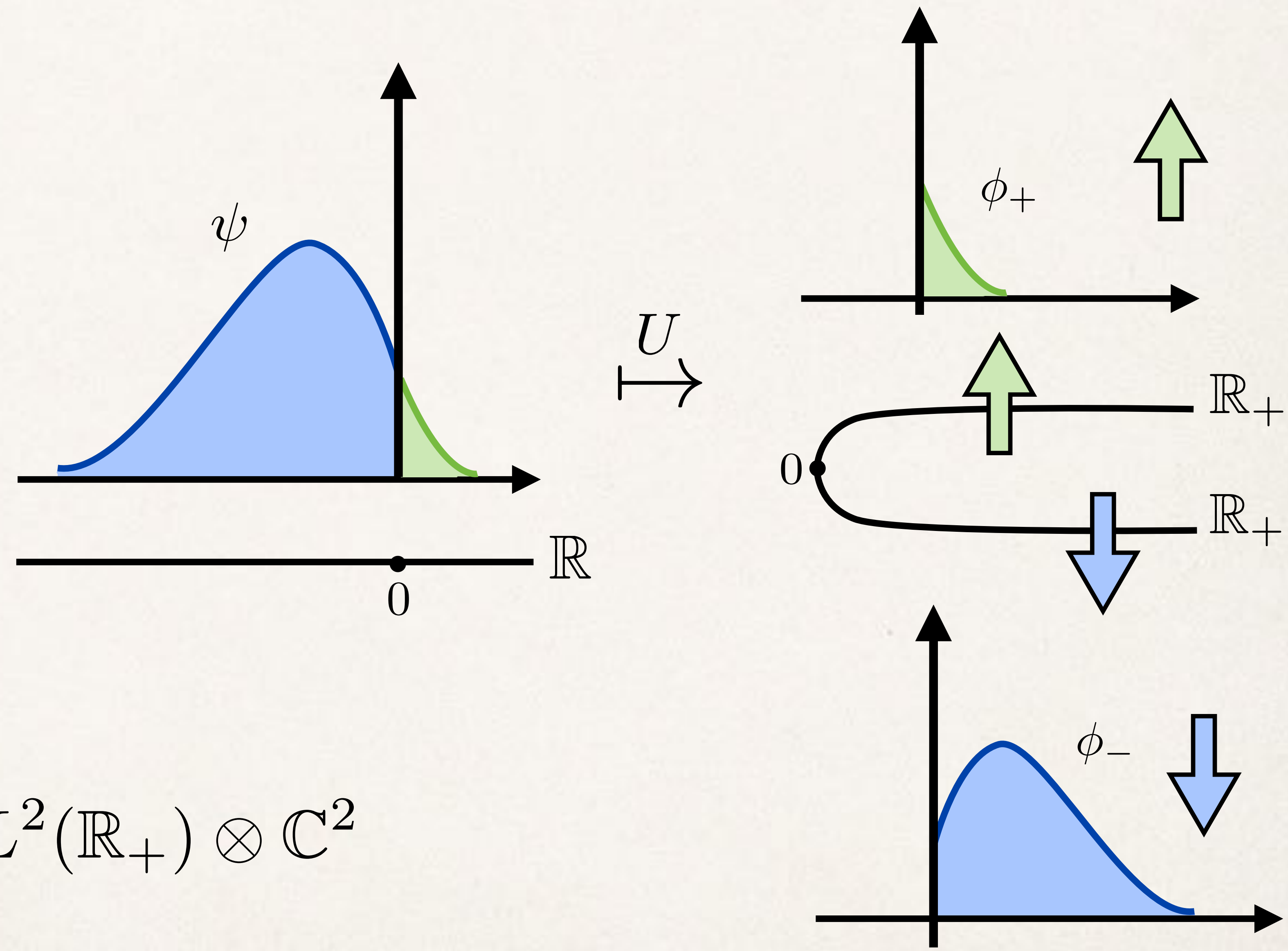
$$D(p) = \mathrm{H}^1(\mathbb{R}) = \{\psi \in L^2(\mathbb{R}) \mid \psi' \in L^2(\mathbb{R})\,\}$$ first Sobolev space

It is self-adjoint.

