Non-trivial Violations of the Area Law

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Plan of the talk

1. Introduction to Entanglement Entropy

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2. Motzkin Spin Chain

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3. Fredkin Spin Chain

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1. Introduction to Entanglement Entropy
2. Motzkin Spin Chain
3. Fredkin Spin Chain
4. Modified Motzkin Spin Chain

Plan of the talk

Entropy : Classical vs Quantum

- For classical systems entropy measures the uncertainty of the microstates.
- In quantum systems we can build reduced density matrix out of pure states that have a non zero von Neumann entropy.

$$S(\rho_B) = -\mathrm{tr}\left[\rho_B \log_2 \rho_B\right]$$

where $\rho_B = \mathrm{tr}_A(\rho)$ with $\rho = |\psi\rangle\langle\psi|$ the density matrix of the full system.
- This is called *entanglement entropy* (EE) and is purely a quantum property, different from the thermal entropy.

Area Law and Locality

- A general quantum many body state can have extensive entropy which implies we require a enormous amount of information to specify it.
- However most systems have local interactions and have ground states that only obey the area law implying the need for fewer parameters to study the state.
- Early signs of this law seen in black hole entropy and the holographic principle. (Bekenstein 1974, 2004; Hawking 1974)
- Gapped systems in one dimensions are shown to obey the area law. (Hastings, 2007)

$$S(\rho_I) \leq S_{max} = c_0 \xi \log(6\xi) \log(d) 2^{6\xi \log(d)}$$

where $\xi = \max(\frac{2v}{\Delta E}, \xi_C)$ and $\xi_C \sim O(1)$.

Numerical Simulation of Many Body Systems

- A classical system of n particles require $O(n)$ parameters to describe it whereas the analogous quantum system requires an exponentially large number, $2^{O(n)}$.
- If the state has little entanglement then we can expect it to be described by fewer parameters.
- In one dimensions this has been shown using the methods of the density-matrix renormalization group (DMRG). (White 1992; Schollwöck 2005)
- Such finitely correlated states can be approximated by a matrix-product state (MPS).

Violations of the Area Law

- For a N-site chain of harmonic oscillators given by

$$H = \frac{1}{2} \sum_{i,j \in L} \left(p_i P_{i,j} p_j + x_i X_{i,j} x_j \right),$$

with X and P real, symmetric and positive matrices. The entanglement entropy of the half chain is

$$S(\rho_I) \leq \frac{1}{2} \log_2 \left(\frac{||X||^{\frac{1}{2}}}{\Delta E} \right),$$

with ΔE the energy gap and $||.||$ the operator norm.
- In the large N limit we obtain the Klein-Gordon field Hamiltonian and the bound on the EE becomes

$$S(\rho_I) \leq \frac{1}{2} \log_2 \left(\frac{2N}{m} \right),$$

where m is the mass of the scalar field. So logarithmic divergence !

More violations .....

- For quantum critical systems it was seen that the EE grows logarithmically in the subsystem size. This is seen for fermionic quasifree models with periodic boundary conditions.

$$H = \frac{1}{2} \sum_{i,j \in L} \left(f_i^\dagger A_{i,j} f_j - f_i A_{i,j} f_j^\dagger + f_i B_{i,j} f_j - f_i^\dagger B_{i,j} f_j^\dagger \right).$$

The EE for $B = 0$ scales as

$$S(\rho_I) = \xi \log_2(n) + O(1)$$

- A more complicated translationally invariant system with local HIlbert space having dimension 21 is shown to have volume law behavior for EE. (S. Irani, 2009)

Do local interactions imply an area law for EE ?
Can we do better than a logarithmic violation ?

Motzkin Spin Chain (P. Shor et. al. 2014)

- The local Hilbert space is given by $\{u^1, u^2, \cdots, u^s, 0, d^1, d^2, \cdots, d^s\}$, where u, d and 0 are dubbed "up", "down" and "flat" steps respectively.
- The system lives on a 1D chain and we can geometrically interpret the above steps as being along the $(1,1)$, $(1,-1)$ and $(1,0)$ directions respectively. s denotes the color of the step.
- For a $2n$-step/link chain the many body states are 2D paths. *Motzkin* walks are paths which start at $(0,0)$, end at $(2n,0)$, and always stays in the positive quadrant.
- The uniform superposition of such paths form the ground state of the Motzkin spin chain and has a half chain EE

$$S = 2\log_2(s)\sqrt{\frac{2\sigma n}{\pi}} + \frac{1}{2}\log_2(2\pi\sigma n) + O(1),$$

with $\sigma = \frac{\sqrt{s}}{2\sqrt{s}+1}$ and γ is Euler constant.

