The Low-Energy Spectrum of Quantum Yang-Mills from Gauge Matrix Model

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Introduction

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1. Pure Yang-Mills Theory
2. A New Matrix Model for Yang-Mills
3. Quantization and Spectrum of YM Matrix Model
4. Variation Estimate of Energies
5. Comparison with Lattice Data

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Review of YM Theory

- What are the physical states of QCD?
- Wide implications: confinement, chiral symmetry breaking, color superconductivity, ...
- Recall that the $SU(N)$ Yang-Mills action is

$$S = -\frac{1}{2g^2}\int d^4x \,\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

$$A_\mu = A_\mu^a T^a, \quad \mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}, \quad a, b = 1, \cdots N^2 - 1.$$

- The gauge symmetry

$$u \cdot A_\mu \mapsto uA_\mu u^{-1} + u\partial_\mu u^{-1}, \quad u(x) \in SU(N)$$

is actually a redundancy, and needs to be fixed.

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Groups(s) of Gauge Transformations

- Our interest is in YM theory on $S^3 \times \mathbb{R}$ (secretly $\mathbb{R}^3 \times \mathbb{R}$).
- Temporal gauge $A_0 = 0$. The configuration space is based on $A_i(x) = A_i^a(x) T^a$.
- Group of all gauge transformations:

$$\mathcal{G} \equiv \{u : \mathbb{R}^3 \to SU(N) \mid u(\vec{r}) \to u_\infty \in SU(N) \text{ as } |\vec{r}| \to \infty\}$$

- Group of asymptotically trivial gauge transformations:

$$\mathcal{G}^\infty \equiv \{u : \mathbb{R}^3 \to SU(N) \mid u(\vec{r}) \to 1 \text{ as } |\vec{r}| \to \infty\}$$

- Group $\mathcal{G}_0^\infty$ of asymptotically and topologically trivial gauge transformations: this is the connected component of $\mathcal{G}^\infty$.

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Gauge Transformations

- Both $\mathcal{G}^\infty$ and $\mathcal{G}_0^\infty$ are normal subgroups of $\mathcal{G}$.
- Gauss law ($\partial_i E_i + [A_i, E_i] = D_i E_i \approx 0$) generates $\mathcal{G}_0^\infty$.
- In fact $\mathcal{G}^\infty / \mathcal{G}_0^\infty \cong \pi_3(SU(N)) = \mathbb{Z}$.
- Representations of this $\mathbb{Z} \ni n \to e^{in\theta}$ give the QCD θ-states.
- The color group is $\mathcal{G}/\mathcal{G}^\infty = SU(N)$.
- The configuration space $\mathcal{C}$ for local observables is $\mathcal{A}/\mathcal{G}$.
- This bundle is twisted.

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Case of $SU(2)$

- The key idea: Narasimhan-Ramadas on $SU(2)$ YM theory on $S^3 \times \mathbb{R}$.
- Their aim: prove rigorously that $\mathcal{G}_0^\infty \to \mathcal{A} \to \mathcal{A}/\mathcal{G}_0^\infty$ is twisted.
- They consider a special subset of left-invariant connections

$$\omega = i(\mathrm{Tr}\, \tau_i u^{-1} du) M_{ij} \tau_j, \quad u \in SU(2), M \in M_3(\mathbb{R}) \equiv \mathcal{M}_0.$$

- This connection is pulled back to spatial S^3 using $S^3 \to SU(2)$.

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- All such ω's are preserved under global $SU(2)$ adjoint action $\omega \to v\omega v^{-1}$, or, equivalently, $M \to MR^T$. (R is in image of v in $SO(3)$.
- The action of $SO(3)$ on $\mathcal{M}_0$ is free for all matrices with rank 2 or 3.
- This gives a fibre bundle $SO(3) \to \mathcal{M}_0 \to \mathcal{M}_0/SO(3)$.
- Narasimhan-Ramadas show that this bundle is twisted, and hence the full gauge bundle is also twisted.
- The matrix model for $SU(2)$ comes from this matrix M_{ia}.