Need a unitary mapping onto the half-line

Folding

Ancillary spin



Unitary:

$$U : L^2(\mathbb{R}) \quad \to \quad L^2(\mathbb{R}_+) \otimes \mathbb{C}^2$$

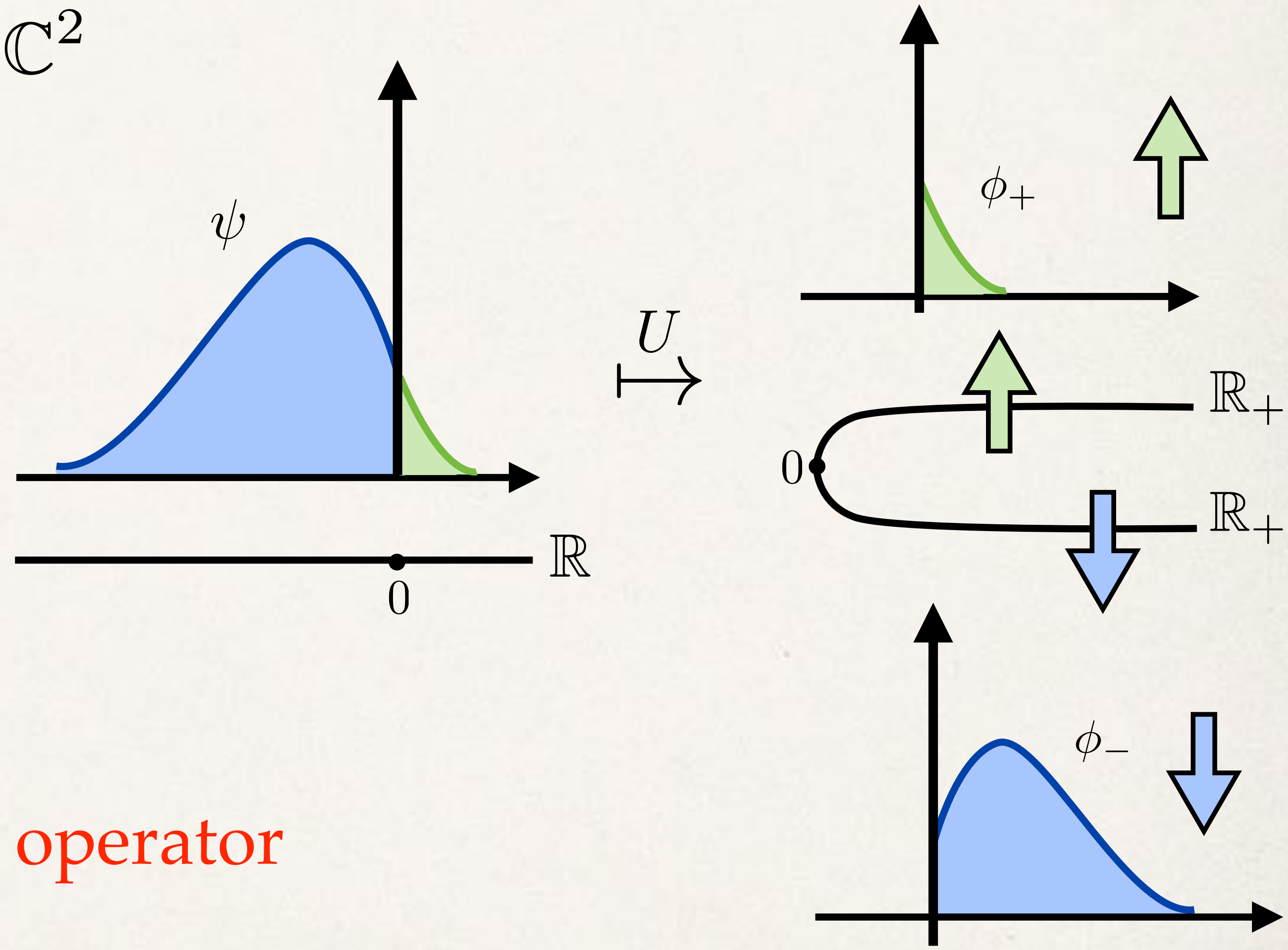
$$\psi(x) \quad \mapsto \quad \Phi(y) = \begin{pmatrix} \phi_+(y) \\ \phi_-(y) \end{pmatrix} = (U\psi)(y) = \begin{pmatrix} \psi(y) \\ \psi(-y) \end{pmatrix}$$

