Dimensional deception: Fuzzy torus vs Horava-Lifshitz

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Outline

- Quantum Gravity => Noncommutative Geometry
- Spectral dimension: mathematical vs physical (scaling)
- Horava-Lifshitz gravity and its spectral dimension
- Tori: commutative $\longrightarrow$ (highly) noncommutative
- A model of NC geometry with a highly nontrivial scaling dimension
- Conclusions

(New part is based on F. Lizzi and AP 2017)

Why do we "deform" geometry?

1) Problems with the quantization of gravity

$[\lambda] = \delta$ in momentum units

$D = d - (d/2 - 1)E - n\delta$

D - superficial degree of divergence

d - space - time dimension

E - numberf of the external legs

n - number of vertices

We can expect renormalizability only when $\delta \geq 0$

Why do we "deform" geometry?

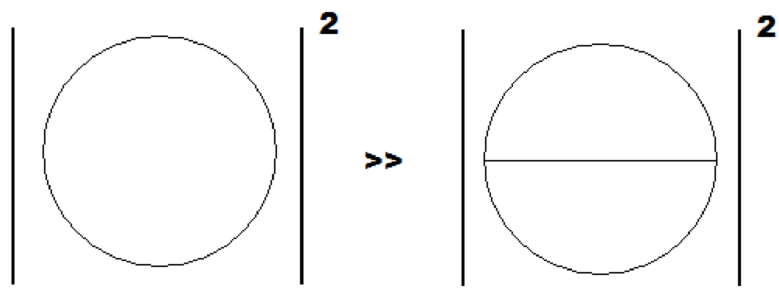
$$S_{EH} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R \quad => \quad \delta \equiv [G] = 2 - d$$

$$\text{for } d = 4, \delta = -2 < 0$$

As the result, the effective dimensionless constant is given by

$$GE^2 := \left(\frac{E}{M_P}\right)^2 \quad \text{where } M_P = \sqrt{\frac{\hbar c}{G}} = 1.22 \times 10^{19}\, GeV$$

$$\text{i.e. when } E << M_p$$



Why do we "deform" geometry?

Possible solutions

i) (Super)string theory: contains a spin-2 massles mode => has to describe gravity. GR is recovered in long-wave regime. But, the predictive power is quite poor: the string theory landscape has 10^{500} vacua.
ii) Loop quantum gravity: one can perform non-perturbative quantization. Among problems, the difficulty of the recovery quasiclassical space.
iii) Some other approaches treat gravity as an emergent phenomenon (e.g., entropic gravity).

Why do we "deform" geometry?

2) General arguments that the notion of a space-time as a classical manifold should be abandoned Doplicher, Fredenhagen and Roberts 1995

Spectral dimension

Weyl theorem:

Let M be a compact Riemannian manifold, D – the standard Dirac operator on it. Then

$$N_D(\lambda) \xrightarrow[\lambda\to\infty]{} \frac{2^m \Omega_n}{n(2\pi)^n} Vol(M)\lambda^n$$

$$N_D(\lambda) = \{\#\,\text{eigenvalues} < \lambda\} \text{ - a counting function}$$

Spectral dimension

- As defined, the spectral dimension is a UV dimension, i.e. assumes that we use some device with an infinite precision.
- This is not the case: physically, the device resolution will introduce some cutoff => physical (scaling dimension) AP 2011

$$d_S(\lambda) := \frac{d(\log N_D(\lambda))}{d\log(\lambda)}$$

- Why is Dirac operator relevant at all?