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- Start with the left-invariant one-form on $SU(3)$:

$$\Omega = \mathrm{Tr}\left(\frac{\lambda}{2}u^{-1}du\right) M_{ab}\lambda_b, \quad u \in SU(3).$$

- Here M is a 8×8 real matrix.
- Map the spatial S^3 diffeomorphically to $SU(2) \subset SU(3)$.
- $X_i \equiv$ vector fields for right action on $SU(3)$ representing λ_i $(i = 1, 2, 3)$, then $[X_i, X_j] = i\epsilon_{ijk}X_k$.
- $\Omega(X_i) = -M_{ib}\frac{\lambda_b}{2}$.
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$SU(3)$ Yang-Mills

- The M's parametrize a submanifold of connections $\mathcal{A}$.
- They have no spatial dependence: we have completely gauge-fixed the "small" gauge transformations.
- Only the global transformations are left – the ones responsible for the Gribov problem.
- Global color $SU(3)$ acts on the vector potential:

$$A_j \to hA_jh^{-1}, \quad \text{or} \quad M \to M(Ad\,h)^T, \quad h \in SU(3)$$

- For $SU(3)$, the M's are 3×8 matrices.
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Configuration space of $SU(3)$ YM Matrix Model

- The configuration space $\mathcal{C}$ for pure $SU(3)$ is $M_{3,8}(\mathbb{R})/Ad\,SU(3)$.
- This space has dimension $3.8 - 8 = 16$ (not so at fixed points).
- Wavefunctions are sections of vector bundles on $\mathcal{C}$ that transform according to representations of $Ad\,SU(N)$.
- Those transforming according to the trivial representation are colorless, which those transforming nontrivially are coloured.

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Quantization of the Matrix Model

- Recall that the YM Hamiltonian is

$$H = \frac{1}{2}\int d^3x \,\mathrm{Tr}\left(g^2 E_i E_i + \frac{1}{g^2} F_{ij}^2\right).$$

- For the matrix model, M_{ia} are the dynamical variables, and the (Legendre transform of) $\frac{dM_{ia}}{dt}$ as the conjugate of M_{ia}.
- We identify this conjugate operator as the matrix model chormoelectric field E_{ia}.
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$$H = \frac{1}{R}\left(\frac{g^2 E_{ia}E_{ia}}{2} + V(M)\right) = \frac{1}{R}\left(-\frac{g^2}{2}\sum_{i,a}\frac{\partial^2}{\partial M_{ia}^2} + V(M)\right)$$

- The overall factor of R comes from dimensional analysis.
- The Gauss' law constraint: $[G_a, \mathcal{O}] \equiv [f_{abc}M_{ib}E_{ic}, \mathcal{O}] = 0$ for all observables $\mathcal{O}$.
- The physical states $|\psi_{phys}\rangle$ are given by $G_a|\psi_{phys}\rangle = 0$.
- The Hilbert space has scalar product

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Spectrum of H

- $H = H_0 + \frac{1}{R} V_{int}(M) = \frac{1}{R}\left(-\frac{1}{2}\frac{\partial^2}{\partial M_{ia}^2} + \frac{1}{2} M_{ia} M_{ia}\right) + \frac{1}{R}\left(-\frac{g}{2}\epsilon_{ijk} f_{abc} M_{ia} M_{jb} M_{kc} + \frac{g^2}{4} f_{abc} f_{ade} M_{ib} M_{jc} M_{id} M_{je}\right)$
- The interaction has a cubic term and a quartic term.
- The potential grows quartically, and is smooth everywhere.
- The spectrum is discrete.
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Zero-point Energy

- The H only accounts for the classical zero-mode sector of the full theory.
- The full QFT contributes an extra constant to the energy.
- It comes from zero-point energy of all the higher, spatially dependent modes.
- We can account for this by working with

$$H + \frac{c(R)}{R}$$

- The R-dependence of c comes from renormalization .
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Variational Computational Scheme

