# Coherent States, Thermofield Dynamics & Gravity

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# Happy Birthday Bal!

And many more happy, healthy, productive years

- Fuzzy spaces (noncommutative spaces) can be modeled by the lowest Landau level of a quantum Hall system.

- Thermofield dynamics gives a way of discussing mixed states in terms of a pure state description. Presumably they are important for gravity. (ISRAEL; MALDACENA; JACOBSON; + *others*)

> Can we put these together to get some insights into gravity on *even* $+ 1$ dimensional noncommutative spacetimes?

- The result is a bit different from the one based on the spectral action principle (CONNES, CHAMSEDDINE, ...).

*Some of the material is joint work with* D. KARABALI; LEI JIUSI.

- Thermofield dynamics can be expressed in terms of a field theory for a quantum Hall system, with a particular limit to be taken at the end.
- We can think of gauge fields enter as a way of defining the large $N$ limit for fuzzy spaces.
- Double the Hilbert space modeling the fuzzy space to $\mathcal{H}_N \otimes \tilde{\mathcal{H}}_N$, with left chirality gravitational fields ($SO(3)_L$ in 3d) on $\mathcal{H}_N$ and right chirality fields ($SO(3)_R$) on $\tilde{\mathcal{H}}_N$.
- This leads to

$$\begin{aligned} S &= -\frac{1}{4\pi} \int \left[ \mathrm{Tr}\left(A\,dA + \tfrac{2}{3}A^3\right)_L - \mathrm{Tr}\left(A\,dA + \tfrac{2}{3}A^3\right)_R \right] \\ &= \text{Einstein - Hilbert action} \end{aligned}$$

- For Minkowski signature, we can use $SO(2,1)_L \times SO(2,1)_R$.
- One can also use appropriate representations of the Virasoro algebra (on the orbits $\widehat{\text{diff}\, S^1}/S^1$ which admit a Kähler structure).
- The large $c$ limit of the latter leads back to $SO(2,1)_L \times SO(2,1)_R$.
- Introduction of quantum matter is not yet clear, but classically, particle dynamics can be described via the Einstein-Infeld-Hoffmann method (*work with* LEI JIUSI).

- For a system with Hilbert space $\mathcal{H}$, the expectation value of observable $\mathcal{O}$ is

$$\langle \mathcal{O} \rangle = \mathrm{Tr}(\rho\, \mathcal{O}) = \frac{1}{Z} \mathrm{Tr}\left(e^{-\beta H} \mathcal{O}\right), \qquad Z = \mathrm{Tr}\left(e^{-\beta H}\right)$$

- We double the Hilbert space to $\mathcal{H} \otimes \tilde{\mathcal{H}}$ and introduce the pure state (called thermofield vacuum)

$$|\Omega\rangle = \frac{1}{\sqrt{Z}} \sum_n e^{-\frac{1}{2}\beta E_n} |n, \tilde{n}\rangle$$

- Then we get

$$\langle \Omega | \, \mathcal{O} \, |\Omega\rangle = \frac{1}{Z} \sum_{m,n} e^{-\frac{1}{2}\beta (E_n + E_m)} \langle m|\mathcal{O}|n\rangle \, \langle \tilde{m}|\tilde{n}\rangle = \mathrm{Tr}(\rho\, \mathcal{O})$$

- The Hamiltonian is taken as

$$\check{H} = H + \tilde{H} = H \otimes \mathbb{1} - \mathbb{1} \otimes H, \qquad \Longrightarrow \quad \check{H}\, |\Omega\rangle = 0$$

- The formalism of thermofield dynamics is very useful for considering time-dependent (nonequilibrium) effects at finite temperature.

- For a quantum system, the density matrix evolves by the Liouville equation

$$i\,\frac{\partial \rho}{\partial t} = H\,\rho - \rho\,H$$

- We can write an "action" for this,

$$S = \int dt\,\mathrm{Tr}\left[\rho_0\,\left(U^\dagger\, i\partial_t U - U^\dagger\, H\, U\right)\right]$$

where $U$'s are to be varied, and $\rho = U\,\rho_0\,U^\dagger$.

- Our first step is to write thermofield dynamics as a field theory functional integral with an action similar to this.

- The transition amplitude for observables $B, C$ is given by

$$\langle T\, B_{t_1}\, C_{t_2} \rangle = \sum (\sqrt{\rho})_{\tilde{m}m} \langle m|\, T\left[B_{t_1}\, C_{t_2}\, e^{-iHt}\right] |n\rangle\, (\sqrt{\rho})_{n\tilde{n}}\, \langle \tilde{n}|\, e^{iH^T t}\, |\tilde{m}\rangle$$

- For the tilde part, we have $H \to -H^T$ as expected for conjugation for unitary matrices.