$$S_{mat} = \langle \psi | D | \psi \rangle$$

Horava-Lifshitz gravity

Lifshitz model

$$S = \int dt d^n x \left(\dot{\phi}^2 - c^2 \phi \Delta \phi + g (\Delta \phi)^2 \right)$$

$$[x] = -1, [t] = -2, [\mathrm{c}] = 1$$

The propagator has the form:

$$G(\omega, \vec{k}) \propto \frac{1}{\omega^2 - c^2 \vec{k}^2 - g \vec{k}^4}$$

Horava-Lifshitz gravity

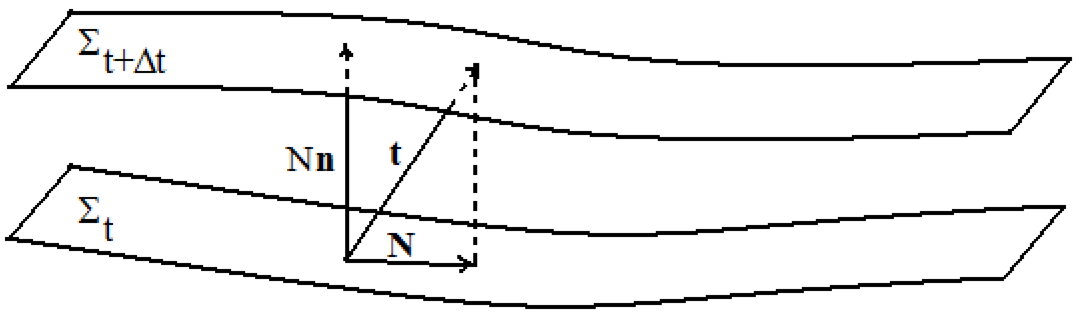
$$\mathrm{UV}: \frac{1}{\omega^2 - c^2\vec{k}^2 - g^2\vec{k}^4} = \frac{1}{\omega^2 - g^2\vec{k}^4} + \frac{1}{\omega^2 - g^2\vec{k}^4} c^2\vec{k}^2 \frac{1}{\omega^2 - g^2\vec{k}^4} + \ldots$$

$$\mathrm{IR}: \frac{1}{\omega^2 - c^2\vec{k}^2 - g^2\vec{k}^4} = \frac{1}{\omega^2 - c^2\vec{k}^2} + \frac{1}{\omega^2 - c^2\vec{k}^2} g^2\vec{k}^4 \frac{1}{\omega^2 - c^2\vec{k}^2} + \ldots$$

I.e. we have two fixed points: UV, which corresponds to z=2 and has significantly improved behavior and IR, in which by the time rescaling we can set c=1 and restore relativistic invariance, z=1

Horava-Lifshitz gravity

Horavas's idea Horava 2009



$$ds^2 = g_{ij}(dx^i + N^i dt)(dx^j + N^j dt) - (Ndt)^2$$

$$S_{EH} = \frac{1}{16\pi G}\int dt d^3x N\sqrt{g}\left(K_{ij}K^{ij} - K^2 + {}^3R\right)$$

where $K_{ij} = \frac{1}{2N}\left(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i\right)$ - second fundamental form

Horava-Lifshitz gravity

We take ADM slicing as fundamental, i.e. instead of considering just a manifold, we endow it with the foliation structure:

$$\tilde{x}^i = \tilde{x}^i(\vec{x}, t), \ \tilde{t} = \tilde{t}(t)$$

These are foliation-preserving diffeos or FDiffs

* It is also interesting to consider the restricted foliation-preserving diffeos, RFDiffs:

$$\tilde{x}^i = \tilde{x}^i(\vec{x}, t), \ \tilde{t} = t + const$$

Horava-Lifshitz gravity

- Projectable FDiff gravity

$$N = N(t), \quad N \to N\frac{\partial t}{\partial \tilde{t}}$$

$$S = \frac{M_P^2}{2}\int d^3x dt\sqrt{g}N\left(K_{ij}K^{ij} - \lambda K^2 - V_P\right)$$

$$V_P = -\xi R + M_*^{-2}\left(A_1 R^2 + A_2 R_{ij}R^{ij} + \cdots\right) + M_*^{-4}\left(B_1 R\Delta R + B_2 R_{ij}R^{jk}R_k^i + \cdots\right)$$