- Trial wavefunctions are linear combinations of eigenstates of H_0.
- Angular momentum ($L_i = \epsilon_{ijk} M_{ja} E_{ka}$) commutes with the Hamiltonian.
- Organize the eigenstates and energies by their spins s.
- We consider 16 variational states with spin-0, 10 triplets with spin-1, and 18 quintuplets with spin-2.
- Express the cubic and quartic interaction terms in terms of the creation/annihilation operators.
- Compute the variational Hamiltonian matrix $\tilde{H}_{ij} = \langle \psi_i | H | \psi_j \rangle$.
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Parity P and Charge Conjugation C

- We need to assign P and C to the variational eigenstates.
- Under $C : M_{ia}T_a \to M_{ia}T_a^*$.
- C is a good symmetry of H and can be assigned unambiguously.
- P poses a slight problem, because $P : M_{ia} \to -M_{ia}$, but the cubic term in H flips in sign under P.
- In the large R limit, the expectation value of P in a variational eigenstate asymptotes to ± 1.
- So P can be assigned in the "flat space" limit.

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The large R limit

- For a given s, the energies are of the form $\mathcal{E}_n[s] = \frac{f_n^{(s)}(g)+c(R)}{R}$, measured in units of R^{-1}.
- Neither R nor the bare coupling g are directly measurable.
- Energy differences depend on g and R, but not on c.
- Ratios of energy differences depend only on g.
- For fixed g, all the $\mathcal{E}_n[s]$ vanish in the 'flat space" limit $R \to \infty$. (an analogous situation occurs in lattice computations as well).
- But masses of physical particles must be computed in this limit!

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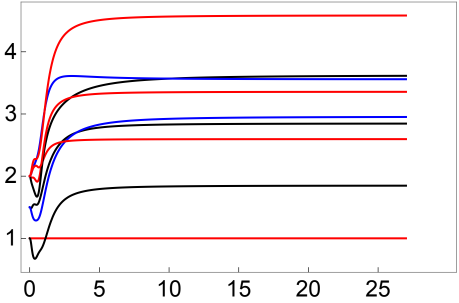
Mass Difference Ratios

- Ratios of mass differences are independent of both $x(g)$ and $c(x)$.

Ratios of mass differences $\frac{\mathcal{E}(X)-\mathcal{E}(0^{++})}{\mathcal{E}(2^{++})-\mathcal{E}(0^{++})}$ as a function of g. (The black, blue and red curves represent spin-0, spin-1 and spin-2 levels respectively.)



- $X(J^{PC}) = 2^{++}, 0^{-+}, 2^{-+}, 0^{*++}, 1^{+-}, 2^{*-+}, 1^{--}, 0^{*-+}, 2^{--}$.

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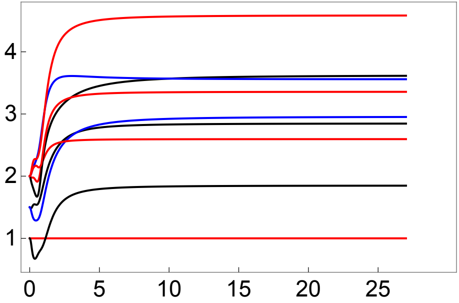

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Integrated Renormalization Group Equation