Momentum on the half-line

$$U : L^2(\mathbb{R}) \quad \to \quad L^2(\mathbb{R}_+) \otimes \mathbb{C}^2$$

Image of the momentum:

$$\tilde{p} = U p U^\dagger$$



$$\tilde{p} = -i\hbar \frac{\mathrm{d}}{\mathrm{d}y} \otimes \sigma_z$$ Dirac operator

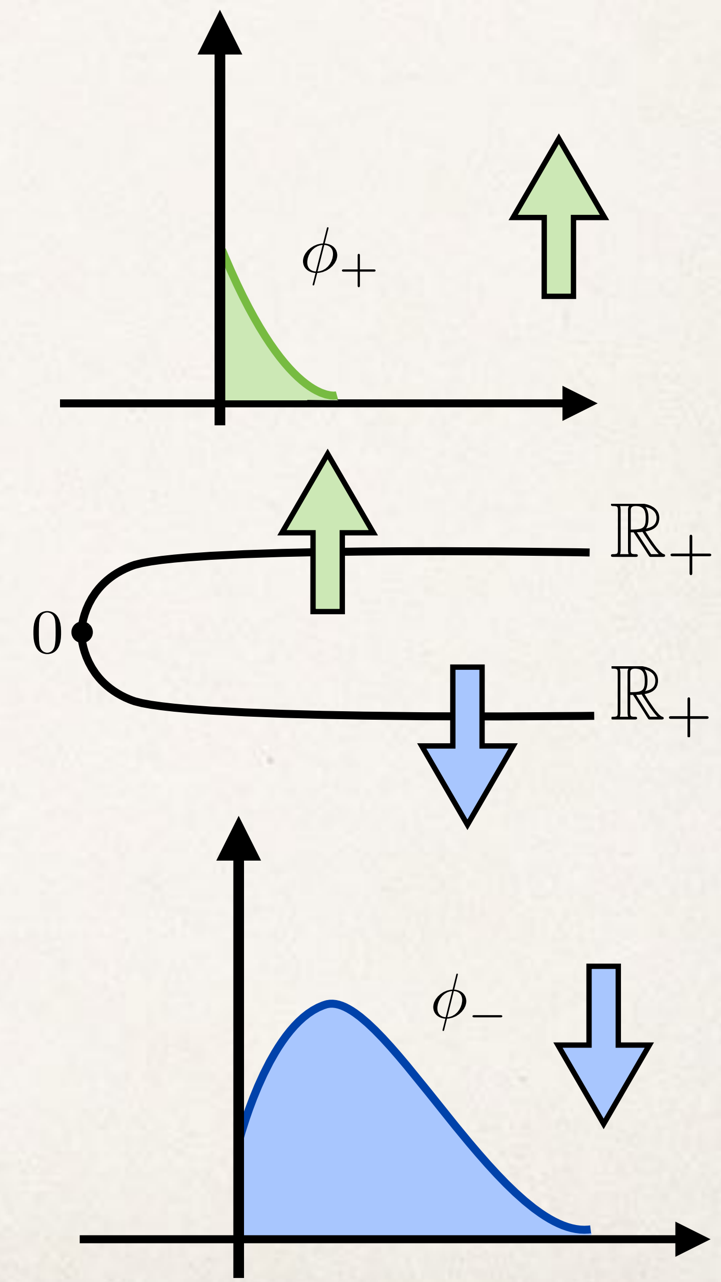
$$D(\tilde{p}) = \{ \Phi \in \mathrm{H}^1(\mathbb{R}_+) \otimes \mathbb{C}^2 \,|\, \Phi(0) = \sigma_x \Phi(0) \,\}$$ QBC

