# The quantum YB equation and noncommutative Euclidean spaces

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Based on: M. Dubois-Violette, GL, arXiv:1706.06930 and arXiv:1801.03410.

Bal' influence at Syracuse:

19.8.1993 – 4.12.1994

1.10.1988 – 30.9.1989

6.2.1988 – 5.5.1988

## Background

A correspondence (a duality) between spaces and algebras of functions on these spaces

The idea of noncommutative geometry: forget the commutativity of the algebras of functions and replace them by appropriate classes of noncommutative associative algebras. These are considered as algebras of functions on some (virtual) noncommutative spaces

For instance, the natural algebras of functions on finite-dimensional vector spaces are the algebras of polynomial functions generated by the coordinates. In these polynomial algebras the coordinates commute.

Given a class of noncommutative associative algebras generalizing the polynomial algebras, one may consider regular algebras generated by coordinates in which they satisfy other relations than the commutation between them and defing thereby noncommutative vector spaces.

In the following, a noncommutative $\mathbb{R}^N$ correspond to a regular complex $*$-algebra generated by hermitian elements $x^k$, $k \in \{1, \dots, N\}$.

Abstract

- noncommutative generalizations of $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$

- noncommutative generalizations of $\mathbb{S}^{N_1+N_2-1}$ and $\mathbb{S}^{N_1+N_2}$

- noncommutative generalizations of $\mathbb{S}^{N_1-1} \times \mathbb{S}^{N_2-1}$

- a quaternionic noncommutative torus $\mathbb{S}^3 \times_{\mathbf{u}} \mathbb{S}^3$, $\mathbf{u} \in \mathbb{S}^2 = \mathrm{SU}(2)/\,\mathrm{U}(1)$

- spherical manifolds : volume forms from top Chern–Connes characters

- noncommutative principal bundles

- spectral triples

## The $\theta$-deformation

$\mathbb{C}^2_\theta$ = Noncommutative space dual to the
$*$-algebra $\mathcal{A}_\theta$ generated by normal elements $z_1, z_2$ with relations

$$z_1 z_2 = e^{i\theta} z_2 z_1, \qquad z_1 z_2^* = e^{-i\theta} z_2^* z_1 \tag{1}$$

$\Rightarrow$ center generated by $z_1 z_1^* = \|z_1\|^2$ and $z_2 z_2^* = \|z_2\|^2$

Real version $\mathbb{C}^2_\theta = (\mathbb{R}^2)^2_\theta = \mathbb{R}^2 \times_\theta \mathbb{R}^2$

$$z_1 = x_1^1 + \mathrm{i} x_1^2, \quad z_2 = x_2^1 + \mathrm{i} x_2^2, \qquad (x_k^\lambda)^* = x_k^\lambda$$

and $\{(1) +$ normality of $z_k\} \Leftrightarrow (2)$

$$\begin{cases} x_1^\lambda x_1^\mu = x_1^\mu x_1^\lambda \quad ; \qquad x_2^\lambda x_2^\mu = x_2^\mu x_2^\lambda \\ \\ x_1^\lambda x_2^\mu = R^{\lambda\mu}_{\nu\rho} x_2^\nu x_1^\rho \end{cases} \tag{2}$$

with

$$R^{\lambda\mu}_{\nu\rho} = \cos(\theta)\, \delta^\lambda_\rho \delta^\mu_\nu + \mathrm{i} \sin(\theta)\, C^\lambda_\rho D^\mu_\nu \qquad C = -D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

## 2-torus and 3-sphere

$$\mathbb{R}^2 \times_\theta \mathbb{R}^2/(\|x_1\|^2 - 1, \|x_2\|^2 - 1) = \mathbb{T}^2_\theta$$

$$\mathbb{R}^2 \times_\theta \mathbb{R}^2/(\|x_1\|^2 + \|x_2\|^2 - 1) = \mathbb{S}^3_\theta$$

nice examples of singular spaces

## The quadratic $*$-algebra $\mathcal{A}_R$

The $*$-algebra $\mathcal{A}_R$ is generated by two sets of hermitian elements $x_1^\lambda$ with $\lambda \in \{1,\dots,N_1\}$ and $x_2^\alpha$ with $\alpha \in \{1,\dots,N_2\}$ with relations

$$\begin{cases} x_1^\lambda x_1^\mu = x_1^\mu x_1^\lambda \quad ; \qquad x_2^\alpha x_2^\beta = x_2^\beta x_2^\alpha \\ \\ x_1^\lambda x_2^\alpha = R^{\lambda\alpha}_{\beta\mu} x_2^\beta x_1^\mu \end{cases} \tag{3}$$

for suitable $R^{\lambda\alpha}_{\beta\mu} \in \mathbb{C}$. In view of the hermiticity of the $x_1^\lambda, x_2^\alpha$ we impose

$$\overline{R}^{\lambda\alpha}_{\beta\mu} R^{\mu\beta}_{\gamma\nu} = \delta^\lambda_\nu \delta^\alpha_\gamma \tag{4}$$