- To get meaningful results, make g a function of R such that all energies have well-defined (and non-zero) values at $R = \infty$.
- Measure the energies in some other units (like, say, MeV), not in units of $1/R$.
- The radius of S^3 is now $x = R/\ell$ in these units.
- Then $\mathcal{E}_n[s] = \left(\frac{f_n^{(s)}(g)}{x} + \frac{c(x)}{x} \right) \frac{1}{\ell}$.
- Make $g = g(x)$ by fixing $\mathcal{E}_0[2] - \mathcal{E}_0[0]$ to the observed (lattice) value.
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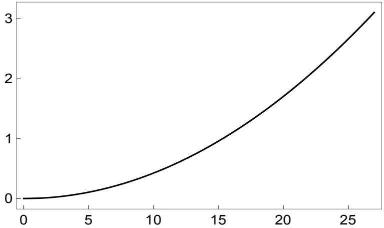
- To get meaningful results, make g a function of R such that all energies have well-defined (and non-zero) values at $R = \infty$.
- Measure the energies in some other units (like, say, MeV), not in units of $1/R$.
- The radius of S^3 is now $x = R/\ell$ in these units.
- Then $\mathcal{E}_n[s] = \left(\frac{f_n^{(s)}(g)}{x} + \frac{c(x)}{x} \right) \frac{1}{\ell}$.
- Make $g = g(x)$ by fixing $\mathcal{E}_0[2] - \mathcal{E}_0[0]$ to the observed (lattice) value.
- This is our integrated renormalization group equation $g(x)$.

Integrated Renormalization Group Equation

- In practice it is easier to make $x(g) = \frac{\mathcal{E}_0[2]-\mathcal{E}_0[0]}{m(2^{++})-m(0^{++})}$.

$x(g)$ versus g.



- Here we have used $m(2^{++}) - m(0^{++}) = 460$ MeV.

- Actual numerical values of masses also need asymptotic $c(x)/x$.
- To fix this, demand that the physical mass of our lowest glueball be fixed to be within the range predicted by lattice simulations $(1580 - 1840$ MeV$)$.
- Choosing 1050 MeV for asymptotic $c(x)/x$, we get the best fit with lattice predictions.

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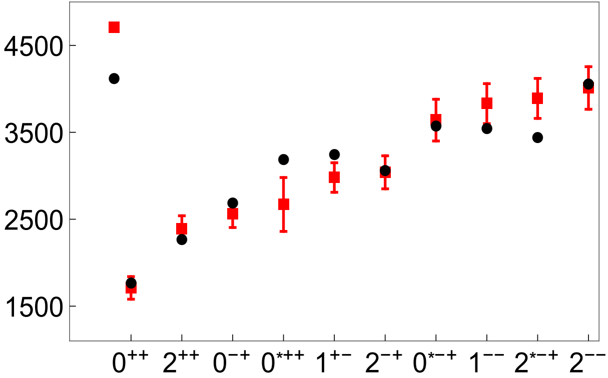
Glueball states J^{PC}	Physical masses from matrix model (MeV)	Physical masses from lattice QCD (MeV)
0^{++}	$1757.08^{\dagger}$	1580 - 1840
2^{++}	$2257.08^{\dagger}$	2240 - 2540
0^{-+}	2681.45	2405 - 2715
0^{*++}	3180.82	2360 - 2980
1^{+-}	3235.41	2810 - 3150
2^{-+}	3054.97	2850 - 3230
0^{*-+}	3568.02	3400 - 3880
1^{--}	3535.66	3600 - 4060
2^{*-+}	3435.75	3660 - 4120
2^{--}	4050.14	3765 - 4255

$^{\dagger} \equiv$ (input)

Glueball Masses (MeV)



■ ≡ Lattice • ≡ Matrix Model. 0^{++} and 2^{++} are used in Matrix Model input.

For 0^{*++}, lattice has poor statistics near the continuum limit, so finite volume effects are substantial.

For 2^{*++}, lattice has large errors due to the presence of two other glueball states in the vicinity.

Summary

- A natural reduction of $SU(N)$ YM on $S^3 \times \mathbb{R}$ to a matrix model.
- It captures the non-trivial topological character of the full gauge bundle.
- The matrix model based on $M_{3,N^2-1}(\mathbb{R})$.
- The canonical quantisation can be carried out, and the spectrum of the full Hamiltonian can be estimated variationally.
- In the large R limit, the eigenvalues tend to non-trivial asymptotic values provided $g(R)$ is chosen appropriately (our RG prescription).
- *THESE ASYMPTOTIC VALUES AGREE WELL WITH LATTICE PREDICTIONS FOR GLUEBALL MASSES*.

