The Shale-Stinespring theorem and topological phases of matter

Balfest 80

Andrés F. Reyes Lega

January 23, 2018

Universidad de los Andes

Table of contents

1. Introduction
2. Shale-Stinespring theorem
3. The Kitaev chain
4. Classical 2D-Ising model

Introduction

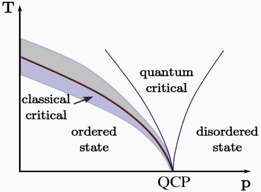
About this talk

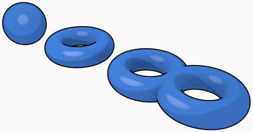
- Partially based on arXiv 1712.05069
- Sebastián Calderón
- Ling Sequera
- Souad Tabban

Shining Bal!

Motivation

Universality, criticality $\overset{?}{\longleftrightarrow}$ Topology





Shale-Stinespring theorem

Shale-Stinespring theorem

The Weyl (or bosonic) algebra

Canonical commutation relations (CCR): $[\hat{q}, \hat{p}] = i\hbar\mathbb{1}$.

Define operators $U(a)$ and $V(b)$, for $a, b \in \mathbb{R}$, acting on wave functions as follows:

$$\begin{aligned} (U(a)\psi)(x) &:= \psi(x - \hbar a), \\ (V(b)\psi)(x) &:= e^{-ibx}\psi(x). \end{aligned} \tag{1}$$

In terms of $\hat{q}$ and $\hat{p}$, we have: $U(a) = e^{-ia\hat{p}}$ and $V(b) = e^{-ib\hat{q}}$.

$U(a)$ and $V(b)$ satisfy the following commutation relations (Weyl form):

$$\begin{aligned} U(a_1)U(a_2) &= U(a_1 + a_2), \\ V(b_1)V(b_2) &= V(b_1 + b_2), \\ U(a)V(b) &= e^{i\hbar ab}V(b)U(a). \end{aligned} \tag{2}$$

For $u = (\alpha, \beta) \in T^*\mathbb{R}^n$, define

$$W(\alpha, \beta) := e^{-i(\alpha\hat{q} + \beta\hat{p})}.$$

These operators satisfy the following identity:

$$W(u)W(v) = e^{-\frac{i}{2}\sigma(u,v)}W(u+v),$$

where u and v denote elements of the *symplectic vector space* $T^*\mathbb{R}^n$, and σ the standard symplectic form.

They are related to U and V through

$$W(\alpha, \beta) = e^{i\frac{\hbar}{2}\alpha\beta}V(\alpha)U(\beta).$$

Definition (Weyl $*$-algebra)

Let V be a real vector space and $\sigma : V \times V \to \mathbb{R}$ a symplectic form. A $*$-algebra $\mathcal{W}(V, \sigma)$ is called a Weyl $*$-algebra of (V, σ) if there is a family $\{W(u)\}_{u \in V}$ of "generators" such that

(*i*) $W(u)W(v) = e^{-\frac{i}{2}\sigma(u,v)}W(u+v)$, $W(u)^* = W(-u)$, $u, v \in V$.

(*ii*) $\mathcal{W}(V, \sigma)$ is *generated* by the family $\{W(u)\}_{u \in V}$, *i.e.*, it is the span of finite linear combinations of finite products of the $W(u)$.

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- Every symplectic vector space (V, σ) determines uniquely a Weyl $*$-algebra, up to $*$-isomorphism.
- $(V_1, \sigma_1) \cong (V_2, \sigma_2) \Rightarrow \mathcal{W}(V_1, \sigma_1) \cong \mathcal{W}(V_2, \sigma_2)$.

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- Every symplectic vector space (V, σ) determines uniquely a Weyl $*$-algebra, up to $*$-isomorphism.
- $(V_1, \sigma_1) \cong (V_2, \sigma_2) \ \Rightarrow \ \mathcal{W}(V_1, \sigma_1) \cong \mathcal{W}(V_2, \sigma_2)$.
- A most important fact is that $\mathcal{W}(V, \sigma)$ can be completed to a C^*-algebra, the Weyl C^*-algebra.

It is a fundamental result, due to Stone and von Neumann, that when the dimension of V is finite (and hence necessarily even), there is essentially only one representation of the CCR, which can be taken to be the standard Schrödinger representation. But in infinite dimensions uniqueness is lost, and so *inequivalent* representations do exist. This fact is closely related to the non-uniqueness of a vacuum state for a free quantum field in a curved spacetime background.

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Theorem (Stone-von Neumann)

If (V, σ) finite dimensional, all irreducible representations of the Weyl algebra are unitarily equivalent.