Thus $\mathcal{A}_R$ is a graded quadratic algebra $\mathcal{A}_R = \oplus_n \mathcal{A}_R^n$

which is connected: $\mathcal{A}_R^0 = \mathbb{C}\mathbb{1}$ ;

the elements $x_1^\lambda, x_2^\alpha$ form a basis of $\mathcal{A}_R^1$ ;

by the requirement (4):
the elements $x_1^\lambda x_1^\mu$ with $\lambda \leq \mu$, $x_2^\alpha x_2^\beta$ with $\alpha \leq \beta$ and $x_1^\lambda x_2^\alpha$ form a basis of $\mathcal{A}_R^2$.

## The Koszul dual $\mathcal{A}^!_R$

A quadratic algebra is an algebra $\mathcal{A}$ of the form

$$\mathcal{A} = T(E)/(\mathcal{R})$$

where $E$ is a finite-dimensional vector space and $(\mathcal{R})$ denotes the ideal of the tensor algebra generated by $\mathcal{R} \subset E \otimes E$.

The Koszul dual $\mathcal{A}^!$ of $\mathcal{A}$ is the quadratic algebra

$$\mathcal{A}^! = T(E^*)/(\mathcal{R}^\perp)$$

where $\mathcal{R}^\perp \subset E^* \otimes E^*$ is the orthogonal of $\mathcal{R}$.

Our $\mathcal{A}^!_R$ is generated by the dual bases $\theta^1_\lambda, \theta^2_\alpha$ of the $x^\lambda_1, x^\alpha_2$ with relations

$$\begin{cases} \theta^1_\lambda \theta^2_\mu = -\theta^2_\mu \theta^1_\lambda \quad ; \qquad \theta^2_\alpha \theta^2_\beta = -\theta^2_\beta \theta^2_\alpha \\ \theta^2_\beta \theta^1_\mu = -R^{\lambda\alpha}_{\beta\mu} \theta^1_\lambda \theta^2_\alpha \end{cases} \tag{5}$$

## An $\mathcal{R}$-matrix

Define $x^a$ for $a \in \{1, \dots, N_1 + N_2\}$ by $\quad x^\lambda = x_1^\lambda, \quad x^{\alpha+N_1} = x_2^\alpha.$

Then the relations (3) together with $x_2^\alpha x_1^\lambda = \overline{R}^{\lambda\alpha}_{\beta\mu} x_1^\mu x_2^\alpha$ reads

$$x^a x^b = \mathcal{R}^{ab}_{cd} x^c x^d \tag{6}$$

In view of (3) and (4) one gets:

$$\mathcal{R}^{\lambda\mu}_{\tau\rho} = \delta^\lambda_\rho\, \delta^\mu_\tau, \qquad \mathcal{R}^{\gamma\delta}_{\alpha\beta} = \delta^\gamma_\beta\, \delta^\delta_\alpha$$
$$\mathcal{R}^{\lambda\alpha}_{\beta\mu} = R^{\lambda\alpha}_{\beta\mu}, \qquad \mathcal{R}^{\alpha\lambda}_{\mu\beta} = \overline{R}^{\lambda\alpha}_{\beta\mu}$$

$$\begin{aligned}
\mathcal{R}^{\lambda\mu}_{\alpha\nu} &= \mathcal{R}^{\lambda\mu}_{\alpha\beta} = \mathcal{R}^{\lambda\mu}_{\nu\beta} = 0, \\
\mathcal{R}^{\alpha\gamma}_{\lambda\beta} &= \mathcal{R}^{\alpha\gamma}_{\lambda\mu} = \mathcal{R}^{\alpha\gamma}_{\beta\mu} = 0, \\
\mathcal{R}^{\lambda\alpha}_{\mu\nu} &= \mathcal{R}^{\lambda\alpha}_{\beta\gamma} = \mathcal{R}^{\lambda\alpha}_{\mu\beta} = 0, \\
\mathcal{R}^{\alpha\lambda}_{\mu\nu} &= \mathcal{R}^{\alpha\lambda}_{\beta\gamma} = \mathcal{R}^{\alpha\lambda}_{\beta\mu} = 0.
\end{aligned}$$

## Yang-Baxter condition

The matrix $\mathcal{R}$ is involutive, i.e. $\mathcal{R}^2 = \mathbb{1} \otimes \mathbb{1}$ or

$$\mathcal{R}^{ab}_{cd} \mathcal{R}^{cd}_{ef} = \delta^a_e \delta^b_f$$

We next impose the Yang-Baxter condition for $\mathcal{R}$

$$(\mathcal{R} \otimes \mathbb{1})(\mathbb{1} \otimes \mathcal{R})(\mathcal{R} \otimes \mathbb{1})^{abc} = (\mathbb{1} \otimes \mathcal{R})(\mathcal{R} \otimes \mathbb{1})(\mathbb{1} \otimes \mathcal{R})^{abc}$$