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Ongoing Work and Outlook

- Investigate the glueball spectrum for $SU(4), SU(5), SU(6), \cdots$.
- Include fermions (quarks), and try to get the masses of light hadrons.
- Include the θ-term, and compute topological susceptibility χ_t.
- Relation between χ_t and the mass of η'.

A much deeper puzzle: why does this model work so well?

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- $$A_{ia} = \frac{1}{\sqrt{2}}\left(M_{ia} + \frac{\partial}{\partial M_{ia}}\right), \quad A^\dagger_{ia} = \frac{1}{\sqrt{2}}\left(M_{ia} - \frac{\partial}{\partial M_{ia}}\right) \implies [A_{ia}, A^\dagger_{jb}] = \delta_{ia}\delta_{jb}$$
- The oscillator vacuum is $\langle M|0\rangle = \frac{1}{\pi^6} e^{-\frac{Tr(M^T M)}{2}}$
- Spin-0:

$$\begin{aligned}
|\psi^0_1\rangle &= |0\rangle \\
|\psi^0_2\rangle &= A^\dagger_{ia}A^\dagger_{ia}|0\rangle \\
|\psi^0_3\rangle &= \epsilon_{ijk} f_{abc} A^\dagger_{ia}A^\dagger_{jb}A^\dagger_{kc}|0\rangle \\
|\psi^0_4\rangle &= A^\dagger_{ia}A^\dagger_{ia}A^\dagger_{jb}A^\dagger_{jb}|0\rangle \\
|\psi^0_5\rangle &= A^\dagger_{ia}A^\dagger_{ib}A^\dagger_{ja}A^\dagger_{jb}|0\rangle \\
|\psi^0_6\rangle &= d_{abe}d_{cde}A^\dagger_{ia}A^\dagger_{ib}A^\dagger_{jc}A^\dagger_{jd}|0\rangle \\
|\psi^0_7\rangle &= \epsilon_{ijk} f_{abc} A^\dagger_{ia}A^\dagger_{jb}A^\dagger_{kc}A^\dagger_{ld}A^\dagger_{ld}|0\rangle \\
|\psi^0_8\rangle &= \epsilon_{ijk} f_{abc} d_{a_1 b_1 e} d_{a_2 c e} A^\dagger_{ia}A^\dagger_{jb}A^\dagger_{ka_1}A^\dagger_{lb_1}A^\dagger_{la_2}|0\rangle \\
|\psi^0_9\rangle &= A^\dagger_{ia}A^\dagger_{ia}A^\dagger_{jb}A^\dagger_{jb}A^\dagger_{kc}A^\dagger_{kc}|0\rangle \\
|\psi^0_{10}\rangle &= A^\dagger_{ia}A^\dagger_{ib}A^\dagger_{jb}A^\dagger_{jc}A^\dagger_{kc}A^\dagger_{ka}|0\rangle \\
|\psi^0_{11}\rangle &= \epsilon_{ijk}\epsilon_{lmn}A^\dagger_{ia}A^\dagger_{la}A^\dagger_{jb}A^\dagger_{mb}A^\dagger_{kc}A^\dagger_{nc}|0\rangle \\
|\psi^0_{12}\rangle &= \epsilon_{i_1 j_1 k_1} f_{a_1 b_1 c_1} \epsilon_{i_2 j_2 k_2} f_{a_2 b_2 c_2} A^\dagger_{i_1 a_1}A^\dagger_{j_1 b_1}A^\dagger_{k_1 c_1}A^\dagger_{i_2 a_2}A^\dagger_{j_2 b_2}A^\dagger_{k_2 c_2}|0\rangle \\
|\psi^0_{13}\rangle &= d_{abc}d_{def}A^\dagger_{ia}A^\dagger_{id}A^\dagger_{jb}A^\dagger_{je}A^\dagger_{kc}A^\dagger_{kf}|0\rangle \\
|\psi^0_{14}\rangle &= d_{b_1 c_1 d} d_{b_2 c_2 d} A^\dagger_{ia}A^\dagger_{ia}A^\dagger_{jb_1}A^\dagger_{jc_1}A^\dagger_{kb_2}A^\dagger_{kc_2}|0\rangle \\
|\psi^0_{15}\rangle &= \epsilon_{i_1 j_1 k_1} f_{a_1 b_1 c_1} \epsilon_{i_2 j_2 k_2} f_{a_2 b_2 c_2} d_{c_1 d_1 e} d_{c_2 d_2 e} A^\dagger_{i_1 a_1}A^\dagger_{j_1 b_1}A^\dagger_{k_1 d_1}A^\dagger_{i_2 a_2}A^\dagger_{j_2 b_2}A^\dagger_{k_2 d_2}|0\rangle \\
|\psi^0_{16}\rangle &= d_{abc}d_{ad_1 e_1}d_{ad_2 e_2}d_{ad_3 e_3}A^\dagger_{id_1}A^\dagger_{ie_1}A^\dagger_{jd_2}A^\dagger_{je_2}A^\dagger_{kd_3}A^\dagger_{ke_3}|0\rangle
\end{aligned}$$