Local Hilbert Space : Colored Motzkin

$|\uparrow\rangle \;\equiv$ 

$|\uparrow^k\rangle \;\equiv$ 

$|\downarrow\rangle \;\equiv$ 

$|\downarrow^k\rangle \;\equiv$ 

$|\rightarrow\rangle \;\equiv$ 

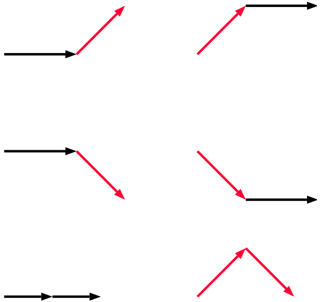
Motzkin Spin Chain Hamiltonian : $H_{Motzkin}$

- The local, frustration free Hamiltonian is built out of projectors to local equivalence moves

$$\begin{aligned} \left|D^k\right\rangle &= \frac{1}{\sqrt{2}}\left[\left|0d^k\right\rangle - \left|d^k 0\right\rangle\right] \\ \left|U^k\right\rangle &= \frac{1}{\sqrt{2}}\left[\left|0u^k\right\rangle - \left|u^k 0\right\rangle\right] \\ \left|F^k\right\rangle &= \frac{1}{\sqrt{2}}\left[\left|00\right\rangle - \left|u^k d^k\right\rangle\right] \end{aligned}$$

$$\Pi_{j,j+1} = \sum_{k=1}^{s}\left[\left|D^k\right\rangle_{j,j+1}\left\langle D^k\right| + \left|U^k\right\rangle_{j,j+1}\left\langle U^k\right| + \left|F^k\right\rangle_{j,j+1}\left\langle F^k\right|\right]$$

Local Equivalences : Colored Motzkin Chain



$H_{Motzkin}$.....

-The boundary term is

$$\Pi_{boundary} = \sum_{k=1}^{s} \left[\left| d^k \right\rangle_1 \left\langle d^k \right| + \left| u^k \right\rangle_{2n} \left\langle u^k \right| \right]$$

- A color balancing term

$$\Pi^{cross}_{j,j+1} = \sum_{k \neq i} \left| u^k d^i \right\rangle_{j,j+1} \left\langle u^k d^i \right|$$

- Finally

$$H_{Motzkin} = \Pi_{boundary} + \sum_{j=1}^{2n-1} \left[\Pi_{j,j+1} + \Pi^{cross}_{j,j+1} \right].$$

This is essentially a spin 1 chain. Model is gapless with gap scaling as n^{-c} with $c \geq 2$.

Fredkin Spin Chain (V. Korepin et. al. 2016)

- The local Hilbert space is spanned by $\{|\uparrow\rangle, |\downarrow\rangle\}$.
- Geometrically we have only "up" and "down" steps and no "flat" steps. The "up" step points along $(1,1)$ and the "down" step points along $(1,-1)$.
- The states on the global Hilbert space are mapped to 2D *Dyck* walks which again start at $(0,0)$ and end at $(2n,0)$ without leaving the first quadrant.
- Notice that this is an uncolored local Hilbert space and the EE scales as

$$S = \frac{1}{2}\log(L) + O(1)$$

Local Hilbert Space : Colored Fredkin Chain

$|\uparrow\rangle \equiv$ 

$|\uparrow^{k}\rangle \equiv$ 

$|\downarrow\rangle \equiv$ 

$|\downarrow^{k}\rangle \equiv$ 

Fredkin Spin Chain Hamiltonian : $H_{Fredkin}$

- The local, frustration free Hamiltonian is built out of projectors to local equivalence moves

$$|U_j\rangle = \frac{1}{\sqrt{2}}\left[|\uparrow_j, \uparrow_{j+1}, \downarrow_{j+2}\rangle - |\uparrow_j, \downarrow_{j+1}, \uparrow_{j+2}\rangle\right],$$
$$|D_j\rangle = \frac{1}{\sqrt{2}}\left[|\uparrow_j, \downarrow_{j+1}, \downarrow_{j+2}\rangle - |\downarrow_j, \uparrow_{j+1}, \downarrow_{j+2}\rangle\right].$$