## Yang-Baxter condition - cont.d

This breaks in a series of conditions:

The cubic YB equations are equivalent to quadratic relations:

$$\begin{cases} R^{\lambda\alpha}_{\gamma\rho} R^{\rho\beta}_{\delta\mu} = R^{\lambda\beta}_{\delta\rho} R^{\rho\alpha}_{\gamma\mu} & (\lambda\alpha\beta) \\ \overline{R}^{\lambda\alpha}_{\gamma\rho} \overline{R}^{\rho\beta}_{\delta\mu} = \overline{R}^{\lambda\beta}_{\delta\rho} \overline{R}^{\rho\alpha}_{\gamma\mu} & (\alpha\beta\lambda) \\ \overline{R}^{\lambda\alpha}_{\gamma\rho} R^{\rho\beta}_{\delta\mu} = R^{\lambda\beta}_{\delta\rho} \overline{R}^{\rho\alpha}_{\gamma\mu} & (\alpha\lambda\beta) \end{cases} \tag{7}$$

and

$$\begin{cases} R^{\lambda\alpha}_{\gamma\nu} R^{\mu\gamma}_{\beta\rho} = R^{\mu\alpha}_{\gamma\rho} R^{\lambda\gamma}_{\beta\nu} & (\lambda\mu\alpha) \\ \overline{R}^{\lambda\alpha}_{\gamma\nu} \overline{R}^{\mu\gamma}_{\beta\rho} = \overline{R}^{\mu\alpha}_{\gamma\rho} \overline{R}^{\lambda\gamma}_{\beta\nu} & (\alpha\lambda\mu) \\ R^{\lambda\alpha}_{\gamma\nu} \overline{R}^{\mu\gamma}_{\beta\rho} = \overline{R}^{\mu\alpha}_{\gamma\rho} R^{\lambda\gamma}_{\beta\nu} & (\lambda\alpha\mu) \end{cases} \tag{8}$$

for the $R^{\lambda\alpha}_{\beta\mu}$ and $\overline{R}^{\lambda\alpha}_{\beta\mu}$ (the components $abc$ of (7) are in the $(\ldots)$).

## Noncommutative product of $\mathbb{R}^{N_1}$ and $\mathbb{R}^{N_2}$

From now on it is assumed that the matrix $R$ of relations (3) for the algebra $\mathcal{A}_R$ satisfies conditions (4), (7) and (8)

The classical (commutative) solution $R = R_0$ is

$$(R_0)^{\lambda\alpha}_{\beta\mu} = \delta^\lambda_\mu \delta^\alpha_\beta$$

and the corresponding algebra $\mathcal{A}_{R_0}$ is the algebra of polynomial functions on the product $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$

This is the reason we define the noncommutative product of $\mathbb{R}^{N_1}$ and $\mathbb{R}^{N_2}$

$$\mathbb{R}^{N_1} \times_R \mathbb{R}^{N_2}$$

to be the "dual" of the algebra $\mathcal{A}_R$ for general $R$.

## Regularity properties of $\mathcal{A}_R$ and $\mathcal{A}^!_R$

Since the relations of $\mathcal{A}_R$ can be written in the form (6) with $\mathcal{R}$ involutive and satisfying the Yang-Baxter equation,

it follows from general results (Gurevich, Wambst) that $\mathcal{A}_R$ is very regular.

In particular $\mathcal{A}_R$ is a Koszul algebra of global dimension $N_1 + N_2$ having the Gorenstein property (an appropriate version of the Poincaré duality property)

In our case, this implies that, in terms of the $x^a$ and the dual basis $\theta_a$:

the $x^{a_1} \dots x^{a_p}$ for $a_1 \leq \dots \leq a_p$ and $p \in \mathbb{N}$ is a basis of $\mathcal{A}_R$

while the $\theta_{a_1} \dots \theta_{a_p}$ for $a_1 < \dots < a_p$ and $p \in \{1, \dots, N_1 + N_2\}$ is a basis of $\mathcal{A}^!_R$.