Momentum on the half-line

$$\tilde{p} = -i\hbar \frac{\mathrm{d}}{\mathrm{d}y} \otimes \sigma_z$$

$$\Phi(0) = \sigma_x \Phi(0)$$

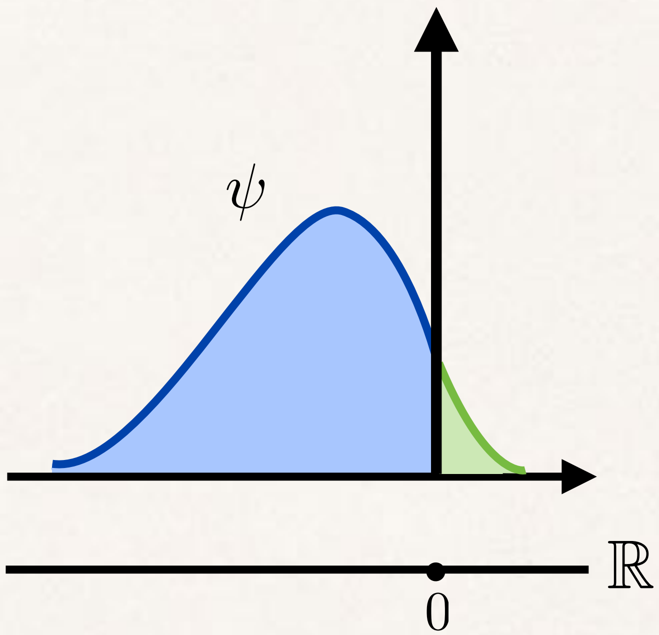
- spin-1/2 particle: Both momentum and spin flips when bouncing off the boundary. Helicity is conserved;
- spinless particle + detector: the detector clicks when the particle bounces off the boundary.

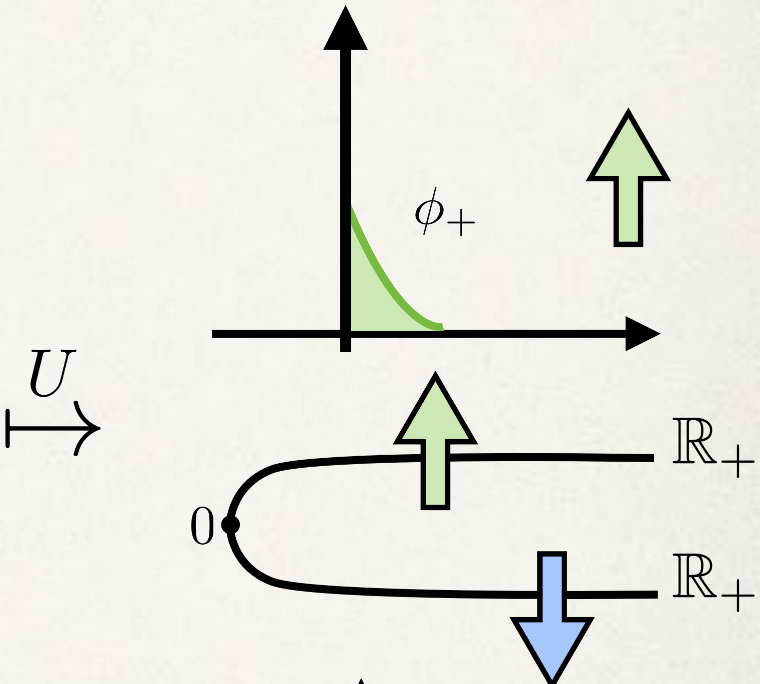


Entanglement vs self-adjointness

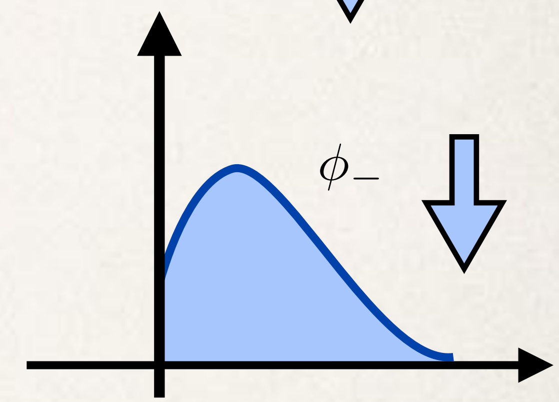
Factorized initial state

$$\phi \otimes \frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}}$$





Entanglement generation!