- Introduce coherent states for some suitable orbit $G/H$ of some Lie group $G$, $f_n(z)$, $h_n(w)$ such that

$$\int_{\mathcal{M}} d\mu(\bar{z}, z)\; f_n^*\, f_m = \delta_{nm}, \qquad \int_{\mathcal{M}} d\mu(\bar{w}, w)\;\, h_n^*\, h_m = \delta_{nm}$$

$\mathcal{M} = G/H$

- There are many choices for the space of $z, \bar{z}$ (and $w, \bar{w}$); the simplest is to use $\mathbb{CP}^{N-1} \sim SU(N)/U(N-1)$.
- The states can be taken for this case as

$$f_N = \frac{1}{\sqrt{1 + \bar{z}\cdot z}}, \qquad f_i = \frac{z_i}{\sqrt{1 + \bar{z}\cdot z}}, \qquad i = 1, 2, \cdots, (N-1)$$

- Another choice could be coherent states for $\mathbb{CP}^1 \sim SU(2)/U(1)$. We can use the rank $r$ representation with

$$f_n(z, \bar{z}) = \left[\frac{(r+1)!}{n!\,(r-n)!}\right]^{\frac{1}{2}} \frac{z^n}{(1 + \bar{z}z)^{r/2}}, \qquad n = 0, 1, \cdots, r$$

- One can use similar formulae for $h_n(w, \bar{w})$.

- The thermofield vacuum is given by $\Omega = \sum h^*_m \sqrt{\rho}_{mn} f_n$.
- There is a way of using $C^*$-algebras to formulate thermofield dynamics.
- A key result from Tomita-Takesaki theory:

  There is an antilinear operation $J$, called 'modular conjugation', and a 'modular operator' $\Delta$.
- These are easily and explicitly realized in terms of the coherent states as

$$J \cdot f_n = h^*_n, \qquad J \cdot h_n = f^*_n, \qquad J \cdot \lambda f_n = \lambda^* h^*_n$$
$$J \cdot \Omega = \Omega$$

- For the thermal state $\Delta = e^{-\beta \check{H}}$.

- The coherent states are of the form $\langle n|U|w\rangle$ for a highest weight state $|w\rangle$ (which has $H$-invariance) and $U$ is a unitary representation of $G$.

- The diagonal coherent state representation of operators also allows us to introduce $A_0(z,\bar{z}) = H(z,\bar{z})$ such that

$$H_{kl} = \int_{\mathcal{M}} d\mu(z,\bar{z})\, f_k^*\, A_0(z,\bar{z})\, f_l$$

  and similarly for the tilde part (and for other operators).

- Introduce fermionic fields $\psi,\ \chi$ on $\mathcal{M}$,

$$\psi(z,\bar{z},t) = \sum_k a_k\, f_k, \qquad \psi^*(z,\bar{z},t) = \sum_k a_k^*\, f_k^*$$

$$\chi(w,\bar{w},t) = \sum_k b_k\, h_k, \qquad \chi^*(w,\bar{w},t) = \sum_k b_k^*\, h_k^*$$

- The transition amplitude can now be written as

$$
\begin{aligned}
F &= \mathcal{N}\int [d\psi d\psi^* \, d\chi d\chi^*] \, e^{iS} \, B_{t_1} \, C_{t_2} \, \Omega^*(t) \, \Omega(0) \\
\Omega(\psi^*, \chi^*) &= \int_{\mathcal{M}} d\mu_z d\mu_w \;\; \psi^*(z) \, \chi^*(w) \;\; (z_k \sqrt{\rho}_{kl} \, w_l) \\
B_{t_1} &= \int d\mu \;\; \psi^* B(z, \bar{z}, t_1) \psi
\end{aligned}
$$

- The action is given by

$$
\begin{aligned}
S &= \int dt \int_{\mathcal{M}} \psi^* \left( i\,\partial_0 - A_0(z,\bar{z}) + \frac{D^2 + E_0}{2m} \right) \psi \\
&\quad + \int dt \int_{\tilde{\mathcal{M}}} \chi^* \left( i\,\partial_0 - A_0(w,\bar{w}) + \frac{D^2 + E_0}{2m} \right) \chi
\end{aligned}
$$

$\tilde{\mathcal{M}}$ has orientation opposite to $\mathcal{M}$, $E_0$ is the lowest eigenvalue of $-D^2$.