- Non-projectable FDiff gravity

$$N = N(t, \vec{x}), \quad a_i := N^{-1}\partial_i N$$

$$S = \frac{M_P^2}{2}\int d^3x dt\sqrt{g}N\left(K_{ij}K^{ij} - \lambda K^2 - V_{NP}\right)$$

$$V_{NP} = V_P - \alpha a_i a^i + M_*^{-2}\left(C_1 a_i \Delta a^i + C_2 (a_i a^i)^2 + C_3 a_i a_j R^{ij} \cdots\right) +$$
$$+ M_*^{-4}\left(D_1 a_i \Delta^2 a^i + D_2 (a_i a^i)^3 + D_3 a_k a^k a_i a_j R^{ij} \cdots\right)$$

HL gravity: spectral dimension

- The choice of the Dirac operator in the form $D=\gamma^{\mu}(\partial_{\mu}+\omega_{\mu})$ is not natural anymore
- The foliation structure dictates the following (schematic) form for D (for $z=3$)
  $D=\partial_t+\sigma^{\mu}\partial_{\mu}\Delta+M_{*}\Delta+M_{*}^{2}\sigma^{\mu}\partial_{\mu}$
- This D should be used to obtain "physical" geometry instead of auxiliary 3+1 dimensional

(AP 2010, Gregory & AP 2012)

HL gravity: spectral dimension

- $M = S^1 \times T^3$, $D^2 = \partial_t^{\,2} + \Delta^3 + M_*^{\,2}\Delta^2 + M_*^{\,4}\,\Delta$
- $sp(D^2) = \{n^2 + (n_1^{\,2} + n_2^{\,2} + n_3^{\,2})^3 + M_*^{\,2}(n_1^{\,2} + n_2^{\,2} + n_3^{\,2})^2$
  $+ M_*^{\,4}(n_1^{\,2} + n_2^{\,2} + n_3^{\,2})\ ,\ n_i \in Z\}$
- $N_D(\lambda) = \{\#\ \text{eigenvalues} < \lambda\}$
- when $\lambda << M_*^{\,6}$ the last term dominates:

$$N_{|D|}(\lambda) \cong \int_0^{\lambda} dn \int_0^{(\lambda^2 - n^2)^{1/2}} 4\pi\rho^2 d\rho \propto \lambda^4 \quad \Rightarrow \quad d = 4$$

when $\lambda >> M_*^{\,6}$ the first term dominates:

$$N_{|D|}(\lambda) \cong \int_0^{\lambda} dn \int_0^{(\lambda^2 - n^2)^{1/6}} 4\pi\rho^2 d\rho \propto \lambda^2 \quad \Rightarrow \quad d = 2$$

Various 2-d tori

- Commutative torus
  algebra (topology) is generated by $u=exp(2\pi ix)$ and $v=exp(2\pi iy)$
  derivatives (geometry) are given on the generators:

$$\nabla_1 u = 2\pi i u, \quad \nabla_1 v = 0$$

$$\nabla_2 u = 0, \quad \nabla_2 v = 2\pi i v$$

- Noncommutative torus
  algebra (topology) is generated by U and V subject to

$$VU = \exp(2\pi i\theta)UV$$

  and the same derivatives

- Matrix approximation?

Matrix (fuzzy) torus

- Approximating a (non)commutative torus by a sequence of some matrix algebras, i.e. realizing it as an AF-algebra is impossible due to the K-theoretical obstruction: $Z \oplus Z \neq 0$
- Still one can construct a 0-dimensional approximation in much less trivial way Lizzi&Szabo 1999; Landi, Lizzi & Szabo 2001
- Instead one can construct an approximation by an inductive limit of some 1-dimensional matrix geometries Elliott&Evans 1993; Landi, Lizzi & Szabo 2004

Matrix (fuzzy) torus

Elliott-Evans construction

- Start with a two towers of the Rieffel-type projectors in the algebra of NC torus:

  $P_{11} := \mathbf{v}^{-q'}\mathbf{g} + \mathbf{f} + \mathbf{g}\mathbf{v}^{q'}$

  $\mathrm{Tr} P_{11} = p' - q'\theta =: \beta$, i.e. P_{11} represents the $(p', -q')$-class in K_0.