$$\Pi_{j,j+1,j+2} = |U_j\rangle\langle U_j| + |D_j\rangle\langle D_j|$$

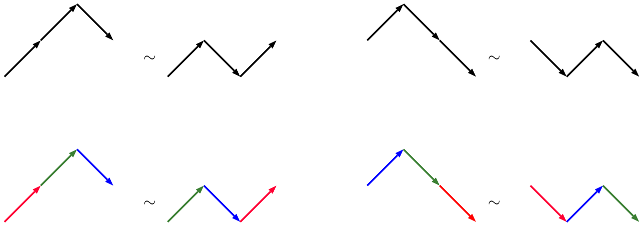
Boundary term is

$$H_{boundary} = \left[|\downarrow_1\rangle\langle\downarrow_1| + |\uparrow_{2n}\rangle\langle\uparrow_{2n}|\right]$$

$$H_{Fredkin} = H_{boundary} + \sum_{j=1}^{2n-2} \Pi_{j,j+1,j+2}.$$

- This is a spin $\frac{1}{2}$ chain. Has global $U(1)$ symmetry.

Local Equivalences : Colored Fredkin Chain



Colored Fredkin Spin Chain : $H_{colored,\,Fredkin}$

- Include s colors to each of the local basis states. The local equivalence moves now become

$$\begin{aligned}\left|U_j^{c_1,\,c_2,\,c_3}\right\rangle &= \frac{1}{\sqrt{2}}\left[\left|\uparrow_j^{c_1},\,\uparrow_{j+1}^{c_2},\,\downarrow_{j+2}^{c_3}\right\rangle-\left|\uparrow_j^{c_2},\,\downarrow_{j+1}^{c_3},\,\uparrow_{j+2}^{c_1}\right\rangle\right],\\ \left|D_j^{c_1,\,c_2,\,c_3}\right\rangle &= \frac{1}{\sqrt{2}}\left[\left|\uparrow_j^{c_2},\,\downarrow_{j+1}^{c_3},\,\downarrow_{j+2}^{c_1}\right\rangle-\left|\downarrow_j^{c_1},\,\uparrow_{j+1}^{c_2},\,\downarrow_{j+2}^{c_3}\right\rangle\right].\end{aligned}$$

$$B_{j,j+1}=\left|\uparrow_j^{c_1},\,\downarrow_{j+1}^{c_2}\right\rangle\left\langle\uparrow_j^{c_1},\,\downarrow_{j+1}^{c_2}\right|$$

$$C_{j,j+1}=\Pi^{\frac{1}{\sqrt{2}}\left[\left|\uparrow_j^{c_1},\downarrow_{j+1}^{c_1}\right\rangle-\left|\uparrow_j^{c_2},\downarrow_{j+1}^{c_2}\right\rangle\right]}.$$

$$S\sim\frac{2}{\sqrt{\pi}}\log(s)\sqrt{\frac{(n+r)(n-r)}{n}}++\frac{1}{2}\ln\frac{(n+r)(n-r)}{n}+O(1).$$

Deformed Motzkin Chain (Z. Zhang et. al. 2016)

$$\begin{aligned} \left|D^k\right\rangle &= \frac{1}{\sqrt{1+t^2}}\left[\left|0d^k\right\rangle - t\left|d^k0\right\rangle\right] \\ \left|U^k\right\rangle &= \frac{1}{\sqrt{1+t^2}}\left[\left|0u^k\right\rangle - t\left|u^k0\right\rangle\right] \\ \left|F^k\right\rangle &= \frac{1}{\sqrt{1+t^2}}\left[\left|00\right\rangle - t\left|u^kd^k\right\rangle\right] \end{aligned}$$

- Ground state is now weighted Motzkin path.
- For $t > 1$ in the colored case the EE scales as the volume $n \log s$ and for $t = 1$ we have the $\sqrt{n}$ scaling for the half chain.