- We consider general fields by regarding the holomorphic states as the lowest Landau level of a mock quantum Hall system (for a constant background), with $m \to 0$ eventually (*work with* KARABALI).

- We can generalize to many copies of these fields. Take the states to be of the form $|k\rangle = |\alpha\, \mathrm{I}\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$ and define a set of fermion fields $\psi_{\mathrm{I}} = \sum_\alpha a_{\alpha \mathrm{I}}\, z_\alpha$.

- We then write

$$
\begin{aligned}
S \;=\; & \int dt \int_{\mathcal{M}} \psi_{\mathrm{I}}^* \left( i\,\partial_0 \delta_{\mathrm{IJ}} - \left( A_0(z,\bar{z})_{\mathrm{IJ}} + \frac{D^2 + E_0}{2m} \delta_{\mathrm{IJ}} \right) \right) \psi_{\mathrm{J}} \\
& + \int dt \int_{\tilde{\mathcal{M}}} \chi_{\mathrm{I}}^* \left( i\,\partial_0 \delta_{\mathrm{IJ}} - \left( A_0(w,\bar{w}) \right)_{\mathrm{IJ}} + \frac{D^2 + E_0}{2m} \delta_{\mathrm{IJ}} \right) \chi_{\mathrm{J}}
\end{aligned}
$$

I, J label some internal symmetry or degrees of freedom.

- Start with a large physical system and consider creation of particles by a perturbation $\sum_i a_i^\dagger \, C_i^\dagger$ within the system.

- Amplitude of interest is

$$\begin{aligned} \mathcal{A} &= \sum \langle a_{i_1} \, C_{i_1} a_{i_2} \, C_{i_2} \cdots U(t, t_1) \, B \, U(t_1) \cdots a_{j_1}^\dagger \, C_{j_1}^\dagger a_{j_2}^\dagger \, C_{j_2}^\dagger \rangle \\ &\sim \sum \langle a_{i_1} \, a_{i_2} \, \cdots U(t, t_1) \, B \, U(t_1) \cdots a_{j_1}^\dagger \, a_{j_2}^\dagger \, \rangle \, \langle C_{i_1} \, C_{i_2} \cdots C_{j_1}^\dagger \, C_{j_2}^\dagger \rangle \end{aligned}$$

- $a \, a \cdots \sim$ particles we study, $B \sim$ some measurement

- We need, at least approximately, this factorization/decomposition of amplitudes to isolate the subsystem under study.

- If $a_i$ evolves as $U_{ik}(t) \, a_k$, then $C_i$ must evolve with $U_{ik}^* \, C_k$ so that the perturbation $a_i C_i$ corresponds to zero energy change.

- This means that we can regard

$$a\text{-part of } \mathcal{A} \leftarrow \text{evolution in } \mathcal{H} \text{ of the subsytem}$$

$$C\text{-part of } \mathcal{A} \rightarrow \text{evolution in } \tilde{\mathcal{H}} \sim \mathcal{H}^*$$

- $C$s interact with many things (environment), so generally

$$\langle C_{i_1}\, C_{i_2} \cdots C^\dagger_{j_1}\, C^\dagger_{j_2} \rangle \sim (\sqrt{\rho})^*_{\alpha\tilde{\alpha}}\, \langle \tilde{\alpha} | e^{iH^T t} | \tilde{\beta} \rangle\, (\sqrt{\rho})_{\beta\tilde{\beta}}$$

$\alpha = (i_1 i_2 \cdots), \qquad \beta = (j_1 j_2 \cdots).$

- $\mathcal{A}$ reduces to the thermofield amplitude.

- If the apparatus is large enough, we may ignore correlations for the $C$s and they may be taken as c-numbers ($\sim \eta, \bar{\eta}$). The amplitude reduces to

$$\mathcal{A} \sim \langle 0|(\bar{\eta}a)\cdots(\bar{\eta}a)\, e^{-iH(t-t_1)}\, B\, e^{-iHt_1}(a^\dagger\eta)\cdots(a^\dagger\eta)|0\rangle$$

$\eta,\ \bar{\eta} \sim$ sources.

- The key point is that the particle "creation operators" are of the form

$$a^\dagger C^\dagger \sim \int \bar{\psi}\chi \qquad \text{(of net zero energy)}$$

- This implies there is parity reversal between $\mathcal{H}$ and $\tilde{\mathcal{H}}$.