- f_{abc} and d_{abc} are the structure constants of $SU(3)$.

Spin-1

$$
\begin{aligned}
|\psi_1^1\rangle &= d_{abc} A^\dagger_{jb} A^\dagger_{jc} A^\dagger_{ia} |0\rangle \\
|\psi_2^1\rangle &= \epsilon_{jkl} d_{ab_1c_1} f_{ab_2c_2} A^\dagger_{ib_1} A^\dagger_{jc_1} A^\dagger_{kb_2} A^\dagger_{lc_2} |0\rangle \\
|\psi_3^1\rangle &= d_{ace} A^\dagger_{ia} A^\dagger_{jb} A^\dagger_{jb} A^\dagger_{kc} A^\dagger_{ke} |0\rangle \\
|\psi_4^1\rangle &= d_{ace} A^\dagger_{ib} A^\dagger_{jb} A^\dagger_{ja} A^\dagger_{kc} A^\dagger_{ke} |0\rangle \\
|\psi_5^1\rangle &= d_{ace} A^\dagger_{ia} A^\dagger_{jb} A^\dagger_{jc} A^\dagger_{ke} A^\dagger_{kb} |0\rangle \\
|\psi_6^1\rangle &= d_{abc} f_{bc_1b_2} f_{cc_2b_1} A^\dagger_{ia} A^\dagger_{jb_1} A^\dagger_{jc_1} A^\dagger_{kb_2} A^\dagger_{kc_2} |0\rangle \\
|\psi_7^1\rangle &= \epsilon_{jkl} d_{abc} f_{ade} A^\dagger_{ib} A^\dagger_{jc} A^\dagger_{kd} A^\dagger_{le} A^\dagger_{i_1a_1} A^\dagger_{i_1a_1} |0\rangle \\
|\psi_8^1\rangle &= \epsilon_{jkl} d_{ab_1c_1} f_{aa_2b_2} A^\dagger_{ia_1} A^\dagger_{i_1a_1} A^\dagger_{i_1b_1} A^\dagger_{jc_1} A^\dagger_{ka_2} A^\dagger_{lb_2} |0\rangle \\
|\psi_9^1\rangle &= \epsilon_{ijk} d_{ab_1c_1} d_{aa_2b_2} A^\dagger_{ja_1} A^\dagger_{i_1a_1} A^\dagger_{i_1b_1} A^\dagger_{kc_1} A^\dagger_{la_2} A^\dagger_{lb_2} |0\rangle \\
|\psi_{10}^1\rangle &= \epsilon_{ijk} d_{ab_1c_1} f_{bb_2c_2} A^\dagger_{i_1b_1} A^\dagger_{i_1c_1} A^\dagger_{la} A^\dagger_{lb} A^\dagger_{jb_2} A^\dagger_{kc_2} |0\rangle
\end{aligned}
$$