As a consequence the Poincaré series are classical :

$$P_{\mathcal{A}_R}(t) := \sum_n \mathrm{dim}(\mathcal{A}^n_R) t^n = \left(\frac{1}{1-t}\right)^{N_1+N_2} \quad \text{and} \quad P_{\mathcal{A}^!_R}(t) = (1+t)^{N_1+N_2} \tag{9}$$

## Regularity properties cont.d

The algebra $\mathcal{A}_R$ is even a Calabi–Yau algebra:

any generator of the top one-dimensional space $((\mathcal{A}^!_R)_{N_1+N_2})^*$

is a cyclic potential for the algebra $\mathcal{A}_R$ (Ginzburg)

( a cyclic pre-regular multilinear form )

## Noncommutative product of Euclidean spaces

**Theorem.** The following conditions (i) (ii) and (iii) are equivalent :

(i) $\sum_{a=1}^{N_1+N_2}(x^a)^2 = \sum_{\lambda=1}^{N_1}(x_1^\lambda)^2 + \sum_{\alpha=1}^{N_2}(x_2^\alpha)^2$ is central in $\mathcal{A}_R$,

(ii) $\sum_{\lambda=1}^{N_1}(x_1^\lambda)^2$ and $\sum_{\alpha=1}^{N_2}(x_2^\alpha)^2$ are in the center of $\mathcal{A}_R$, (10)

(iii) $\sum_{\lambda=1}^{N_1} R^{\lambda\gamma}_{\beta\nu} R^{\lambda\beta}_{\alpha\mu} = \delta^\gamma_\alpha \delta_{\mu\nu}$ and $\sum_{\alpha=1}^{N_2} R^{\lambda\alpha}_{\beta\rho} R^{\rho\alpha}_{\gamma\mu} = \delta^\lambda_\mu \delta_{\beta\gamma}$

We take $R$ to satisfy also (10) and define the the noncommutative product of the Euclidean space $\mathbb{R}^{N_1}$ with the Euclidean space $\mathbb{R}^{N_2}$ to be dual of $\mathcal{A}_R$.

Clearly, the relations (11) are satisfied by the classical $R = R_0$;

$\mathcal{A}_R$ generalizes the algebra of polynomial functions on the product $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$

## Restrictions on the structure of $R$

By using (4) and (10) one obtains

$$R^{\lambda\beta}_{\alpha\mu} = R^{\mu\alpha}_{\beta\lambda} = \overline{R}^{\mu\beta}_{\alpha\lambda} = (R^{-1})^{\beta\mu}_{\lambda\alpha} \tag{11}$$

In turn this implies that relations (7) and (8) reduce to

$$R^{\lambda\alpha}_{\beta\rho} R^{\rho\delta}_{\gamma\mu} = R^{\lambda\delta}_{\gamma\rho} R^{\rho\alpha}_{\beta\mu} \qquad \text{and} \qquad R^{\lambda\alpha}_{\gamma\nu} R^{\mu\gamma}_{\beta\rho} = R^{\mu\alpha}_{\gamma\rho} R^{\lambda\gamma}_{\beta\nu} \tag{12}$$

that is the first relation of (7) and the first relation of (8).

**Corollary.** Relations (3) define (the algebra of) a noncommutative product of a $N_1$-dimensional with a $N_2$-dimensional Euclidean spaces if and only if the $R^{\lambda\alpha}_{\beta\mu}$ satisfy relations (11) and (12).

## Noncommutative product of spheres

The elements $\sum_{\lambda=1}^{N_1}(x_1^\lambda)^2=\|x_1\|^2$ and $\sum_{\alpha=1}^{N_2}(x_2^\alpha)^2=\|x_2\|^2$ of $\mathcal{A}_R$ being central one may consider the quotient algebra

$$\mathcal{A}_R/(\{\|x_1\|^2-\mathbb{1},\|x_2\|^2-\mathbb{1}\}) \qquad \leftrightarrow \qquad \mathbb{S}^{N_1-1}\times_R \mathbb{S}^{N_2-1}$$

This defines by duality the noncommutative product of the classical spheres $\mathbb{S}^{N_1-1}$ and $\mathbb{S}^{N_2-1}$.

Indeed, for $R=R_0$, the above quotient is the restriction to $\mathbb{S}^{N_1-1}\times\mathbb{S}^{N_2-1}$ of the polynomial functions on $\mathbb{R}^{N_1+N_2}$.

## Noncommutative spheres

With $\|x\|^2$ denoting the central element $\sum_{a=1}^{N_1+N_2}(x^a)^2 = \|x_1\|^2 + \|x_2\|^2$, one may also consider the quotient of $\mathcal{A}_R$

$$\mathcal{A}_R/(\|x\|^2 - \mathbb{1}) \qquad \leftrightarrow \qquad (\mathbb{S}^{N_1+N_2-1})_R$$

This defines (dualy) the noncommutative $(N_1+N_2-1)$-sphere $(\mathbb{S}^{N_1+N_2-1})_R$ (a subspace of the noncommutative product of $\mathbb{R}^{N_1}$ with $\mathbb{R}^{N_2}$)

This is a noncommutative spherical manifold in the sense of Connes–Landi and Connes–Dubois-Violette (see below).