- For quantum Hall effect on a compact space $\mathcal{M}$, the lowest Landau level defines a Hilbert space $\mathcal{H}_N$.
- Observables restricted to the lowest Landau level $\in Mat_N$.
- So the lowest Landau level of QHE can be used to model a fuzzy space, giving us $\mathcal{M}_F \equiv (\mathcal{H}_N, Mat_N, \Delta_N)$.
- Phase spaces with symplectic structure $\omega$ and $\omega + dA$ correspond to the same Hilbert space,

$$\int \left(\frac{\omega}{2\pi}\right)^k = \int \left(\frac{\omega + dA}{2\pi}\right)^k$$

- There is ambiguity in which phase space we obtain as $N \to \infty$.

- Starting from $\mathcal{H}_N$, this shows up in the wave functions used to take the large $N$ limit via the symbols

$$O(x,t) = \frac{1}{N} \sum_{m,l} \Psi_m(x)\, \hat{O}_{mn}(t)\, \Psi_n^*(x)$$

The wave functions $\Psi_l^*(x)$ are sensitive to $A$.

- The spatial components of the gauge fields characterize how the large $N$ limit is taken.

- Further, for a space $G/H$, the "magnetic" fields for QHE are in $\underline{H}$, which generates part of the isometry group ($G$) of the space.

Spatial components of gauge fields $\sim$ Gravitational perturbations

- For gravity on a noncommutative space ( $even + 1$ dimensional)
  - Use lowest Landau level of QHE to model the space.
  - Use thermofield dynamics for amplitude calculations, because the state describing space itself is highly entangled.
  - Gauge fields for the frame fields and spin connection emerge as part of defining the large $N$ or “classical” limit.
  - Gravitational fields couple to $\mathcal{H}$ and $\tilde{\mathcal{H}}$ with parity reversal, so we model $\mathcal{H}$ by left chiral fermions, $\tilde{\mathcal{H}}$ by right chiral fermions.
  - The action for TFD in terms of the fermion fields allows for a straightforward calculation of the effective action in the large $N$ or “classical” limit.

- For the gravitational part of $\mathcal{H} \otimes \tilde{\mathcal{H}}$, $SO(3)_L$ fields couple to $\mathcal{H}$ while $SO(3)_R$ fields couple to $\tilde{\mathcal{H}}_R$. i.e., $A_L \sim SO(3)_L$, $A_R \sim SO(3)_R$.

- The large $N$ action is

$$S = k(C.S._L - C.S._R) = -\frac{k}{4\pi\, l} \int d^3x \,\det e \,\left[R - \frac{2}{l^2}\right] + \text{total derivative}$$

$$A^a_{L,R} = \left(-\tfrac{1}{2}\epsilon^a{}_{bc}\,\omega^{bc} \pm (e^a/l)\right), \qquad k = (l/4G) = 1$$

- $A_i$ are auxiliary fields introduced for simplicity of representing the transformation. It is also not clear what $A_0$ should be for gravity.

- So we could try to "optimize" the large $N$ limit by eliminating them via equations of motion.

Optimization of large $N$ limit = Field equations for gravity

- We can do a similar analysis in 4+ 1 dimensions to obtain the effective action

$$\begin{aligned} S &= k\,(C.S._L - C.S._R) \\ &= -i\frac{k}{24\pi^2\, l}\int \mathrm{Tr}\left[3\, e\, R^2 + \frac{2}{l^2}\, e^3\, R + \frac{3}{5\, l^4} e^5 + \frac{e\, De\, De}{l^2}\right] \end{aligned}$$

- This is, of course, not Einstein gravity.

- Regard point-particles as singularities of the solutions for the gravitational field as in Einstein-Infeld-Hoffmann.

- General solution is of the form $A = g^{-1} dg$ where $g$ can have point-like singularities at $\vec{x}_\alpha$ (nonsingular on $\mathcal{M} - \{\vec{x}_\alpha\}$).

$$A = g^{-1}\, a\, g + g^{-1}\, dg, \qquad da = \sum_{\alpha=1}^{N} q_\alpha\, \delta^{(2)}(x - x_\alpha), \qquad a_0 = 0$$

- The action reduces to

$$S \;=\; -\frac{k}{4\pi} \int dt \sum_\alpha \Big[ q_{L\alpha} \mathrm{Tr}(M_0\, g_{L\alpha}^{-1}\, \dot{g}_{L\alpha}) - q_{R\alpha} \mathrm{Tr}(N_0\, g_{R\alpha}^{-1}\, \dot{g}_{R\alpha}) \Big]$$

$M_0$, $N_0$ = diagonal generators of $SO(3)_L$, $SO(3)_R$.