Shale-Stinespring theorem

The CAR (or fermionic) algebra

The CAR algebra

The CAR algebra

Consider the unital, involutive complex algebra $\mathcal{A}_{\text{CAR}}$ generated by elements $a_1, a_2, \ldots, a_n, \mathbb{1}$, subject to the following *canonical anticommutation relations (CAR)*:

$$a_i a_j^* + a_j^* a_i = \delta_{ij} \mathbb{1}, \quad a_i a_j + a_j a_i = 0. \tag{3}$$

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$\hookrightarrow$ The only possible choice is $\|a_i\| = 1 \quad \forall i \in \{1, \dots, n\}$.

The previous example can be generalized to the Hilbert space context, as follows:

Let $(\mathcal{H}, \langle \ , \ \rangle)$ be a Hilbert space. For each pair $u, v \in \mathcal{H}$, define generators $a(u), a(v)$ subject to the CAR relations:

$$\{a(u), a(v)^*\} = \langle u, v\rangle \mathbb{1}, \quad \{a(u), a(v)\} = 0.$$

Here, as well, there is a unique choice for $\|a(u)\|$.

In this way we obtain a C^*-algebra, that we denote with: $\mathcal{A}_{\mathsf{CAR}}(\mathcal{H}, \langle \cdot, \cdot \rangle)$

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In the infinite-dimensional case, the choice of a polarization/complex structure plays an important role for quantum field theory.

Complex structures

- Consider a real vector space V with $\dim_{\mathbb{R}}(V) = 2n$.
- Let $g(\cdot, \cdot)$ be a positive, symmetric bilinear form on V and J an *orthogonal complex structure*.
- Use J to construct a complexification of V, call it V_J ($\neq V^{\mathbb{C}}$).

If we define an inner product in V_J by

$$\langle u, v \rangle_J \coloneqq g(u, v) + i g(Ju, v), \tag{5}$$

we obtain a complex Hilbert space $(V_J, \langle \cdot, \cdot \rangle_J)$ with complex dimension n.

- The (complex) Clifford algebra $\mathbb{C}\ell(V)$ acts naturally on the exterior algebra $\bigwedge^\bullet V^{\mathbb{C}}$, but the resulting representation is not irreducible.
- But *it is irreducible* on $\mathcal{F}_J(V) := \bigwedge^\bullet V_J$.
- As Clifford and CAR algebras are closely related, we also obtain an irreducible representation of $\mathcal{A}_{\mathsf{CAR}}(V_J, \langle\cdot,\cdot\rangle_J)$.

In this representation, creation and annihilation operators $a_J(v)$ and $a_J^\dagger(v)$ acting on $\mathcal{F}_J(V)$ are given by:

$$
\begin{aligned}
a_J^\dagger(v)(u_1 \wedge \cdots \wedge u_k) &= v \wedge u_1 \wedge \cdots \wedge u_k, \\
a_J(v)(u_1 \wedge \cdots \wedge u_k) &= \sum_{j=1}^{k} (-1)^{j-1} \langle v, u_j \rangle_J \, u_1 \wedge \cdots \wedge \hat{u}_j \wedge \cdots \wedge u_k,
\end{aligned}
\tag{6}
$$

for $v \in V$ and $u_1, \ldots, u_k \in V_J$.

Defining

$$\pi_J(v) := a_J^\dagger(v) + a_J(v), \tag{7}$$

we obtain a representation of the (real) Clifford algebra $C\ell(V)$ on $\mathcal{F}_J(V)$.

The vacuum in $\mathcal{F}_J(V)$ can also be characterized as a gaussian state ω_J with a two-point function given by

$$\langle 0_J | a_J(u) a_J^\dagger(v) | 0_J \rangle \equiv \omega_J(a_J(u) a_J^\dagger(v)) = \langle u, v \rangle_J. \tag{8}$$

In fact, this representation can be obtained from ω_J (regarded as an algebraic state) through the GNS construction.

Representations

A most important fact is the possibility (when $\mathrm{dim} V = \infty$) of having inequivalent representations. A very useful characterization of the vacuum state $|0_J\rangle$ in the J-induced representation is obtained if we extend all operators from V to $V^{\mathbb{C}}$, as explained below.

The Clifford generators $\pi_J(v)$, as well as the creation/annihilation operators $a_J(v)^\dagger$, $a_J(v)$ can be regarded as real linear maps from V to $\mathcal{L}(\mathcal{F}_J(V))$. These can be extended to complex linear maps

$$\tilde{\pi}_J,\ \tilde{a}_J,\ \tilde{a}_J^\dagger : V^{\mathbb{C}} \longrightarrow \mathcal{L}(\mathcal{F}_J(V)), \tag{9}$$

Since the complex structure on $\mathcal{F}_J(V)$ is determined by J, we also have (for v in V):

$$a_J^\dagger(Jv) = ia_J^\dagger(v),\ \ a_J(Jv) = -ia_J(v). \tag{10}$$

The minus sign can be traced back to equations (5) and (6) above.