Deformed Fredkin Chain (O. Salberger et. al. 2016)

-Introduce a deforming parameter t in the local equivalence move

$$\left|U_j^{c_1,c_2,c_3}\right\rangle = \frac{1}{\sqrt{1+t^2}}\left[\left|\uparrow_j^{c_1},\uparrow_{j+1}^{c_2},\downarrow_{j+2}^{c_3}\right\rangle - t\left|\uparrow_j^{c_2},\downarrow_{j+1}^{c_3},\uparrow_{j+2}^{c_1}\right\rangle\right],$$

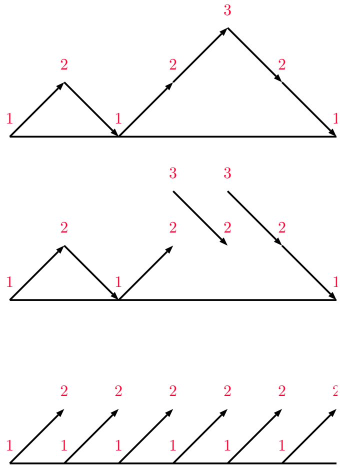
$$\left|D_j^{c_1,c_2,c_3}\right\rangle = \frac{1}{\sqrt{1+t^2}}\left[\left|\uparrow_j^{c_2},\downarrow_{j+1}^{c_3},\downarrow_{j+2}^{c_1}\right\rangle - t\left|\downarrow_j^{c_1},\uparrow_{j+1}^{c_2},\downarrow_{j+2}^{c_3}\right\rangle\right].$$

- Ground state is now weighted Dyck path.
- For $t > 1$ in the colored case the EE scales as the volume $n \log s$ and for $t = 1$ we have the $\sqrt{n}$ scaling for the half chain.

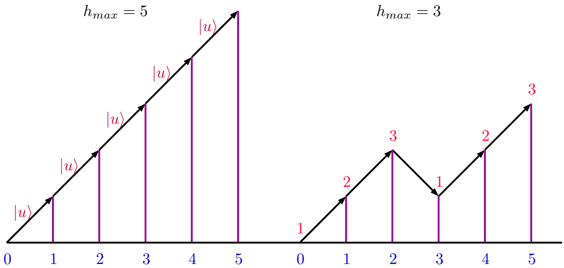
A Modification of the Motzkin Spin Chain (F.Sugino, PP, 2017)

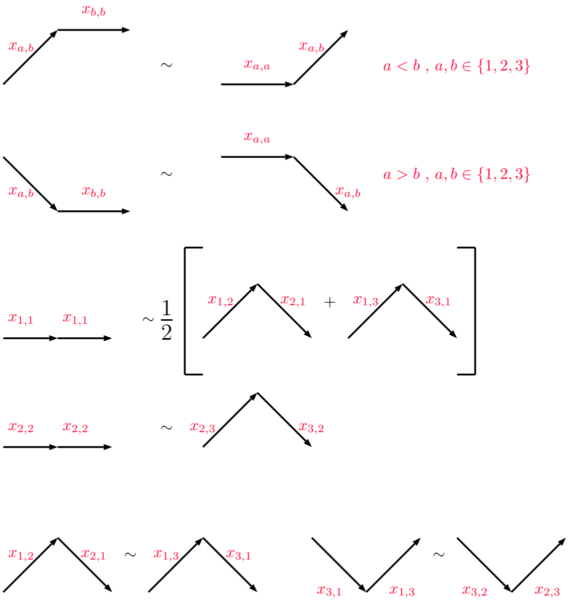
- Change the local Hilbert space to $\{|x_{a,b}\rangle;\ a, b \in \{1,2,3\}\}$. The "up" steps pointing along $(1,1)$ occur when $a < b$, "down" steps pointing along $(1,-1)$ occur when $a > b$ and the "flat" steps pointing along $(1,0)$ occur when $a = b$. These new indices can be thought of as arrow indices or more mathematically they are known as semigroup indices.
- This introduces different kinds of paths, *fully connected*, *partially connected* and *disconnected* paths.
- The maximum heights reached in a path is now smaller.

Different Kinds of Paths



Maximum Heights





x_{b,b}
x_{a,b}
~
x_{a,a}
x_{a,b}
a < b , a, b ∈ {1, 2, 3}
x_{a,a}
x_{a,b}
x_{b,b}
~
x_{a,b}
a > b , a, b ∈ {1, 2, 3}
x_{1,1}
x_{1,1}
~ 1/2
x_{1,2}
x_{2,1}
+
x_{1,3}
x_{3,1}
x_{2,2}
x_{2,2}
~
x_{2,3}
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x_{1,2}
x_{2,1}
~
x_{1,3}
x_{3,1}
x_{3,1}
x_{1,3}
~
x_{3,2}
x_{2,3}