## The (generalized) Clifford algebra $C\ell(\mathcal{A}_R)$

The $*$-algebra $C\ell(\mathcal{A}_R)$ is generated by two sets of hermitian elements $\Gamma^1_\lambda$ with $\lambda \in \{1, \dots, N_1\}$ and $\Gamma^2_\alpha$ with $\alpha \in \{1, \dots, N_2\}$ with relations

$$\begin{cases} \Gamma^1_\lambda \Gamma^1_\mu + \Gamma^1_\mu \Gamma^1_\lambda = 2\delta_{\lambda\mu}\mathbb{1} \\ \Gamma^2_\alpha \Gamma^2_\beta + \Gamma^2_\beta \Gamma^2_\alpha = 2\delta_{\alpha\beta}\mathbb{1} \\ \Gamma^2_\beta \Gamma^1_\mu + R^{\lambda\alpha}_{\beta\mu} \Gamma^1_\lambda \Gamma^2_\alpha = 0 \end{cases} \tag{13}$$

**Proposition** In the algebra $C\ell(\mathcal{A}_R) \otimes \mathcal{A}_R$ one has :

$$(\Gamma^1_\lambda \otimes x^\lambda_1)^2 = \mathbb{1} \otimes \|x_1\|^2, \qquad (\Gamma^2_\alpha \otimes x^\alpha_2)^2 = \mathbb{1} \otimes \|x_2\|^2$$

and

$$(\Gamma^1_\lambda \otimes x^\lambda_1)(\Gamma^2_\alpha \otimes x^\alpha_2) + (\Gamma^2_\alpha \otimes x^\alpha_2)(\Gamma^1_\lambda \otimes x^\lambda_1) = 0$$

## Structure of $C\ell(\mathcal{A}_R)$

Last proposition is equivalent to

$$(\Gamma(x))^2 = \mathbb{1} \otimes \|x\|^2 \tag{14}$$

with $\Gamma(x) = \Gamma_a \otimes x^a = \Gamma^1_\lambda \otimes x^\lambda_1 + \Gamma^2_\alpha \otimes x^\alpha_2$ and $\|x\|^2 = \sum_{a=1}^{N_1+N_2}(x^a)^2$.

The algebra $C\ell(\mathcal{A}_R)$ is nonhomogeneous quadratic with $\mathcal{A}^!_R$ as homogeneous part. It is not $\mathbb{N}$-graded but only $\mathbb{Z}_2$-graded and filtered with

$$\mathcal{F}^n = F^n(C\ell(\mathcal{A}_R)) = \{\text{elements of degree in } \Gamma \leq n\}$$

One has a surjective canonical homomorphism of graded algebra

$$\text{can} : \mathcal{A}^!_R \to \ \text{gr}(C\ell(\mathcal{A}^!_R)) = \oplus_{n\in\mathbb{N}}\mathcal{F}^n/\mathcal{F}^{n-1} \tag{15}$$

which induce the isomorphism of vector spaces

$$(\mathcal{A}^!_R)^1 \simeq \mathcal{F}^1/\mathcal{F}^0$$

## Structure of $C\ell(\mathcal{A}_R)$ - cont.d

The fact that $\|x\|^2 = \sum(x^a)^2$ is central in $\mathcal{A}_R$ and the Koszulity of $\mathcal{A}^!_R$ imply the following PBW property (via the duality of Positselski).

**Proposition** The homomorphism (15) is an isomorphism of graded algebras.

Thus $C\ell(\mathcal{A}_R)$ is a Koszul nonhomogeneous quadratic algebra since $\mathcal{A}^!_R$ is Koszul, (cf. Dubois-Violette). This implies

$$\dim C\ell(\mathcal{A}_R) = \dim \mathcal{A}^!_R = 2^{N_1+N_2}$$

One has the following isomorphisms :

$$\begin{cases} C\ell(N_1) \simeq \text{ subalgebra of } C\ell(\mathcal{A}) \text{ generated by the } \Gamma^1_\lambda \\ C\ell(N_2) \simeq \text{ subalgebra of } C\ell(\mathcal{A}) \text{ generated by the } \Gamma^2_\alpha \end{cases}$$

and

$$C\ell(\mathcal{A}_R) \simeq C\ell(N_1 + N_2) \tag{16}$$

## The general solution

Let us introduce a two-index $\binom{\lambda}{\alpha}$ with $\lambda \in \{1, \ldots, N_1\}$ and $\alpha \in \{1, \ldots, N_2\}$. Then the conditions (11) can be read as symmetry conditions:

$$R^{\binom{\lambda}{\alpha}}{}_{\binom{\mu}{\beta}} = R^{\binom{\mu}{\beta}}{}_{\binom{\lambda}{\alpha}} = \overline{R}^{\binom{\mu}{\alpha}}{}_{\binom{\lambda}{\beta}} = (R^{-1})_{\binom{\lambda}{\beta}}{}^{\binom{\mu}{\alpha}}.$$