Summarizing, we have the following important identities ($v \in V$):

$$\tilde{a}_J^\dagger(iv) = ia_J^\dagger(v) \equiv Ja_J^\dagger(v), \qquad \tilde{a}_J(iv) = ia_J(v) \equiv Ja_J(v), \tag{11}$$

$$a_J^\dagger(Jv) = ia_J^\dagger(v) \equiv Ja_J^\dagger(v), \qquad a_J(Jv) = -ia_J(v) \equiv -Ja_J(v). \tag{12}$$

Consider the linear extension of J to $V^{\mathbb{C}}$ and define

$$P_{\pm J} := \frac{1}{2}(1 \mp iJ), \quad \text{as well as} \quad W_{\pm J} := P_{\pm J}(V^{\mathbb{C}}). \tag{13}$$

Then, using $\langle\langle w, z\rangle\rangle := 2g_{\mathbb{C}}(\overline{w}, z)$ as the inner product for $V^{\mathbb{C}}$, we obtain $W_{-J} = W_J^{\perp}$, so that

$$V^{\mathbb{C}} = W_J \oplus W_J^{\perp}. \tag{14}$$

Furthermore, restricting $\langle\langle \cdot, \cdot \rangle\rangle$ to W_J, we obtain:

$$(V_J, \langle \cdot, \cdot \rangle_J) \cong (W_J, \langle\langle \cdot, \cdot \rangle\rangle). \tag{15}$$

Vacuum condition

It can be shown that the condition

$$\tilde{\pi}_J(u)|0_J\rangle = 0 \iff u \in W_J^{\perp}, \tag{16}$$

provides a full characterization of the vacuum $|0_J\rangle$.

The Shale-Stinespring theorem

Theorem (Shale-Stinespring)

Let J, K be two orthogonal complex structures. Then π_J and π_K are unitarily equivalent iff $J - K$ is Hilbert-Schmidt.

- Any $h \in O(V, g)$ can be decomposed into linear and antilinear parts:

$$h = p_h + q_h, \qquad p_h := \tfrac{1}{2}(h - JhJ), \quad q_h := \tfrac{1}{2}(h + JhJ) \tag{17}$$

- $O_J(V) = \{h \in O(V) \mid [J, h] \text{ is Hilbert-Schmidt}\}$
- Equivalence problem = implementability problem = cyclic vector s.t.

$$\big(a_J(p_h v) + a_J^\dagger(q_h v)\big)\Phi = 0. \tag{18}$$

$$\Phi = u_1 \wedge \cdots \wedge u_n \wedge f_G, \tag{19}$$

with f_G a *gaussian* and $n = \dim \ker p_h < \infty$.

$(V, g, J), \qquad (V_J, \langle\ ,\ \rangle_J)$

Irreducible representation of $\mathbb{C}l(V)$ on $\mathcal{F}_J = \overline{\bigwedge^\bullet V_J}$

Clifford generators $\pi_J(v) = a^\dagger(v) + a(v)$

Theorem (Shale-Stinespring)

Let $h \in O(V, g)$ and put $K \equiv hJh^{-1}$. Then, the following statements are equivalent:

(i) *The Bogoliubov automorphism θ_h is unitarily implementable.*

(ii) *The representation π_J and π_K are unitarily equivalent.*

(iii) *$K - J$ is a Hilbert-Schmidt operator.*

The $\mathbb{Z}_2$-index

Let $h \in O_J(V)$ and set $J_h \coloneqq hJh^{-1}$. Then, we get an index map

$$\begin{aligned} \mathrm{index} : \mathcal{J} &\longrightarrow \mathbb{Z}_2 \\ J_h &\longmapsto (-1)^{\frac{1}{2}\dim\ker(J+J_h)}. \end{aligned} \tag{20}$$

We will see below that this is precisely the topological $\mathbb{Z}_2$-index (Pfaffian invariant) used in condensed matter physics.