Evolution:

$$e^{-itp\otimes\sigma_z}\left(\phi(x)\otimes\frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}}\right) = \phi(x-t)\otimes\frac{|\uparrow\rangle}{\sqrt{2}} + \phi(x+t)\otimes\frac{|\downarrow\rangle}{\sqrt{2}}$$

Momentum on an interval

Consider the momentum on the circle

$$p = -i\hbar \frac{\mathrm{d}}{\mathrm{d}x}$$

$$D(p) = \mathrm{H}^1(\mathbb{S}) = \{\psi \in \mathrm{H}^1[-\pi, \pi] \mid \psi(-\pi) = \psi(\pi)\}$$

first Sobolev space

It is self-adjoint.

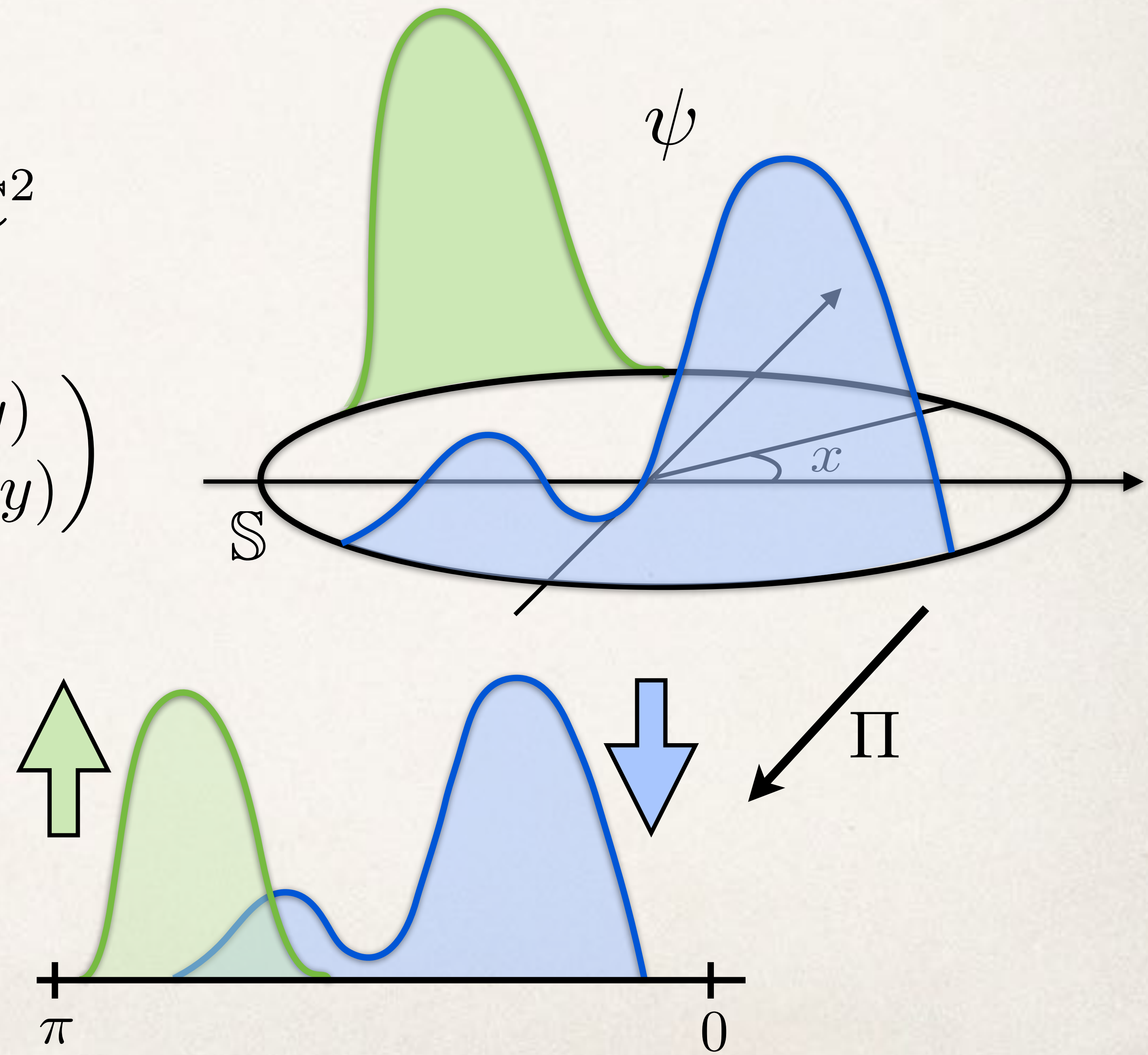
Need a unitary mapping onto the interval