Spin-2

$$
\begin{aligned}
|\psi^2_1\rangle &= (A^\dagger_{ia}A^\dagger_{ja} - \tfrac{1}{3}\delta_{ij}A^\dagger_{la}A^\dagger_{la})|0\rangle \\
|\psi^2_2\rangle &= A^\dagger_{i_1a_1}A^\dagger_{i_1a_1}(A^\dagger_{ia_2}A^\dagger_{ja_2} - \tfrac{1}{3}\delta_{ij}A^\dagger_{i_2a_2}A^\dagger_{j_2a_2})|0\rangle \\
|\psi^2_3\rangle &= (A^\dagger_{ia_1}A^\dagger_{i_1a_1}A^\dagger_{i_1b_1}A^\dagger_{jb_1} - \tfrac{1}{3}\delta_{ij}A^\dagger_{la_1}A^\dagger_{i_1a_1}A^\dagger_{i_1b_1}A^\dagger_{lb_1})|0\rangle \\
|\psi^2_4\rangle &= d_{abc}d_{ade}A^\dagger_{i_1b}A^\dagger_{i_1c}(A^\dagger_{id}A^\dagger_{je} - \tfrac{1}{3}\delta_{ij}A^\dagger_{ld}A^\dagger_{le})|0\rangle \\
|\psi^2_5\rangle &= A^\dagger_{i_1a_1}A^\dagger_{i_1a_1}(A^\dagger_{ia}A^\dagger_{ja} - \tfrac{1}{3}\delta_{ij}A^\dagger_{la}A^\dagger_{la})|0\rangle \\
|\psi^2_6\rangle &= \tfrac{1}{2}d_{abc}(\epsilon_{ikl}A^\dagger_{ja_1}A^\dagger_{ka_1} + \epsilon_{jkl}A^\dagger_{ia_1}A^\dagger_{ka_1})A^\dagger_{la}A^\dagger_{mb}A^\dagger_{mc}|0\rangle \\
|\psi^2_7\rangle &= \tfrac{1}{2}d_{abc}(\epsilon_{ikl}A^\dagger_{ja} + \epsilon_{jkl}A^\dagger_{ia})A^\dagger_{kb}A^\dagger_{la_1}A^\dagger_{ma_1}A^\dagger_{mc}|0\rangle \\
|\psi^2_8\rangle &= \epsilon_{klm}f_{abc}d_{da_1a}d_{da_2b_2}A^\dagger_{ka_1}A^\dagger_{lb}A^\dagger_{mc}(A^\dagger_{ia_2}A^\dagger_{jb_2} - \tfrac{1}{3}\delta_{ij}A^\dagger_{i_2a_2}A^\dagger_{i_2b_2})|0\rangle \\
|\psi^2_9\rangle &= A^\dagger_{i_1a_1}A^\dagger_{i_1a_1}A^\dagger_{i_2a_2}A^\dagger_{i_2a_2}(A^\dagger_{ia}A^\dagger_{ja} - \tfrac{1}{3}\delta_{ij}A^\dagger_{la}A^\dagger_{la})|0\rangle \\
|\psi^2_{10}\rangle &= A^\dagger_{i_1a_1}A^\dagger_{i_1a_1}A^\dagger_{i_2a_2}A^\dagger_{i_2a_1}(A^\dagger_{ia}A^\dagger_{ja} - \tfrac{1}{3}\delta_{ij}A^\dagger_{la}A^\dagger_{la})|0\rangle \\
|\psi^2_{11}\rangle &= d_{ab_1c_1}d_{ab_2c_2}A^\dagger_{i_1b_1}A^\dagger_{i_1c_1}A^\dagger_{i_2b_2}A^\dagger_{i_2c_2}(A^\dagger_{ia}A^\dagger_{ja} - \tfrac{1}{3}\delta_{ij}A^\dagger_{la}A^\dagger_{la})|0\rangle \\