While, the quadratic conditions in (12) can be written as

$$R^{\binom{\lambda}{\alpha}}{}_{\binom{\rho}{\beta}} R^{\binom{\rho}{\gamma}}{}_{\binom{\mu}{\delta}} = R^{\binom{\lambda}{\gamma}}{}_{\binom{\rho}{\delta}} R^{\binom{\rho}{\alpha}}{}_{\binom{\mu}{\beta}} \qquad \text{and} \qquad R^{\binom{\lambda}{\alpha}}{}_{\binom{\nu}{\gamma}} R^{\binom{\mu}{\gamma}}{}_{\binom{\rho}{\beta}} = R^{\binom{\mu}{\alpha}}{}_{\binom{\rho}{\gamma}} R^{\binom{\lambda}{\gamma}}{}_{\binom{\nu}{\beta}}.$$

**Theorem** The general solution is of the form

$$R^{\lambda\beta}_{\alpha\mu} = S^{\lambda\beta}_{\mu\alpha} + \mathrm{i}\, A^{\lambda\beta}_{\mu\alpha}$$

$$\text{with} \qquad [S, A] = 0 \qquad S^2 + A^2 = \mathbb{1}_{N_1} \otimes \mathbb{1}_{N_2}.$$

$S$ and $A$ are real matrices with $S$ standing for symmetric while $A$ antisymmetric for exchange in indices $\lambda \leftrightarrow \mu$ and $\alpha \leftrightarrow \beta$

## The ansatz $A$ $B$ $C$ $D$

Nontrivial realizations of the $R^{\lambda\alpha}_{\beta\mu}$ are given by the following.

**Proposition** Let $A$ and $C$ be two commuting real $N_1 \times N_1$-matrices with $A$ symmetric and $C$ antisymmetric and let $B$ and $D$ be two commuting real $N_2 \times N_2$-matrices with $B$ symmetric and $D$ antisymmetric. Assume that

$$A^2 \otimes B^2 + C^2 \otimes D^2 = \mathbb{1}_{N_1} \otimes \mathbb{1}_{N_2}$$

then the $R^{\lambda\alpha}_{\beta\mu}$ given by

$$R^{\lambda\alpha}_{\beta\mu} = A^{\lambda}_{\mu} B^{\alpha}_{\beta} + \mathrm{i}\, C^{\lambda}_{\mu} D^{\alpha}_{\beta} \tag{17}$$

satisfy the assumptions (11), (12).

## Enter the quaternions

An $SO(4)$-invariant decomposition of $M_4(\mathbb{R})$

$$q = x^0 1 + x^k e_k \in \mathbb{H} \quad \longleftrightarrow \quad x = (x^0, x^1, x^2, x^3) \in \mathbb{R}^4 \qquad \text{Euclidean}$$

a right and a left action

$$e_k q \leftrightarrow J_k^{(+)} x, \qquad q e_k \leftrightarrow -J_k^{(-)} x$$

$$J_{k\mu\nu}^{(\pm)} = \mp(\delta_{0\mu}\delta_{k\nu} - \delta_{0\nu}\delta_{k\mu}) - \varepsilon_{k\ell m}\delta_\mu^\ell \delta_\nu^m$$

$$J_k^{(\pm)} J_\ell^{(\pm)} = -\delta_{k\ell}\mathbb{1} + \sum_m \varepsilon_{k\ell m} J_m^{(\pm)}, \; J_k^{(+)} J_\ell^{(-)} = J_\ell^{(-)} J_k^{(+)}$$

$$M_4(\mathbb{R}) = \mathbb{R}^4 \otimes \mathbb{R}^4 = \mathbb{R}\mathbb{1} \oplus \wedge^2_{(+)}\mathbb{R}^4 \oplus \wedge^2_{(-)}\mathbb{R}^4 \oplus \mathbb{S}^2_0\mathbb{R}^4$$

Orthornormal basis $\mathbb{1}$, $J_k^{(+)}$, $J_\ell^{(-)}$, $J_r^{(+)} J_s^{(-)}$

$(+)$ = antisymmetric self-dual, $\qquad (-)$ = antisymmetric anti-self dual.

## Noncommutative quaternionic planes

Use last theorem for $N_1 = N_2 = 4$

$A = \mathbb{1}, \quad B = u^0 \mathbb{1}, \quad C = J_1^{(\pm)}, \quad D = u^1 J_1^{(\pm)} + u^2 J_2^{(\pm)}$

with $(u^0)^2 + (u^1)^2 + (u^2)^2 = 1$. This gives:

$$R^{\lambda\alpha}_{\beta\mu} = u^0 \delta^\lambda_\mu \delta^\alpha_\beta + \mathrm{i}(J_1^{(\pm)})^\lambda_\mu (u^1 J_1^{(\pm)} + u^2 J_2^{(\pm)})^\alpha_\beta \tag{18}$$

By using the $J_k^{(\mp)}$ one defines an action of $\mathbb{H}$.