  $P_{ii} := (\alpha_{\mathrm{e}^{2\pi\,\mathrm{i}\,p/q},1})^{i-1}(P_{11}) =: \alpha^{i-1}(P_{11})$

  and the same with the obvious exchange of **u** and **v**.

- Complete the system of projectors by P_{ij} to form the algebra of matrix units:

$$\forall i, j = \overline{1, q} \;\; P_{ij}P_{ls} = \delta_{jl}P_{is} \;, \; \sum_{i=1}^{q} P_{ii} = P =: \mathbb{1}_q$$

Matrix (fuzzy) torus

Elliott-Evans construction

- For the consistency add one more unitary element for each tower:

$$\alpha^{q-1}(P_{21}) = zP_{1q}$$

- Enjoy the best possible approximation:

$$\mathrm{U} := \mathcal{C}_q \oplus \mathcal{S}_{q'}(z')\ ,\quad \mathrm{V} := \mathcal{S}_q(z) \oplus \bar{\mathcal{C}}_{q'}$$

where

$$\mathcal{S}_q(z) := \sum_{k=0}^{q-2} P_{2+k,1+k} + zP_{1q}\ ,\ \ \mathcal{C}_q := \sum_{k=0}^{q-1} \xi^{-k} P_{k+1,k+1}$$

which can be isomorphically represented as the elements of $\mathrm{Mat}_q(\mathcal{C}^\infty(\mathbb{S}^1))$

Matrix (fuzzy) torus

Elliott-Evans construction

$$\mathcal{C}_q := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \xi^{-1} & 0 & \cdots & 0 \\ 0 & 0 & \xi^{-2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \xi^{1-q} \end{pmatrix} , \ \mathcal{S}_q(z) := \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & z \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} , \ \xi = \mathrm{e}^{2\pi\,\mathrm{i}\,\frac{p}{q}}$$

- The choice of *q,q',p,p'*

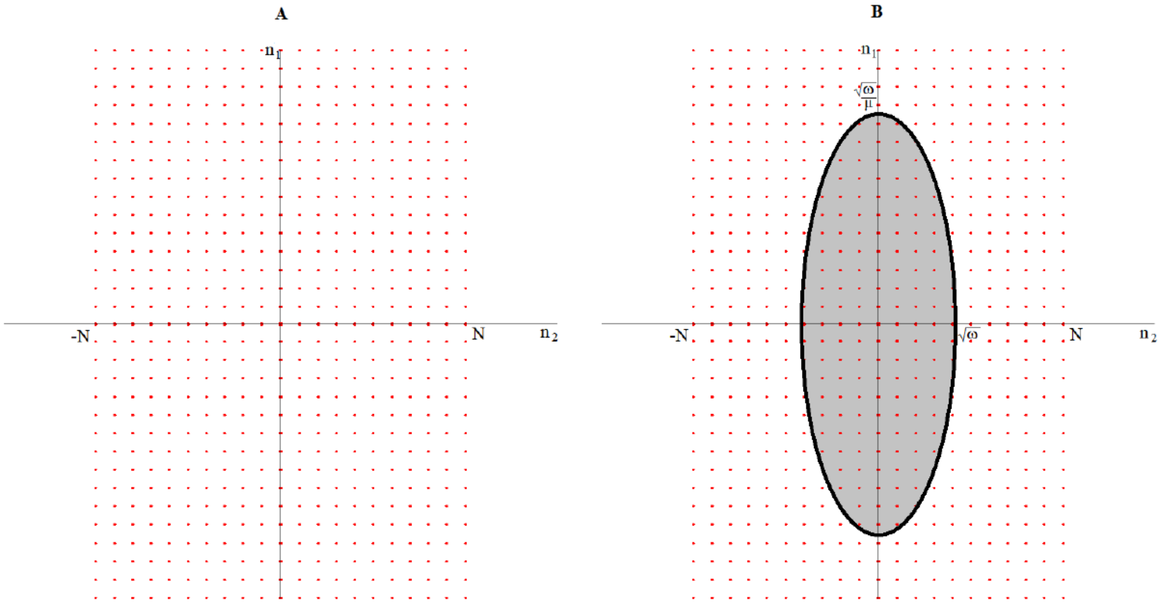
$$\frac{p}{q} < \theta < \frac{p'}{q'}$$