|\psi^2_{12}\rangle &= A^\dagger_{i_1a_1}A^\dagger_{i_1a_1}(A^\dagger_{ia_2}A^\dagger_{i_2a_2}A^\dagger_{i_2b_2}A^\dagger_{jb_2} - A^\dagger_{la_2}A^\dagger_{i_2a_2}A^\dagger_{i_2b_2}A^\dagger_{lb_2})|0\rangle \\
|\psi^2_{13}\rangle &= d_{aa_2b_2}d_{ac_2e_2}A^\dagger_{i_1a_1}A^\dagger_{i_1a_1}A^\dagger_{i_2a_2}A^\dagger_{i_2b_2}(A^\dagger_{ic_2}A^\dagger_{jd_2} - \tfrac{1}{3}\delta_{ij}A^\dagger_{lc_2}A^\dagger_{ld_2})|0\rangle \\
|\psi^2_{14}\rangle &= \tfrac{1}{2}(\epsilon_{ikl}A^\dagger_{jb}A^\dagger_{kb} + \epsilon_{jkl}A^\dagger_{ib}A^\dagger_{kb})\epsilon_{mnp}d_{ab_1c_1}f_{bb_2c_2}A^\dagger_{lb_1}A^\dagger_{mc_1}A^\dagger_{nb_2}A^\dagger_{pc_2}|0\rangle \\
|\psi^2_{15}\rangle &= d_{ab_1c_1}d_{ab_2c_2}A^\dagger_{lb_1}A^\dagger_{lc_1}A^\dagger_{mb_2}A^\dagger_{mc_2}(\tfrac{1}{2}(A^\dagger_{ia}A^\dagger_{jb} + A^\dagger_{ja}A^\dagger_{ib}) - \tfrac{1}{3}\delta_{ij}A^\dagger_{la_2}A^\dagger_{lc_2})|0\rangle \\
|\psi^2_{16}\rangle &= d_{ab_1c_1}d_{bb_2c_2}A^\dagger_{i_1a}A^\dagger_{i_1b}A^\dagger_{j_1b_1}A^\dagger_{j_1c_1}(A^\dagger_{ia}A^\dagger_{jb} - \tfrac{1}{3}\delta_{ij}A^\dagger_{la}A^\dagger_{lb})|0\rangle \\
|\psi^2_{17}\rangle &= d_{aa_2b_2}d_{bc_2a_1}A^\dagger_{i_1a_1}A^\dagger_{i_1a_2}A^\dagger_{j_1b_2}A^\dagger_{j_1c_2}(A^\dagger_{ia}A^\dagger_{jb} - \tfrac{1}{3}\delta_{ij}A^\dagger_{la}A^\dagger_{lb})|0\rangle \\
|\psi^2_{18}\rangle &= d_{ab_1c_1}d_{aa_2b_2}f_{bb_2c_2}A^\dagger_{i_1b_1}A^\dagger_{i_1c_1}A^\dagger_{i_2c_2}A^\dagger_{i_2d_2}(A^\dagger_{ia_2}A^\dagger_{je_2} - \tfrac{1}{3}\delta_{ij}A^\dagger_{la_2}A^\dagger_{le_2})|0\rangle
\end{aligned}
$$

New Identities

We discovered some (new?) identities involving 3×8 matrices:

$$
\begin{aligned}
\mathrm{Tr}(M^T M D_a M^T M D_a) &= -\frac{1}{2}\,\mathrm{Tr}(M^T M D_a)\,\mathrm{Tr}(M^T M D_a) \\
&+ \frac{2}{3}\,\mathrm{Tr}(M^T M M^T M) + \frac{1}{3}\,\mathrm{Tr}(M^T M)^2 \\
\epsilon_{ijk} f_{abc} M_{ia} M_{jb} (M M^T M)_{kc} &= \frac{1}{3} \epsilon_{ijk} f_{abc} M_{ia} M_{jb} M_{kc}\,\mathrm{Tr}(M^T M)
\end{aligned}
$$

where $(D_a)_{bc} \equiv d_{abc}$.