The choice of the direction 1 and of the plane (1 2) is immaterial since one can change them into an arbitrary direction $\vec{n}$ and an arbitrary plane which contains $\vec{n}$ by a rotation of $SO_3^{(\pm)}$.

The exchange $(+) \leftrightarrow (-)$ is induced for instance by the exchange $x^0 \leftrightarrow -x^0$ and therefore does not change the algebra $\mathcal{A}_R$ for $R$ given by (18).

## Noncommutative quaternionic planes cont.d

The solution given by (18) generalizes $\mathbb{C}^2_\theta$ for $\mathbb{C} \to \mathbb{H}$;

For the $\theta$-deformation the parameter is in fact

$$\mathbb{S}^1/\mathbb{S}^0 = U_1(\mathbb{C})/U_1(\mathbb{R}) = P_1(\mathbb{R}).$$

The parameter here

$$\mathbf{u} \in \mathbb{S}^2 = \mathbb{S}^3/\mathbb{S}^1 = U_1(\mathbb{H})/U_1(\mathbb{C}) = P_1(\mathbb{C})$$

and for $u^0 = 1$ ($\Rightarrow u^1 = u^2 = 0$), this gives the classical $\mathbb{H}^2$.

## Noncommutative quaternionic tori

The N.C. product of $\mathbb{H}$ by $\mathbb{H}$ corresponding to $\mathcal{A}_R$ is denoted $\mathbb{H}^2_{\mathbf{u}}$.

Tori obtained by the quotient by the ideal generated by $\{\|x_1\|^2 - 1, \|x_2\|^2 - 1\}$:

$$\mathcal{A}(\mathbb{T}^{\mathbb{H}}_{\mathbf{u}}) = \mathcal{A}(\mathbb{H}^2_{\mathbf{u}}) / < ||x_1||^2 - 1, ||x_2||^2 - 1 >$$

$$\mathbb{T}^{\mathbb{H}}_{\mathbf{u}} \simeq \mathbb{S}^3 \times_{\mathbf{u}} \mathbb{S}^3$$

an $SU(2) \times SU(2)$ action

## Additional strata: other N.C. products of 4-dim. Euclidean spaces

Other $\mathcal{A}_R$ with $N_1 = N_2 = 4$ using the ansatz $A\ B\ C\ D$ with $J_k^{(\pm)}$

1. $A = \mathbb{1}, \quad B = \cos(\theta)\mathbb{1}, \quad C = J_1^{(\pm)}, \quad D = \sin(\theta) J_1^{(\mp)}$.

   The direction 1 for $J_\cdot^{(\pm)}$ (resp $J_\cdot^{\mp)}$) can be changed by acting with $SO_3^{(\pm)}$ (resp. $SO_3^{(\mp)}$). For $\theta = 0$ it corresponds to the classical $\mathbb{R}^4 \times \mathbb{R}^4$.

2. $A = C \cdot (v^k J_k^{(\mp)}), \quad B = D \cdot (w^k J_k^{(\mp)}),$
   $C = J_1^{\pm}, \quad D = u^1 J_1^{\pm} + u^2 J_2^{\pm}$ with $((u^1)^2 + (u^2)^2)(1 + \vec{v}^2 \vec{w}^2) = 1$.

3. $A = C \cdot (v^k J_k^{(\mp)}), \quad B = D \cdot (w^k J_k^{\pm}),$
   $C = J_1^{(\pm)}, \quad D = u J_1^{(\mp)}$ with $u^2 + \vec{v}^2 \vec{w}^2 = 1$.

Solutions 2 and 3 do not contain the classical $\mathbb{R}^4 \times \mathbb{R}^4$.

These solutions are noncommutative for any parameters and cannot be connected with solution 1 and with $\mathbb{H}^2_{\mathbf{u}}$.

## Spherical conditions

A projection $p \in M_{2^n}(\mathcal{A}(S^{2n}_R))$ ,

$$p = \frac{1}{2}\,(\mathbb{1} + \Gamma_a\, x^a + \Gamma x)$$

$$ch_k(p) = 0, \qquad 0 \le k \le n-1 \qquad\qquad ch_n(p) \quad \text{the volume form}$$

A unitary $U \in M_{2^{n-1}}(\mathcal{A}(S^{2n-1}_R))$ ,

$$U = \mathbb{1}\, x^0 + \Sigma_j\, x^j$$

$$ch_{k-\frac{1}{2}}(U) = 0, \qquad 0 \le k \le n-1 \qquad\qquad ch_{n-\frac{1}{2}}(U) \quad \text{the volume form}$$

$$\Gamma_\mu = \begin{pmatrix} 0 & \Sigma_\mu \\ \Sigma_\mu, & 0 \end{pmatrix}$$

A matter of computation for the present examples

## Principal bundles

Consider the two quaternions.

