Curvature in Hamiltonian Mechanics

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The story begins with Hamilton in 1837

- In the paper **W. R. Hamilton, Trans. Roy. Irish Acad., 17, 1–144 (1837)** he considered infinitesimal pertrubations of the eikonal (Hamilton-Jacobi) equation. This would describe the propagation of nearby rays.
- These ideas have been extended by L. Klimes, Journal of Electromagnetic Waves and Applications, 27,1589(2013).
- Some decades later Riemann discovered the notion of curvature
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Riemannian Geometry is a Special Case of Hamiltonian Mechanics

- Geodesics are the solutions of Hamilton's equations with $H = \frac{1}{2} g^{ij} p_i p_j$
- Is there a generalization of Riemannian geometry to more general hamiltonians?
- We want notions of volume, distance and curvature.
- Boltzmann gives a natural volume measure on phase space: $e^{-H} d^n p d^n q$. It reduces to Riemannian volume measure on configuration space in that special case:

$$d^n q \int e^{-\frac{1}{2} g^{ij} p_i p_j} \frac{d^n p}{(2\pi)^{\frac{n}{2}}} = \sqrt{g} d^n q$$

- The "reduced action" or "eikonal" $\sigma_E(Q, Q') = \int_Q^{Q'} p_i dq^i$ along a trajectory of energy E is a generalization of distance. **But**, in general it depends on E; is not positive; or a symmetric function $\sigma(Q, Q') \neq \sigma(Q', Q)$.
- Curvature is hardest to generalize.

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Curvature Depends on Choice of Lagrangian Submanifold

- There is a no notion of curvature in symplectic geometry: Darboux says there are local co-ordinates with $\omega = dp_i \wedge dq^i$
- Even given a hamiltonian there cannot be a notion of curvature: Birkhoff says that near a stable equilibrium point $H(q,p) = \frac{1}{2}\sum_k \left[p_k + \omega_k^2 q_k^2\right] + \mathrm{O}(N)$ for any $N \geq 3$.
- Given a manifold Γ, a symplectic form ω on it, a hamiltonian H **and** a Lagrangian sybmanifold M , we will find a a notion of curvature.
- Curvature is a local notion. So we can identify $\Gamma = T^*M$ and use canonical co-ordinates.

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Second Variation of Action

- The solutions of Hamilton's equation (orbits) are extrema of the action

$$S = \int p_i \dot{q}^i dt - \int H(q(t), p(t)) dt$$

- An infinitesimal deviation of an orbit $q^i \mapsto q^i + \epsilon \xi^i, p_i \mapsto p_i + \epsilon \pi_i$ satisfies the associated equations

$$\dot{\xi}^i = H^i_j \xi^j + H^{ij} \pi_j, \quad \dot{\pi}_i = -H_{ij} \xi^j - H^j_i \pi_j$$

where $H^{ij} = \frac{\partial H}{\partial p_i \partial p_j}, H^i_j = \frac{\partial H}{\partial p_i \partial q^j}, H_{ij} = \frac{\partial^2 H}{\partial q^i \partial q^j}$.

- These are extrema of the second variation:

$$\mathcal{S} = \int \left[\pi_i \dot{\xi}^i - \mathcal{H} \right] dt, \quad \mathcal{H} = \frac{1}{2} \left[H_{ij} \xi^i \xi^j + 2 H^j_i \xi^i \pi_j + H^{ij} \pi_i \pi_j \right]$$

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Lagrangian Version of the Second Variation

- We will assume from now on that the second derivative of H w.r.t. p has an inverse :

$$H^{ij} G_{jk} = \delta^i_k$$

- This $G_{jk}(q,p)$ depends on both position and momentum! In Riemannian geometry, it reduces to the metric tensor.
- Then we can eliminate π_i to get a "Lagrangian" version of the action for deviations

$$\mathcal{S}_1 = \int \left[\frac{1}{2} G_{ij}\dot{\xi}^i\dot{\xi}^j - \dot{\xi}^i\xi^j G_{ik} H^k_j + \frac{1}{2}\xi^i\xi^j \left\{-H_{ij} + H^k_i H^l_j G_{kl}\right\}\right] dt$$

- Although $H^{ij}$and G_{jk} transform as tensors, H^i_j and H_{ij} do not. For example, if we transform $q^i \to \tilde{q}^i(q)$,$\tilde{p}_i = \frac{\partial q^k}{\partial \tilde{q}^i} p_k$,

$$\tilde{H}^i_j = \frac{\partial q^b}{\partial \tilde{q}^j}\frac{\partial \tilde{q}^i}{\partial q^a} H^a_b + \frac{\partial q^c}{\partial \tilde{q}^j}\frac{\partial^2 \tilde{q}^i}{\partial q^c \partial q^a} H^a + \frac{\partial q^c}{\partial \tilde{q}^j}\ \frac{\partial \tilde{q}^i}{\partial q^a}\ \frac{\partial^2 \tilde{q}^k}{\partial q^c \partial q^b} H^{ab}\tilde{p}_k$$

etc.