Momentum on an interval

Unitary

$$U : L^2(\mathbb{S}) \quad \to \quad L^2(0,\pi) \otimes \mathbb{C}^2$$

$$\psi(x) \quad \mapsto \quad (U\psi)(y) = \begin{pmatrix} \psi(y) \\ \psi(-y) \end{pmatrix}$$



Ancillary spin

Momentum on an interval

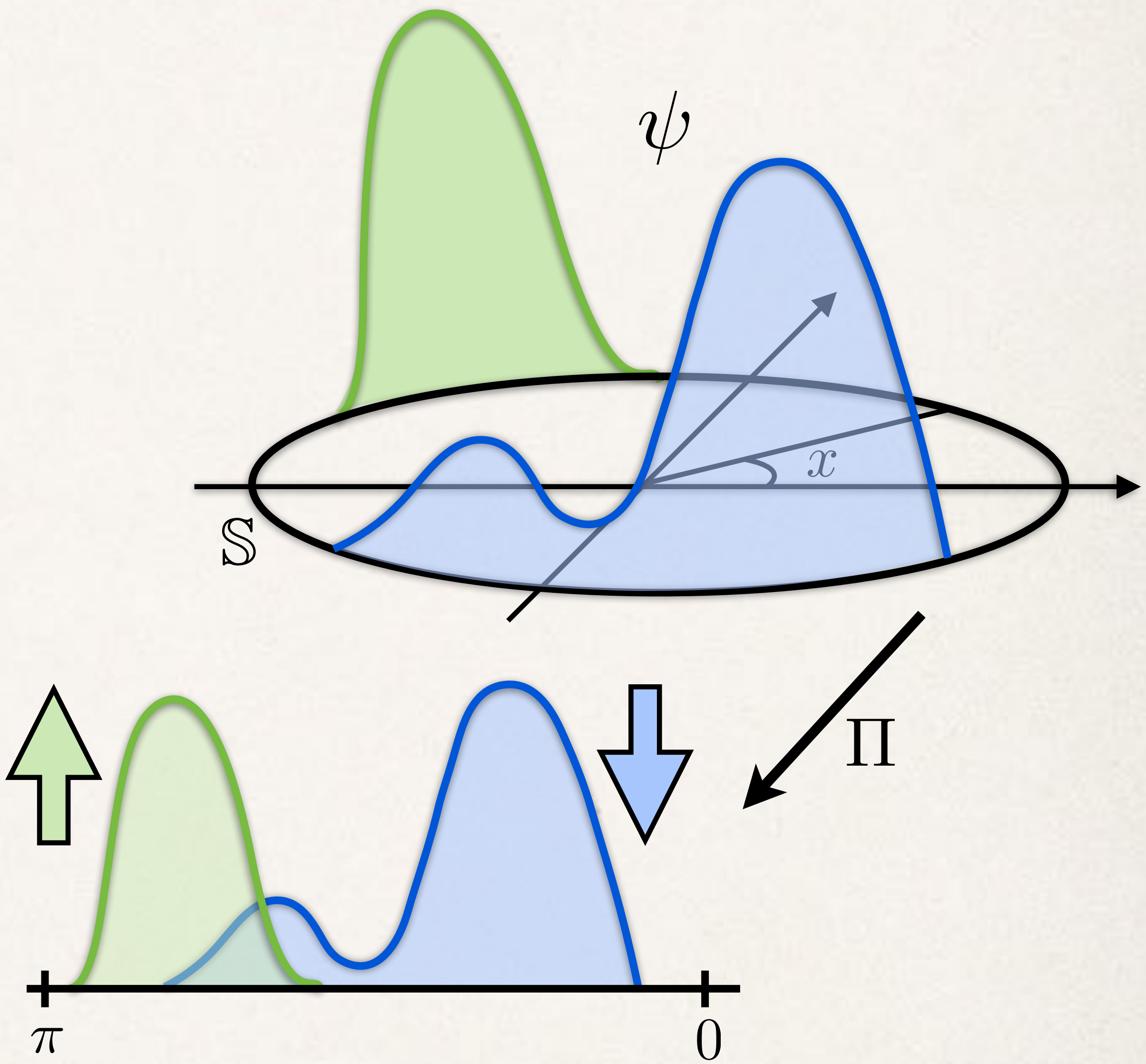
Unitary

$$U : L^2(\mathbb{S}) \quad \to \quad L^2(0,\pi) \otimes \mathbb{C}^2$$