$$x_1 = x_1^{\mu} e_{\mu} \quad x_2 = x_2^{\alpha} e_{\alpha}$$

with commutation relations governed by a matrix $R^{\lambda\alpha}_{\beta\mu}$. When restricting to the sphere $\mathbb{S}^7_R$, we get a normalised vector valued function

$$|\psi\rangle = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} \qquad \langle \psi, \psi \rangle = ||x_1||^2 + ||x_2||^2 = \mathbb{1}$$

and thus a projection: ( $p = p^* = p^2$ )

$$p = |\psi\rangle \langle \psi| = \begin{pmatrix} x_2 x_2^* & x_2 x_1^* \\ x_1 x_2^* & x_1 x_1^* \end{pmatrix},$$

Define coordinate functions $Y = Y^0 e_0 + Y^k e_k$ and $Y^4$ by

$$Y^4 = ||x_2||^2 - ||x_1||^2 \quad \text{and} \quad \tfrac{1}{2} Y = x_2 x_1^*$$

so that

$$p = |\psi\rangle \langle \psi| = \frac{1}{2} \begin{pmatrix} 1 + Y^4 & Y \\ Y^* & 1 - Y^4 \end{pmatrix}.$$

The condition $p^2 = p$ leads to

$$YY^* + (Y^4)^2 = 1 \quad \text{and} \quad Y^*Y + (Y^4)^2 = 1$$

$$YY^4 = Y^4Y \quad \text{and} \quad Y^*Y^4 = Y^4Y^*.$$

Thus the coordinate function $Y^4$ is central while comparing the first two conditions requires $YY^* = Y^*Y$ and that this is a multiple of the identity. These translates to the following conditions

$$\sum_{\mu=0}^{3}(Y^{\mu*}Y^{\mu} - Y^{\mu}Y^{\mu*}) = 0,$$
$$-(Y^{0*}Y^{k} - Y^{k*}Y^{0}) + \varepsilon_{kmn}Y^{m*}Y^{n} = 0,$$
$$Y^{0}Y^{k*} - Y^{k}Y^{0*} + \varepsilon_{kmn}Y^{m}Y^{n*} = 0$$

for $k, r, m = 1, 2, 3$ and totally antisymmetric tensor $\varepsilon_{krm}$.

Then the first condition at top of the page reduces to a four-sphere relation

$$\sum_{\mu=0}^{3} Y^{\mu*}Y^{\mu} + (Y^4)^2 = 1 = \sum_{\mu=0}^{3} Y^{\mu*}Y^{\mu} + (Y^4)^2 = 1. \tag{19}$$

The elements $Y^\mu$ generate the $*$-algebra $\mathcal{A}(\mathbb{S}^4_R)$ of a 4-sphere $\mathbb{S}^4_R$.

The algebra inclusion $\mathcal{A}(\mathbb{S}^4_R) \hookrightarrow \mathcal{A}(\mathbb{S}^7_R)$ is a principal SU(2) bundle:

A unit quaternion $w \in \mathbb{H}_1 \simeq$ SU(2) act on $\mathbb{S}^7_R$ as

$$\alpha_w(|\psi\rangle) = |\psi\rangle\, w = \begin{pmatrix} x_2 w \\ x_1 w \end{pmatrix}.$$

leaving the projection $p$ and then the algebra $\mathcal{A}(\mathbb{S}^4_R)$ invariant.

If $H = \mathcal{A}(\mathrm{SU}(2))$ we have dually a co-action $\delta$ of $H$ on $\mathcal{A}(\mathbb{S}^7_R)$ with algebra of co-invariant element again the subalgebra $\mathcal{A}(\mathbb{S}^4_R)$.

Then the canonical map

$$\chi : \mathcal{A}(\mathbb{S}^7_R) \otimes_{\mathcal{A}(\mathbb{S}^4_R)} \mathcal{A}(\mathbb{S}^7_R) \to \mathcal{A}(\mathbb{S}^7_R) \otimes H, \qquad \chi(p' \otimes p) = p'\delta(p)$$

is bijective.

Indeed,

$$\chi\big( \langle\psi| \otimes_{\mathcal{A}(\mathbb{S}^4_R)} |\psi\rangle \big) = \langle\psi|\, \delta(|\psi\rangle) = \langle\psi, \psi\rangle \otimes w = \mathbb{1} \otimes w,$$

showing surjectivity of $\chi$. This is enough since ... ( $H$ is classical ).

## The next fibration

A similar ( if more involved ) algebra inclusion

$$\mathcal{A}(\mathbb{S}^8_R) \hookrightarrow \mathcal{A}(\mathbb{S}^{15}_R)$$

(coming from an octonionic matrix $R^{\lambda\alpha}_{\beta\mu}$)

which is a $\mathbb{S}^7$-bundle

# Spectral geometry

coming up

thank you