The Kitaev chain

Quadratic Hamiltonians

$$H = \sum_{i,j=1}^{N} \left[a_i^\dagger A_{ij} a_j + \frac{1}{2} \left(a_i^\dagger B_{ij} a_j^\dagger - a_i \overline{B}_{ij} a_j \right) \right], \tag{21}$$

$$H = \frac{1}{2} (a^\dagger, a) \begin{pmatrix} A & B \\ -\overline{B} & -\overline{A} \end{pmatrix} \begin{pmatrix} a \\ a^\dagger \end{pmatrix} + \text{constant}. \tag{22}$$

Bogoliubov transformation

Introduce new operators

$$c_k = \sum_{i=1}^{N} \left(g_{ki} a_i + h_{ki} a_i^\dagger \right), \qquad c_k^\dagger = \sum_{i=1}^{N} \left(\bar{g}_{ki} a_i^\dagger + \bar{h}_{ki} a_i \right), \tag{23}$$

where g and h are $N \times N$ matrices to be chosen so that

(i) The new operators satisfy the same CAR algebra:

$$\{c_k, c_l^\dagger\} = \delta_{kl}, \;\; \{c_k, c_l\} = 0 = \{c_k^\dagger, c_l^\dagger\}. \tag{24}$$

(ii) The Hamiltonian becomes *diagonal* in the new basis:

$$H = \sum_k \Lambda_k c_k^\dagger c_k + \text{constant.} \tag{25}$$

the requirement (24) leads to the following conditions:

$$\begin{aligned} gg^\dagger + hh^\dagger &= \mathbb{1}_N, \\ gh^t + hg^t &= 0. \end{aligned} \tag{26}$$

Consistency between the two expressions for H implies:

$$g_{ki}\,\Lambda_k = \sum_{j=1}^{N} \left(g_{kj}A_{ji} - h_{kj}B_{ji}\right), \quad h_{ki}\,\Lambda_k = \sum_{j=1}^{N} \left(g_{kj}B_{ji} - h_{kj}A_{ji}\right). \tag{27}$$

In order to solve this eigenvalue problem, it proves convenient to introduce new matrices Φ and Ψ, as follows:

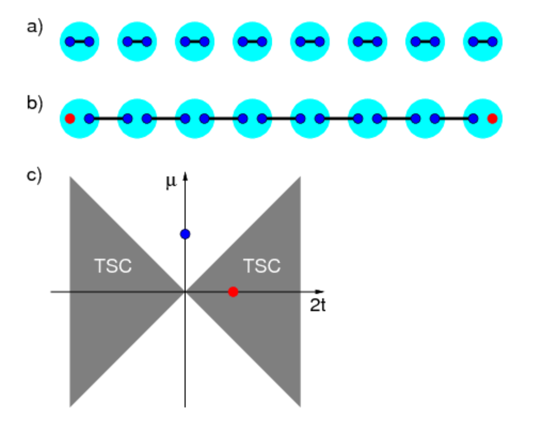
$$\Phi := g + h, \qquad \Psi := g - h.$$

If we now define for each k a vector $|\Phi_k\rangle$, the i^{th} component of which is given by Φ_{ki}, and similarly for Ψ, we find that (27) can be written as follows:

$$(A - B)|\Psi_k\rangle = \Lambda_k|\Phi_k\rangle, \quad (A + B)|\Phi_k\rangle = \Lambda_k|\Psi_k\rangle,$$

Kitaev chain

$$H = \sum_{i=l}^{N} t(a_i^\dagger a_{i+1} + a_{i+1}^\dagger a_i) + \Delta(a_i^\dagger a_{i+1}^\dagger - a_i a_{i+1}) - 2\mu a_i^\dagger a_i. \qquad (28)$$



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  $O(2) \ni h = \begin{pmatrix} \cos\alpha & \sigma\sin\alpha \\ -\sin\alpha & \sigma\cos\alpha \end{pmatrix},\ \sigma = \pm 1.$
  $hJh_\sigma^{-1} = hJh^t = \sigma J, \quad h \in U(V_J) \Leftrightarrow [J,h] = 0 \Leftrightarrow \sigma = 1.$
- $h = \frac{1}{2}(h - JhJ) + \frac{1}{2}(h + JhJ) \equiv p_h + q_h.$

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$\mathbb{Z}_2$-index

$$i(h) := \mathrm{dim}_{\mathbb{C}}(\ker p_h) = \begin{cases} 0, & \sigma = 1, \\ 1, & \sigma = -1. \end{cases}$$

$u, v \in V: \quad \langle u, v\rangle_J := g(u,v) + ig(Ju, v), \quad \{a(u), a^\dagger(v)\} = \langle u, v\rangle_J$

Initial vacuum:

$a(v)|0\rangle = 0, v \in V$.

Bogoliubov transformation:

$c(v) = a(p_h v) + a^\dagger(q_h v)$.

Solve $c(v)|\Omega\rangle = 0$ (for all $v \in V$)...