Image of the momentum

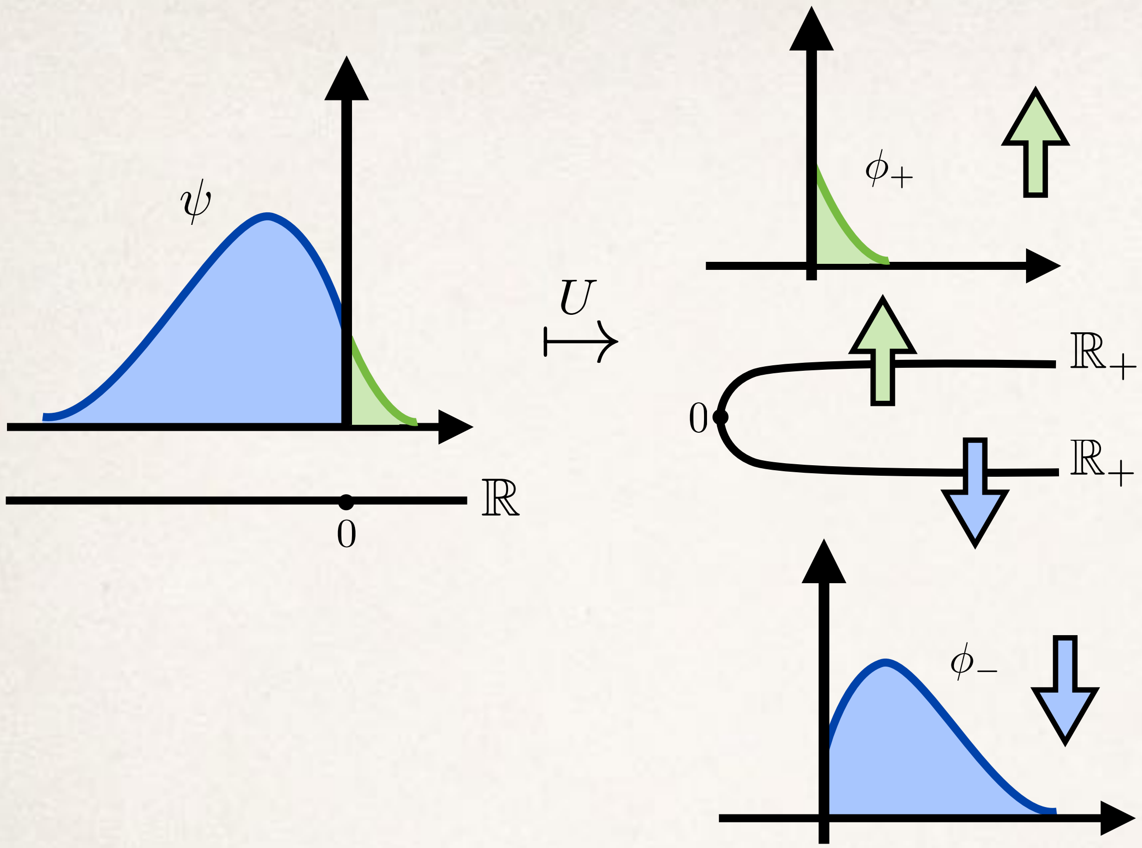
$$\tilde{p} = U p U^\dagger$$

$$\tilde{p} = -i\hbar \frac{\mathrm{d}}{\mathrm{d}y} \otimes \sigma_z$$



Quantum boundary conditions

$$D(\tilde{p}) = \{\Phi \in \mathrm{H}^1[0,\pi] \otimes \mathbb{C}^2 \,|\Phi(0) = \sigma_x\, \Phi(0)\,,\ \Phi(\pi) = \sigma_x\, \Phi(\pi)\}$$



Happy birthday Bal!