Projectors to the Modified Local Equivalence Moves

$$U_{j,j+1} = \sum_{a,b=1;a<b}^{3} \Pi_{\frac{1}{\sqrt{2}}\left[\left|\left(x_{a,b}\right)_j,\left(x_{b,b}\right)_{j+1}\right\rangle - \left|(x_{a,a})_j,\left(x_{a,b}\right)_{j+1}\right\rangle\right]},$$

$$D_{j,j+1} = \sum_{a,b=1;a>b}^{3} \Pi_{\frac{1}{\sqrt{2}}\left[\left|\left(x_{a,b}\right)_j,\left(x_{b,b}\right)_{j+1}\right\rangle - \left|(x_{a,a})_j,\left(x_{a,b}\right)_{j+1}\right\rangle\right]},$$

$$F_{j,j+1} = \Pi_{\sqrt{\frac{2}{3}}\left[\left|(x_{1,1})_j,(x_{1,1})_{j+1}\right\rangle - \frac{1}{2}\left(\left|(x_{1,2})_j,(x_{2,1})_{j+1}\right\rangle + \left|(x_{1,3})_j,(x_{3,1})_{j+1}\right\rangle\right)\right]} + \Pi_{\frac{1}{\sqrt{2}}\left[\left|(x_{2,2})_j,(x_{2,2})_{j+1}\right\rangle - \left|(x_{2,3})_j,(x_{3,2})_{j+1}\right\rangle\right]},$$

$$W_{j,j+1} = \Pi_{\frac{1}{\sqrt{2}}\left[\left|(x_{1,2})_j,(x_{2,1})_{j+1}\right\rangle - \left|(x_{1,3})_j,(x_{3,1})_{j+1}\right\rangle\right]} + \mu\Pi_{\frac{1}{\sqrt{2}}\left[\left|(x_{3,1})_j,(x_{1,3})_{j+1}\right\rangle - \left|(x_{3,2})_j,(x_{2,3})_{j+1}\right\rangle\right]}.$$

Boundary, Balancing and Bulk, Disconnected Terms

$$
\begin{aligned}
H_{left} &= \Pi^{\left|(x_{2,1})_1\right\rangle} + \Pi^{\left|(x_{3,1})_1\right\rangle} + \Pi^{\left|(x_{3,2})_1\right\rangle}, \\
H_{right} &= \Pi^{\left|(x_{1,2})_n\right\rangle} + \Pi^{\left|(x_{1,3})_n\right\rangle} + \Pi^{\left|(x_{2,3})_n\right\rangle}.
\end{aligned}
$$

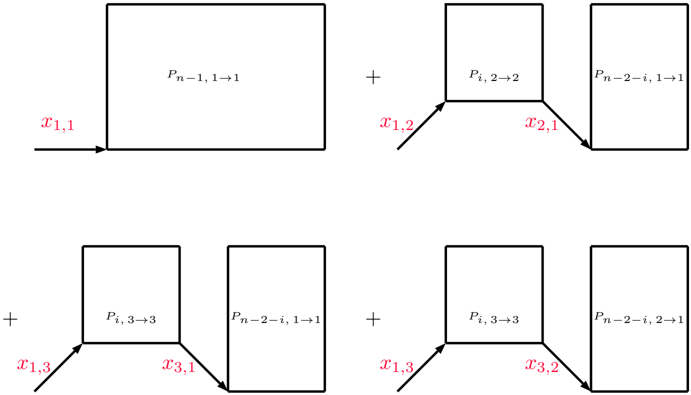
$$
B_{j,j+1} = \Pi^{\left|(x_{1,3})_j,(x_{3,2})_{j+1}\right\rangle} + \Pi^{\left|(x_{2,3})_j,(x_{3,1})_{j+1}\right\rangle}.
$$

$$
H_{bulk,\,disconnected} = \sum_{j=1}^{n-1} \sum_{a,b,c,d=1;b\neq c}^{3} \Pi^{\left|\left(x_{a,b}\right)_j,\left(x_{c,d}\right)_{j+1}\right\rangle}.
$$

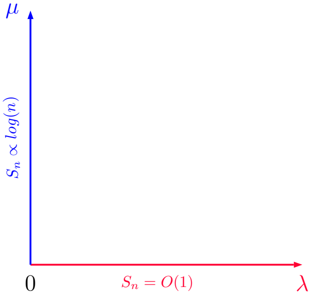
$$
H_{\mathcal{S}_1^3,\,Motzkin} = H_{left} + H_{right} + H_{bulk} + \lambda \sum_{j=1}^{2n-1} B_{j,j+1} + H_{bulk,\,disconnected}.
$$

Ground States

-This system has a ground state degeneracy (GSD) of 5 given by the equivalence classes, $\{11\}$, $\{12\}$, $\{21\}$, $\{22\}$ and $\{33\}$.
- We can use techniques from enumerative combinatorics to compute the normalization of these states.