- With this choice, **U** and **V** go to **u** and **v** as q,q'→∞

Spectral dimension Lizzi&AP 2017

- What do we expect? (A super toy model)
  Consider a (not well-defined) fuzzy geometry:

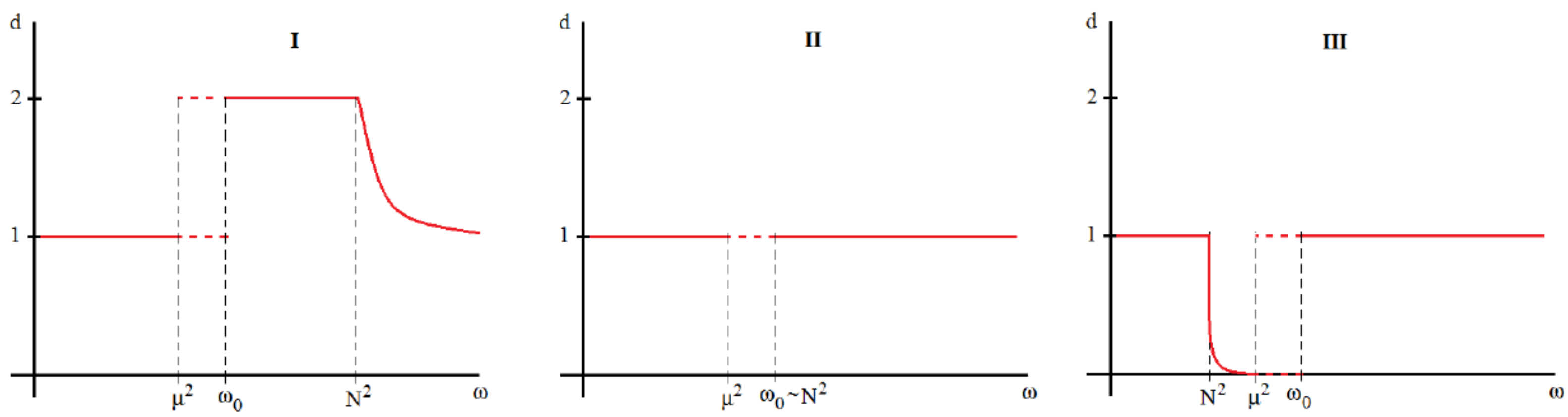
$$Spec(\triangle_{1fuzzy}) = \left\{ \frac{n_1^2}{r^2} + \frac{n_2^2}{R^2} \;,\; n_1, n_2 \in \mathbb{Z} \;,\; |n_2| \leq N \right\}$$



Spectral dimension

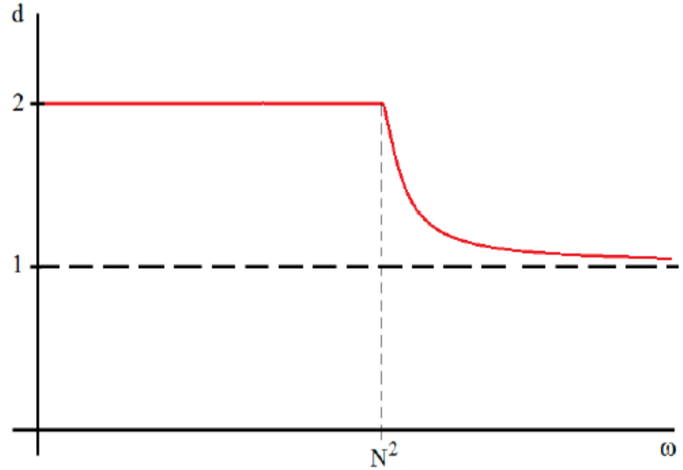
Applying Weyl's formula (in the physical sense), we get various regimes depending on the interplay between the parameters:

i) When $R >> r$



Spectral dimension

i) When $R \sim r$



In both cases the transition 2⟶1 is given by

$$d(\omega) = 2\left(1 + \frac{N}{\sqrt{\omega}R}\frac{\sqrt{1-\frac{N^2}{\omega R^2}}}{\arcsin\left(\frac{N}{\sqrt{\omega}R}\right)}\right)^{-1}$$

Spectral dimension

- Back to the fuzzy torus.
  Unfortunately there are no natural derivatives, satisfying both, the Leibnitz rule and defined nicely on the generators. For our purpose some choice will suffice and we use some slight modification of the one from Landi, Lizzi & Szabo 2004

  1) Firstly, introduce a truncation map. This is a map from the algebra of a NC torus to $\mathrm{Mat}_q(\mathcal{C}^\infty(\mathbb{S}^1)) \oplus \mathrm{Mat}_q(\mathcal{C}^\infty(\mathbb{S}^1))$

  $\Gamma_n \colon \mathcal{A}_\theta \to \mathcal{A}_n$ I.e. for an arbitrary element of $\mathcal{A}_\theta$

$$\forall a \in \mathcal{A}_\theta \ , \ \Gamma_n(a) := \sum_{(l,m)\in\mathbb{Z}^2} a(l,m)\, \mathbf{U}_n^l \mathbf{V}_n^m$$

Spectral dimension

Or in terms of the earlier introduced generalized clock and shift matrices:

$$\Gamma_n(a) = \left(\sum_{m,r=0}^{q_{2n}-1} \sum_{l \in \mathbb{Z}} a^{(n)}(m, r; l) z^l (\mathcal{C}_{q_{2n}})^{m - \left[\frac{q_{2n}}{2}\right]} (\mathcal{S}_{q_{2n}}(z))^r \right) \oplus$$

$$\oplus \left(\sum_{m', r'=0}^{q_{2n-1}-1} \sum_{l' \in \mathbb{Z}} a'^{(n)}(m', r'; l') z'^{l'} (\mathcal{S}_{q_{2n-1}}(z'))^{m'} (\bar{\mathcal{C}}_{q_{2n-1}})^{r' - \left[\frac{q_{2n-1}}{2}\right]} \right)$$

2) Secondly, define the deformed derivatives in such a way that they are diagonal in the above representation and

$$\nabla_i \Gamma_n(a) := \Gamma_n(\partial_i a) + \mathcal{O}(\cdots)$$

Spectral dimension

This leads to

$$\nabla_1 \Gamma_n(a) := 2\pi\,\mathrm{i}\left(\sum_{m,r=0}^{q_{2n}-1}\sum_{l\in\mathbb{Z}}\left(m-\left[\frac{q_{2n}}{2}\right]\right) a^{(n)}(m,r;l) z^l (\mathcal{C}_{q_{2n}})^{m-\left[\frac{q_{2n}}{2}\right]} (\mathcal{S}_{q_{2n}}(z))^r \oplus \right.$$

$$\left. \oplus \sum_{m',r'=0}^{q_{2n-1}-1}\sum_{l'\in\mathbb{Z}} (l' q_{2n-1} + m')\, a'^{(n)}(m',r';l') z'^{l'} (\mathcal{S}_{q_{2n-1}}(z'))^{m'} (\bar{\mathcal{C}}_{q_{2n-1}})^{r'-\left[\frac{q_{2n-1}}{2}\right]} \right) ,$$