What is $|\Omega\rangle$?

$$\begin{aligned} \sigma = 1 \quad &\Leftrightarrow |\Omega\rangle = |0\rangle \quad &\Leftrightarrow \quad i(h) = 0 \\ \sigma = -1 \quad &\Leftrightarrow |\Omega\rangle = a_1^\dagger|0\rangle &\Leftrightarrow \quad i(h) = 1 \end{aligned}$$

Parity of transformed vacuum gives the $\mathbb{Z}_2$-index

2-site Kitaev chain

Let $V = \mathbb{R}^4$ with $g_{\mathsf{E}}(\cdot,\cdot)$ the standard Euclidean metric. For $e_1, \ldots, e_4$ the standard basis vectors, introduce the following complex structure:

$$J = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{29}$$

Notice that we have $e_3 = Je_1$ and $e_4 = Je_2$. Consider now a two-site Kitaev chain (OBC):

$$H = t(a_1^\dagger a_2 + a_2^\dagger a_1) + \Delta(a_1^\dagger a_2^\dagger - a_1 a_2) - 2\mu(a_1^\dagger a_1 + a_2^\dagger a_2). \tag{30}$$

Introduce now the following parameters:

$$\alpha = \sqrt{\Delta^2 + 4\mu^2}, \tag{31}$$

$$\beta_\pm = \sqrt{(\alpha \pm \Delta)/(2\alpha)}, \tag{32}$$

$$\sigma = \mathrm{sgn}(\alpha - t). \tag{33}$$

The Bogoliubov transformation that diagonalizes H is induced by the orthogonal transformation

$$h = \begin{pmatrix} \Phi & 0 \\ 0 & \Psi \end{pmatrix}, \quad \Phi = \begin{pmatrix} \beta_+ & \beta_- \\ -\beta_- & \beta_+ \end{pmatrix}, \quad \Psi = \begin{pmatrix} \sigma\beta_- & \sigma\beta_+ \\ -\beta_+ & \beta_- \end{pmatrix}. \tag{34}$$

For the real maps $p_h, q_h : V \to V$, expressed in block form, we find:

$$p_h = \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}, \quad q_h = \begin{pmatrix} f & 0 \\ 0 & -f \end{pmatrix}, \tag{35}$$

where $g = (1/2)(\Phi + \Psi)$ and $f = (1/2)(\Phi - \Psi)$.

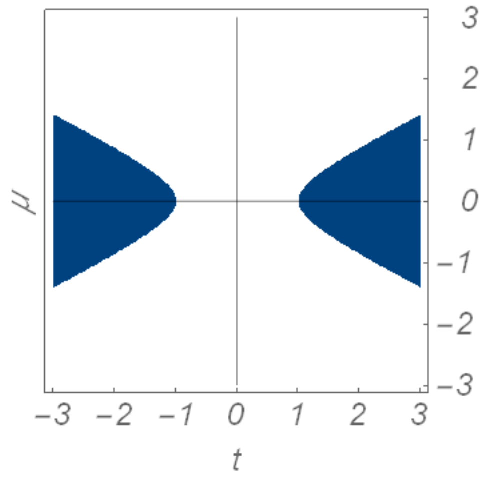
For the orthogonal complex structure we obtain:

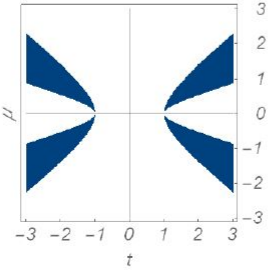
$$J_h = hJh^{\mathsf{T}} = \frac{1}{\sqrt{\Delta^2 + 4\mu^2}} \begin{pmatrix} 0 & 0 & -2\sigma\mu & \Delta \\ 0 & 0 & -\sigma\Delta & -2\mu \\ 2\sigma\mu & \sigma\Delta & 0 & 0 \\ -\Delta & 2\mu & 0 & 0 \end{pmatrix}. \tag{36}$$

The $\mathbb{Z}_2$-index is given by:

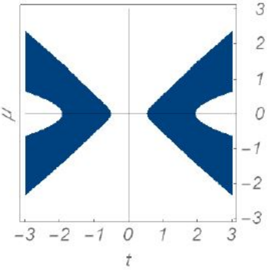
$$\mathrm{index}(h) := (-1)^{\frac{1}{2}\dim\ker(J + J_h)} = \det h = \sigma. \tag{37}$$