Quantum Phase Transition



Colored $\mathcal{S}_1^3$ Motzkin Chain

- We introduce a color degree of freedom to each of the basis states, $\left|x_{a,b}^k\right\rangle, \ k \in \{1, 2\}$.

$$H^{balanced} = \mu \sum_{i=1}^{n} C_j + \sum_{j=1}^{n-1} \Big[U_{j,j+1} + D_{j,j+1} + F_{j,j+1}^{balanced} + W_{j,j+1}^{balanced} + R_{j,j+1}^{balanced} + H_{left} + H_{right} \Big]$$

with new equivalence moves

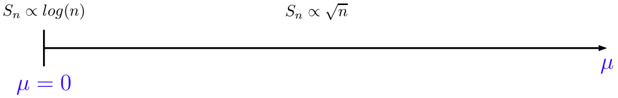
$$C_j = \sum_{a=1}^{3} \Pi^{\frac{1}{\sqrt{2}}\left[\left|(x_{a,a}^1)_j\right\rangle - \left|(x_{a,a}^2)_j\right\rangle\right]},$$

$$R_{j,j+1}^{balanced} = \sum_{a,b,c=1;\, b>a,c}^{3} \left[\Pi^{\left|(x_{a,b}^1)_j,(x_{b,c}^2)_{j+1}\right\rangle} + \Pi^{\left|(x_{a,b}^2)_j,(x_{b,c}^1)_{j+1}\right\rangle} \right].$$

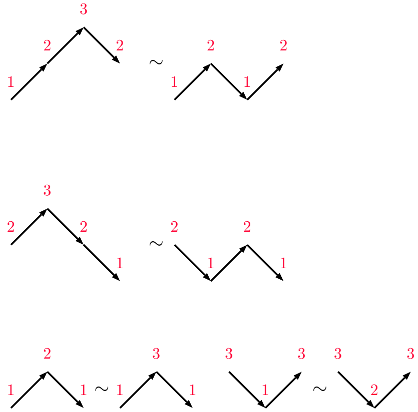
Quantum Phase Transition

$$H_{\mathcal{S}_1^3,\,colored\ Motzkin} = H^{balanced} + H_{bulk,disconnected}.$$

$$S_{A,\,1\to 1} = (2\ln 2)\sqrt{\frac{2\sigma n}{\pi}} + \frac{1}{2}\ln n + \frac{1}{2}\ln(2\pi\sigma) + \gamma - \frac{1}{2} + \ln\frac{3}{2^{1/3}} + (\text{terms vanishing as } n \to \infty)$$



Modified Fredkin Chain (F.Sugino, PP, V.Korepin, 2018)



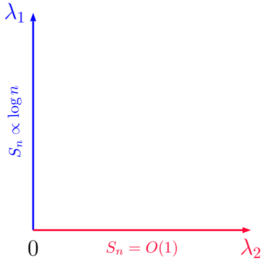
Modified Fredkin Chain Hamiltonian

$$
\begin{aligned}
U_{j,j+1,j+2} &= \Pi \tfrac{1}{\sqrt{2}}\left[\left|(x_{1,2})_j,(x_{2,3})_{j+1},(x_{3,2})_{j+2}\right\rangle - \left|(x_{1,2})_j,(x_{2,1})_{j+1},(x_{1,2})_{j+2}\right\rangle\right] \\
D_{j,j+1,j+2} &= \Pi \tfrac{1}{\sqrt{2}}\left[\left|(x_{2,3})_j,(x_{3,2})_{j+1},(x_{2,1})_{j+2}\right\rangle - \left|(x_{2,1})_j,(x_{1,2})_{j+1},(x_{2,1})_{j+2}\right\rangle\right] \\
W_{j,j+1} &= \Pi \tfrac{1}{\sqrt{2}}\left[\left|(x_{1,2})_j,(x_{2,1})_{j+1}\right\rangle - \left|(x_{1,3})_j,(x_{3,1})_{j+1}\right\rangle\right] \\
&\quad + \lambda_1 \Pi \tfrac{1}{\sqrt{2}}\left[\left|(x_{3,1})_j,(x_{1,3})_{j+1}\right\rangle - \left|(x_{3,2})_j,(x_{2,3})_{j+1}\right\rangle\right],
\end{aligned}
$$