$$\nabla_2 \Gamma_n(a) := 2\pi\,\mathrm{i}\left(\sum_{m,r=0}^{q_{2n}-1}\sum_{l\in\mathbb{Z}} (l q_{2n} + r)\; a^{(n)}(m,r;l) z^l (\mathcal{C}_{q_{2n}})^{m-\left[\frac{q_{2n}}{2}\right]} (\mathcal{S}_{q_{2n}}(z))^r \oplus \right.$$

$$\left. \oplus \sum_{m',r'=0}^{q_{2n-1}-1}\sum_{l'\in\mathbb{Z}} \left(r' - \left[\frac{q_{2n-1}}{2}\right]\right) a'^{(n)}(m',r';l') z'^{l'} (\mathcal{S}_{q_{2n-1}}(z'))^{m'} (\bar{\mathcal{C}}_{q_{2n-1}})^{r'-\left[\frac{q_{2n-1}}{2}\right]} \right) . \quad (5.3)$$

Spectral dimension

- Now we are ready to analyze the spectral (scaling, physical) dimension of this fuzzy geometry. For this we define the deformed Laplacian in the most natural way:

$$\Delta_{(n)} = -\nabla_1^2 - \nabla_2^2$$

The spectrum can be found exactly:

$$Spec(\Delta_{(n)}) = \left\{ 4\pi^2(k^2+s^2)\ ,\ k \in \overline{-\left[\frac{q_{2n}}{2}\right], \left[\frac{q_{2n}}{2}\right]}\ ,\ s \in \mathbb{Z} \right\} \bigcup$$
$$\bigcup \left\{ 4\pi^2(k'^2+s'^2)\ ,\ k' \in \overline{-\left[\frac{q_{2n-1}}{2}\right], \left[\frac{q_{2n-1}}{2}\right]}\ ,\ s' \in \mathbb{Z} \right\}$$

Spectral dimension

- We will analyze the spectral dimension in two regemes, IR and UV

  IR) By IR we mean that the cut-off scale ω is below the characteristic quantum geometric scale. In the case of a toy model this scale was controlled by the number of the states along the R-direction. In the present case, this means that $\omega < q^2_{2n-1}$, it does not even have to be much smaller. Then we immediately have for the counting function

$$N_\Delta(\omega) \sim \text{degeneracy} \times \int\limits_{m^2+s^2\leq\frac{\omega}{4\pi^2}} dm\, ds = \text{const} \times \omega$$

leading to $d_{IR}=2$

Spectral dimension

UV) This is the case opposite to the previous one, i.e. many of the winding modes are excited. This means that the hypothetical experiment can probe the physics up to the cut-off $\omega \gg q_{2n}^2, q_{2n-1}^2$. For the spectrum we have

$$4\pi^2 \left(\left(r' - \left[\frac{q_{2n-1}}{2}\right]\right)^2 + (q_{2n-1}l' + m')^2 \right) = 4\pi^2 q_{2n-1}^2 l'^2 \left(1 + \mathcal{O}\left(\frac{1}{l'}\right)\right)$$

resulting in the counting function

$$N_\Delta(\omega) \rightarrow \text{degeneracy} \times \iint^{q_{2n-1}} dm\, dr \int_{-\frac{\sqrt{\omega}}{2\pi q_{2n-1}}}^{\frac{\sqrt{\omega}}{2\pi q_{2n-1}}} dk = \text{const} \times q_{2n-1}\sqrt{\omega}$$

and leading to $d_{UV}=1$

Conclusions

- Though HL gravity seems to provide UV completion for GR there several conceptual problems: the existence of the preferred direction (time) and complete ignorance about the fundamental (UV) degrees of freedom.
- Though our model is not supported (yet) by any fundamental theory, it is an example of a geometry with completely different d.o.f. in UV and IR (and one can identify those!), which does not have a preferred direction.
- In principle, the model can be generalized to a more realistic 4d case.
- Work/discussion in progress: the choice of a "natural" Dirac operator; coupling geometry to matter (spectral action), etc.

Happy birthday, Bal!!!