$\mathbf{t^2 = \Delta^2 + 4\mu^2}$

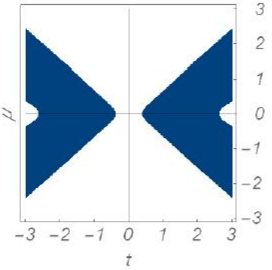




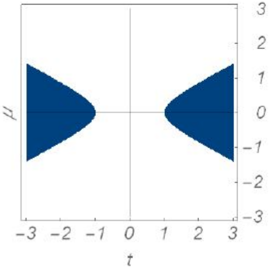
(a) $N = 4, r = 0$



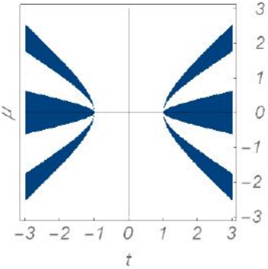
(b) $N = 4, r = 0.1$



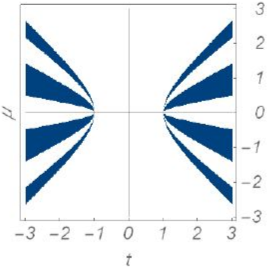
(c) $N = 4, r = 0.2$



(d) $N = 2, r = 0$



(e) $N = 6, r = 0$



(f) $N = 8, r = 0$

Classical 2D-Ising model

The Quantum-Classical mapping

Thermal expectation values of the classical model in dimension $d+1$ correspond to vacuum expectation values of a quantum system in dimension d (transfer matrix formalism):

$$\langle f \rangle_\beta := tr(\hat{\rho}\hat{O}_{f,\beta}), \tag{38}$$

f is a classical observable, $\beta = (k_B T)^{-1}$, $\hat{\rho}$ a density matrix and $\hat{O}_{f,\beta}$ a (quantum) observable associated to f.

For the 1D-classical Ising model, we have:

$$\langle f \rangle_\beta := Z_\beta^{-1} \sum_{\{s\}} f(s) e^{-\beta H_\Lambda(s)} = tr(\hat{\rho}\hat{f}) \tag{39}$$

where $Z_\beta = \sum_{\{s\}} e^{-\beta H_\Lambda(s)}$, $H_\Lambda(s) = -J \sum_{\langle i,j \rangle_\Lambda} s_i s_j$, $J > 0$, and $\hat{\rho} = Z^{-1} e^{-\beta \hat{H}_q}$. The quantum Hamiltonian obtained by this operation is:

$$H_q = -\tfrac{1}{2\xi}\sigma_x. \tag{40}$$

Algebraic approach

For d=2, the mapping gives rise to the 1-dimensional quantum Ising chain:

$$H_\lambda = -\sum_j \sigma_x^j \sigma_x^{j+1} - \lambda \sum_j \sigma_z^j \tag{41}$$

- Using the Jordan-Wigner transformation, we can describe this system by means of a quadratic fermionic Hamiltonian.
- Araki and Matsui have shown (1980s) that the classical phase transition can be characterized in terms of equivalence classes of representations.
- Going through their (very technical) proof, one recognizes that the same index discussed today is present in this case (Tabban, Sequera, AFRL).
- Extension to the quantum-critical region (work in progress).

THANKS, BAL!

$\mathbb{Z}_2$-action

- An automorphism of A is an invertible linear map $\theta : A \to A$ satisfying
$$\theta(ab) = \theta(a)\theta(b), \qquad \theta(a^*) = \theta(a)^* \tag{42}$$
- A $\mathbb{Z}_2$-action on A is an automorphism $\theta : A \to A$ with $\theta^2 = \mathbb{I}$. An algebra A carrying a $\mathbb{Z}_2$-action decomposes as:
$$A = A_+ + A_-, \qquad A_\pm = \{a \in A \mid \theta(a) = \pm a\} \tag{43}$$
Example: Let $u = H \to H$ be a unitary operator $u^2 = 1$, $A = B(H)$, then
$$\theta(a) = uau^* \tag{44}$$
defines a $\mathbb{Z}_2$ action on A, so $A_\pm = \{a \in A \mid au \mp ua = 0\}$

Pauli and Fermionic algebras

- $I_L = [-L, L]$, $L \in \mathbb{Z}_+$
- Pauli Algebra $A_L^P \simeq \bigotimes_{I_L} M_2$ generated by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_0 \equiv \mathbb{I} \tag{45}$$

- Fermionic Algebra $A_L^F \simeq A^{CAR}\big(l^2(I_L)\big)$ generated by

$$\{a_i, a_j^*\} = \delta_{ij}, \quad \{a_i, a_j\} = 0 = \{a_i^* a_j^*\} \tag{46}$$

- Let θ be a $\mathbb{Z}_2$-action such that for $i \in I_L$

$$\begin{aligned} \theta(\sigma_x^i) = -\sigma_x^i, \quad \theta(\sigma_y^i) = -\sigma_y^i, \quad \theta(\sigma_z^i) = \sigma_z^i, \qquad & A^P = A_+^P + A_-^P \\ \theta(a_i) = -a_i, \qquad & A^F = A_+^F + A_-^F \end{aligned} \tag{47}$$