- This gives multiparticle dynamics as representations of the isometry group with

$$\begin{aligned} \text{mass} &= m = (k/8\pi l)\,(q_L + q_R) = (q_R + q_L)/32\pi G \\ \text{spin} &= s = (k/4)\,(q_L - q_R) \end{aligned}$$

- In 4+1 dimensions, we need to consider point-like instantons for particle dynamics. For canonical embedding of $SU(2)$ in the $SO(4,2)$, we get the co-adjoint orbit action with

$$m = \frac{k}{2\,l}(Q_\alpha^{(1)} - Q_\alpha^{(2)})$$

$Q_\alpha^{(1)}$, $Q_\alpha^{(2)}$ = instanton numbers. (General case is being studied (*with* LEI JIUSI). )

- Unitary representations of Virasoro group, for $c > 1$, are obtained by quantizing $\widehat{\text{diff}\, S^1}/S^1$ and $\widehat{\text{diff}\, S^1}/SL(2,\mathbb{R})$ (BOWICK& RAJEEV; WITTEN; *others*).

- Introduce the unitary operator $U = \exp(\sum_n \bar{w}_n L_n - w_n L_{-n})$ with

$$U^1 dU = \sum_n \left(\mathcal{E}^n L_{-n} - \bar{\mathcal{E}}^n L_n + (\mathcal{E}^0 - \bar{\mathcal{E}}^0)\, L_0 + (\mathcal{E} - \bar{\mathcal{E}})\mathbb{1}\right)$$

- Using homogeneity of the orbit, we can choose $(\bar{w}_n, w_n)$ as functions of complex coordinates $(\bar{s}_n, s_n)$ such that

$$\mathcal{E}^n = \mathcal{E}^n_k\, ds_k = (1,0)\text{-form}, \qquad \bar{\mathcal{E}}^n = (0,1)\text{-form}$$

$$(\mathcal{E}^0 - \bar{\mathcal{E}}^0)\, L_0 + (\mathcal{E} - \bar{\mathcal{E}})\mathbb{1} = -\tfrac{1}{2} \sum_n (s_n d\bar{s}_n - \bar{s}_n ds_n) \left[2nL_0 + \frac{c}{12}(n^3 - n)\right] + \cdots$$

- Further there is a left-invariant symplectic structure (WITTEN). If we use an integration measure $d\mu$ for this, we can define coherent state wave functions $\Psi_a$ such that

$$\begin{aligned} \Psi_a &= \langle 0|U^\dagger|a\rangle, \qquad L_0|0\rangle = h\,|0\rangle, \quad L_n|0\rangle = 0 \\ \int d\mu\, \Psi_a^* \Psi_b &= \delta_{ab} \quad \text{(Normalization)} \\ \int d\mu\, U|0\rangle\langle 0|U^\dagger &= \mathbb{1} \qquad \text{(Completeness)} \end{aligned}$$

- Define the symbol for an operator as

$$(A) = \langle 0|U^\dagger\, A\, U|0\rangle$$

Then we can obtain the $*$-product

$$(AB) = (A) * (B)$$

- Using completeness

$$
\begin{aligned}
(AB) &= \langle 0|U^\dagger A B U|0\rangle = \langle 0|U^\dagger A(U\mathbb{1}U^\dagger)BU|0\rangle \\
U\mathbb{1}U^\dagger &= U\left(|0\rangle\langle 0| + L_{-1}|0\rangle\frac{1}{2h}\langle 0|L_1 + \sum |i\rangle (M^{(2)})^{-1}_{ij}\langle j| + \cdots\right)U^\dagger \\
U\,L_{-n}|0\rangle &= (\mathcal{E}^{-1})^k_n\left(\frac{\partial}{\partial s_k} - \mathcal{E}^0_k\,h\right)\,U|0\rangle \\
\langle 0|L_n\,U^\dagger &= (\bar{\mathcal{E}}^{-1})^k_n\left(\frac{\partial}{\partial \bar{s}_k} - \bar{\mathcal{E}}^0_k\,h\right)\,\langle 0|U^\dagger
\end{aligned}
$$

- Key results:
  - $h,\ c$ characterize the orbits.
  - When $c \to \infty$, with $h$ finite, the results reduce to the case of just $SL(2,\mathbb{R})$.

- A similar analysis can be done for any *even* $+ 1$ dimensions, although it is not Einstein gravity.
- The level number is $1$ so far, we need multiplicity $(l/8G)$ for a large level number.
- Continuation to Minkowski space seems possible using the field theory representation for the TFD.
- One can use the orbits of the Virasoro group to carry out a similar construction. (Large $c, h$ needed to simplify the action; may connect to WITTEN, MALONEY, + *others*)
- Point-particles with nontrivial dynamics or coupling of matter fields is being explored.

THANK YOU