$$
H_F = H_{left} + H_{bulk,\,connected} + H_{right} + \lambda_2 \sum_{j=1}^{n-1} B_{j,j+1} + H_{bulk,\,disconnected}.
$$

Quantum Phase Transition

-The GSD is 4, we no longer have the $\{33\}$ equivalence class.
$\lambda_1 = \lambda_2 = 0$ is a special phase where there is an extensive GSD in each equivalence class.
- When $\lambda_1, \lambda_2 > 0$ the Hamiltonian is no longer frustration free and is not shown in the figure.



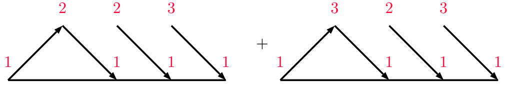
Excitations

- There are three kinds of excitations in these systems, fully connected, partially connected and disconnected excitations.
- The partially connected excitations are localized both in the low energy and high energy sector.

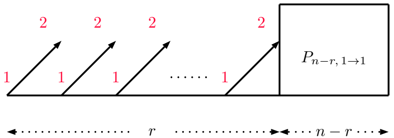
$$|x_{2,3}\rangle_i \langle x_{1,2}| \triangleright |P_{n,\,1\to 1}\rangle = \sum_{h=0}^{h_{max,i}} \left[\left| P^{(0\to h)}_{i-1,\,1\to 1} \right\rangle \otimes |x_{2,3}\rangle_i \otimes \left| P^{(h+1\to 0)}_{n-i,\,2\to 1} \right\rangle \right].$$

Partially Connected Excitations

A low energy example



A high energy example



Localization

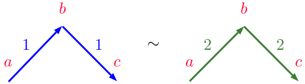
- The partially connected excitations are localized as can be seen by computing connected 2-point correlation functions.

$$\langle pce|\theta_i(t)\theta_j(0)|pce\rangle - \langle pce|\theta_i(t)|pce\rangle\langle pce|\theta_j(0)|pce\rangle = 0,$$

$$\theta_i(0) = |x_{a_1,b_1}\rangle_i\langle x_{a_2,b_2}|,\ a_1 \neq a_2 \text{ and } b_1 \neq b_2,$$

$$\theta_i(0) = \sum_{a,b} k_{a,b}|x_{a,b}\rangle_i\langle x_{a,b}|,\ a, b \in \{1,2,3\}.$$

Modified Colored Fredkin Chain

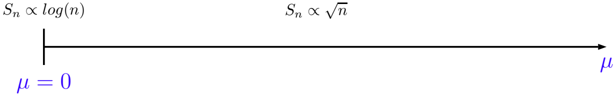


$$H^{balanced} = \sum_{j=1}^{n-1} \left[\mu C_{j,j+1} + W_{j,j+1} + R_{j,j+1}\right] + \sum_{j=1}^{n-2} \left[U_{j,j+1,j+2} + D_{j,j+1,j+2}\right] + H_{left} + H_{right},$$

Quantum Phase Transition

$$
\begin{aligned}
S_{A,\,1\to 1} &= \frac{2\ln 2}{3}\sqrt{\frac{(n+r)(n-r)}{\pi n}} + \frac{1}{2}\ln\frac{(n+r)(n-r)}{n} \\
&\quad + \frac{1}{2}\ln\frac{\pi}{4} + \gamma - \frac{1}{2} - \frac{1}{3}\ln 2 + (\text{terms vanishing as } n\to\infty).
\end{aligned}
$$

- Same as colored Fredkin chain with $s = 2^{\frac{1}{3}}$!



Outlook

- These models can be generalized for an arbitrary number of arrow indices and colors. In general they are higher spin models.
- Extend to higher dimensional random walks.
- Continuum limits.
- EE scaling in local models as n^p with p a fraction other than $\frac{1}{2}$?

I wish Bal a Happy 80th and many more fun filled years in physics !

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Thank you !