Jordan-Wigner transformation

The JWT is an isomorphism $\alpha_L : A_L^P \to A_L^F$ ($L < \infty$)

$$\sigma_x^j = TS_j(a_j + a_j^*), \quad \sigma_y^j = iTS_j(a_j - a_j^*), \quad \sigma_z^j = 1 - 2a_j^* a_j \tag{48}$$

where $T = \prod_{k=-L}^{0} \sigma_z^k$ and $TS_j = \prod_{k=-L}^{j-1} \sigma_z^k$. Since the tail T depends on L, the diagram is not commutative

$$\begin{array}{ccc} A_L^P & \xrightarrow[\alpha_L]{} & A_L^F \\ \bigcap & & \bigcap \\ A_{L+1}^P & \xrightarrow[\alpha_{L+1}]{} & A_{L+1}^F \end{array} \tag{49}$$

A_{L+}^P is generated by σ_z^i and $\sigma_x^j \sigma_x^{j+1}$, since $T^2 = 1$, the restriction of α_L to the even subalgebra is not L-dependent, so $\lim_{L\to\infty} \alpha_L|_{A_{L+}^P}$ gives an isomorphism of A_+^P with A_+^F.

Local observables and ground state

- Hilbert space by site $H = \mathbb{C}^2$
- $H(\Lambda) = \bigotimes_{x \in \Lambda} H_x$, (dim$H(\Lambda) = \big(\text{dim}H\big)^{|\Lambda|}$)
- Local observables $A(\Lambda) = B(H(\Lambda))$, $\Lambda \subset \Lambda' \Rightarrow A(\Lambda) \hookrightarrow A(\Lambda')$
- Heisenberg equation:

$$\frac{da(t)}{dt} = i[H_\Lambda, a(t)] \tag{50}$$

Setting $t = 0$, this defines a derivation

$$\begin{array}{rcccl} \delta_\Lambda & : & A(\Lambda) & \rightarrow & A(\Lambda) \\ & & a & \mapsto & \delta_\Lambda(a) = i[H_\Lambda, a] \end{array} \tag{51}$$

- For each $a \in A(\Lambda)$, $\delta(a) = i \lim_{\Lambda \uparrow \mathbb{Z}^d} [H_\Lambda, a]$ exists.
- A ground state is a state $\omega_0 : A \to \Lambda$ such that

$$-i\omega_0(a^* \delta(a)) \geq 0, \qquad \forall a \in A \tag{52}$$

- An even state on A^P or A^F is one which is invariant under θ.

- Let $\omega : A \to \mathbb{C}$ be a state on a C^*-algebra A. There exists a cyclic representation π_ω of A on a Hilbert space H_ω with cyclic unit vector Ω_ω such that

$$\omega(a) = \langle \Omega_\omega, \pi_\omega(a)\Omega_\omega \rangle, \quad \forall a \in A \tag{53}$$

- The GNS representation $\pi_\omega(A)$ is irreducible iff ω is pure.
- Let $(H_\omega, \pi_\omega, \Omega_\omega)$ be the GNS triple of $\omega : A \to \mathbb{C}$, due to θ-invariance of ω

$$H_\omega = H_+ \oplus H_-, \quad H_\pm = \overline{\pi_\omega(A_\pm)\Omega_\omega} \tag{54}$$

- Suppose A carries a $\mathbb{Z}_2$-action θ and consider a state $\omega : A \to \mathbb{C}$ that is $\mathbb{Z}_2$-invariant in the sense that $\omega\big(\theta(a)\big) = \omega(a)$ for all $a \in A$. We write this as $\theta^*\omega = \omega$, with $\theta^*\omega := \omega \circ \theta$. Then there is a unitary operator $u : H_\omega \to H_\omega$ satisfacing $u^2 = 1$, $u\Omega = \Omega$ and $u\pi_\omega(a)u^* = \pi_\omega\big(\theta(a)\big)$ for each $a \in A$.
- A symmetry $\theta : A \to A$ is implementable in an Hilbert space $\mathcal{H}$ iff there is an unitary operator U such that:

$$U\pi(a)U^* = \pi(\theta a), \forall a \in A \tag{55}$$

Consider the fermionic algebra $A = Span\{a^*, a, a^*a, \mathbb{I}\} \cong M_2(\mathbb{C})$, in terms of Pauli matrices $\sigma_\pm = \sigma_x \pm i\sigma_y$, $a = \sigma_-$, $a^* = \sigma_+$. The $\mathbb{Z}_2$ action can be implemented by the unitary σ_z, so $A_+ = Span\{a^*a, \mathbb{I}\}$ and $A_- = Span\{a^*, a\}$, then:

$$A_+ = \left\{ \begin{pmatrix} z_+ & 0 \\ 0 & z_- \end{pmatrix}, \ z_\pm \in \mathbb{C} \right\}; \quad A_- = \left\{ \begin{pmatrix} 0 & z_1 \\ z_2 & 0 \end{pmatrix}, \ z_1, z_2 \in \mathbb{C} \right\} \tag{56}$$