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Covariant Time Derivative

- By explicit calculation, we can verify that the following quantity transforms as a tensor:

$$\overset{\circ}{\xi}{}^k = \dot{\xi}^k + \gamma^k_j \xi^j$$

where

$$\gamma^k_j = \frac{1}{2}\left[-H^k_j + H^{km} G_{jl} H^l_m + H^{kl} \dot{G}_{lj}\right]$$

- We do **not** attempt to define a covariant derivative in a general direction, only along the orbit.
- The second variation can be written as (after some integration by parts etc.)

$$\mathcal{S}_1 = \int \left[\frac{1}{2} G_{ij}(q,p) \overset{\circ}{\xi}{}^i \overset{\circ}{\xi}{}^j - \frac{1}{2} \xi^i \xi^j \mathcal{R}_{ij}(q,p)\right] dt$$

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Curvature

- The tensor $\mathcal{R}_{ij}(q,p)$ appearing here is the generalization of the Riemann tensor . By explict calculation, we see that it depends on up to four derivatives of the hamiltonian:

$$\mathcal{R}_{ij} = G_{kl}\gamma_i^k\gamma_j^l - \frac{1}{2}\left\{H, G_{ik}H_j^k + G_{jk}H_i^k\right\}$$

$$-\frac{1}{2}\{H,\{H,G_{ij}\}\} + H_{ij} - H_i^k H_j^l G_{kl}$$

- In Riemannian geometry its dependence on momentum is quardatic and

$$\mathcal{R}_{ij}(q,p) = -g_{im}p^k p^l R^m{}_{klj}(q)$$

- In general there will be lower and higher order terms in the momenta.

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Analogue of Ricci Tensor and the Einstein-Hilbert Action

- The contraction $\mathcal{R}(q,p) = H^{ij}\mathcal{R}_{ij}(q,p)$ is the analogue of the Ricci **tensor**.
- Again, in Riemannian geometry it is a quadratic function of momenta, $\mathcal{R} = p^k p^l R_{kl}$.
- By integrating this quantity over all of phase space we get an action principle for the hamiltonian:

$$L(H) = \int H^{ij}\mathcal{R}_{ij}(q,p)e^{-H}d^nq\frac{d^np}{(2\pi)^n}$$

- In Riemannian geometry, this reduces to the Einstein-Hilbert action

$$L(H) = \int p^k p^l R_{kl}(q)e^{-\frac{1}{2}g^{ij}p_ip_j}d^nq\frac{d^np}{(2\pi)^n} = \int g^{ij}R_{ij}d^nq.$$

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- We do not know yet if it is ghost free. Might be worth studying further.

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Einstein-Maxwell-Dilaton Theory

- We can consider the special case $H = \frac{1}{2} g^{kl} [p_k - A_k] [p_l - A_l] + \phi$
- The above variational principle reduces to

$$L(g, A, \phi) = \int \left[R + \frac{1}{4} F_{ik} F_{jl} g^{kl} g^{ij} + \Delta\phi \right] e^{-\phi} \sqrt{g} d^n q$$

- After the field redefiniton $\tilde{g}_{ij} = e^{2\alpha\phi} g_{ij}, \quad \alpha = -\frac{1}{n-2}$ this becomes recognizable:

$$\int \left[\sqrt{\tilde{g}} \tilde{R} + \frac{1}{4} F_{ik} F_{jl} \tilde{g}^{kl} \tilde{g}^{ij} \sqrt{\tilde{g}} e^{-\frac{2}{n-2}\phi} + \frac{2n-1}{n-2} \sqrt{\tilde{g}} \tilde{g}^{ij} \partial_i \phi \partial_j \phi \right] d^n q$$

- The fieldϕ can now be identified with the dilaton; $\tilde{g}$ is the metric of GR and $F_{kl} = \partial_k A_l - \partial_l A_k$is the Maxwell tensor.

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Other Special Cases

- With flat metric and no electro-magnetic field $H = \frac{1}{2}p_k p_k + \phi(q)$ the curvature is just the Hessian of ϕ.

$$\mathcal{R}_{ij} = \frac{\partial^2\phi}{\partial q^i \partial q^j}$$

- In particular, the simple harmonic oscillar has constant positive curvature.
- If you add a constant magnetic field

$$\mathcal{R}_{ij} = \frac{1}{4}F_{ik}F_{jk} + \partial_i\partial_j\phi$$

- A Penning trap is a subtle case. The potential ϕ is harmonic, so cannot have a stable equilibrium . A strong enough magnetic field oriented along the unstable direction of ϕ can restore stability.
- A sufficient condition or stability is that $\mathcal{R}_{ij}$ above is a positive matrix. It is not necessary, however.

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Thank You Bal!

- What is unique about Bal as a scientist is audacity: he would jump into any problem that interested him, swimming against the current if necessary.
- The week I arrived in Syracuse, Bal placed two stacks of papers on my desk: one on the Skyrme model and another on String Theory.
- Both were totally out of fashion in the early Eighties. But my mid-Eighties, they were all the rage; partly because of Witten's charismatic leadership.
- The timing could not have been better for me. I worked on those two themes for a decade with Bal and others (Schechter, Nair,Michelson, Bowick..).
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