- $\Omega = (1,0)$, $\omega(a) := \langle \Omega, a\Omega \rangle$, $\sigma_z \Omega = \Omega$.
- $\pi_\omega(A)$ is the defining representations of $M_2(\mathbb{C})$ on $H_\omega = \mathbb{C}^2$, $\Omega_\omega = \Omega$.
- $H_+ = \{(z,0), \ z \in \mathbb{C}\}$ and $H_- = \{(0,z), \ z \in \mathbb{C}\}$.
- Let $\pi_\pm$ be the restriction of $\pi_\omega(A_+)$ to $H_\pm$

$$\pi_\pm \begin{pmatrix} z_+ & 0 \\ 0 & z_- \end{pmatrix} = z_\pm \tag{57}$$

Theorem
Suppose A carries a $\mathbb{Z}_2$-action θ as well as a $\mathbb{Z}_2$-invariant state $\omega : A \to \mathbb{C}$. suppose the representation $\pi_+(A_+)$ on H_+ is irreducible. Then also the representation $\pi_-(A_+)$ on H_- is irreducible, and there are the following two possibilities for the representation $\pi_\omega(A)$ on $H = H_+ \oplus H_-$

- *$\pi_\omega(A)$ is irreducible (and hence ω is pure) iff $\pi_\pm(A_+)$ are inequivalent;*
- *$\pi_\omega(A)$ is reducible (and hence ω is mixed) iff $\pi_\pm(A_+)$ are equivalent.*

Self-dual formalism

Diagonalization of quadratic Hamiltonians:

- $K = H \oplus H$
- We have two conjugations: $S : H \to H,\ S^* = S,\ S^2 = 1$ and $\Gamma : K \to K$
- $B(h) = a^*(f) + a(Sg)$
- $B^*(h) = B(h)^* = B(\Gamma h)$

Theorem

There is a bijective correspondence between basis projections $P : K \to K$ ($\Gamma P \Gamma = 1 - P$) and states ω_P on A^F that satisfy

$$\omega_P(B(h)^* B(h)) = \langle h \mid Ph \rangle, \quad \forall h \in K \tag{58}$$

Such a state (quasi-free) ω_P is pure (so that the corresponding GNS representation π_P is irreducible).

$$H_\lambda = -\sum_j \sigma_x^j \sigma_x^{j+1} - \lambda \sum_j \sigma_z^j, \tag{59}$$

- For each $|\lambda| \neq 1$ we have $\pi_{\omega_0^F}(A^F) \cong \pi_{\theta_-^* \omega_0^F}(A^F)$ which implies, the ground state ω_0^P is pure on A^P.
- Let $W_- : K \to K$ be the $\mathbb{Z}_2$-action on K defining the $\mathbb{Z}_2$-action θ_- on A^F and let E_+ be the projection onto the positive energy space for H_{SD} in K, then

$$\begin{aligned} \pi_{\omega_0^F} &= \pi_{E_+} \\ \pi_{\theta_-^* \omega_0^F} &= \pi_{W_- E_+ W_-} \end{aligned} \tag{60}$$

Theorem (Araki-Matsui)

The unique $\mathbb{Z}_2$-invariant ground state ω_0 of the Hamiltonian of the Ising model is pure (and hence forms the unique ground state) iff both of the following hold

1. $E_+ - W_- E_+ W_- \in B_2(K)$;
2. *dim*$(E_+ K \cap (1 - W_- E_+ W_-)K)$ *is even*

- $\mathbb{Z}_2$-index between two basis projections E_1, E_2

$$\sigma(E_1, E_2) = (-1)^{\mathrm{dim} E_1 \cap (1 - E_2)} \tag{61}$$

- For the Ising model

$$\sigma\big(E_+, (1 - W_- E_+ W_-)\big) = \begin{cases} +1, & |\lambda| \geq 1 \quad \omega_0 \text{ is pure} \\ -1, & |\lambda| < 1 \quad \omega_0 = \frac{1}{2}(\omega_0^+ + \omega_0^-) \end{cases} \tag{62}$$

where $\omega_0^\pm$ are pure and transform under the $\mathbb{Z}_2$-action θ as $\omega_0^\pm \circ \theta = \omega_0^\mp$