Quantum Physics: Fields,Particles, and Information Geometry

In honour of A. P. Balachandran on the occasion of his 80th birthday.

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Quantum Hall Effect on Odd-dimensional Spheres

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A Short Timeline

- Haldane (1983) considered QHE on S^2.
- Hu & Zhang (2000) obtained a generalization of QHE on S^4.
- Karabali & Nair (2002) formulated QHE on $\mathbb{C}P^N$.
- QHE on even-dimensional spheres, S^{2k}, Kimura & Hasebe (2004), on S^8 Bernevig et.al. (2004).
- QHE on S^3 considered by Nair & Daemi (2004), and by Hasebe (2014).
- We formulated QHE on $\mathbf{Gr}_2(\mathbb{C}^{\mathbb{N}})$ (2014).

Early and Recent Motivations

- In $2D$ edge excitations give spin zero particles (massless chiral bosons), in $4D$ edge excitations have higher spin particles like photons and gravitons. However, other massless higher-spin states also occur.
- For S^4 and $\mathbb{C}P^2$, effective Abelian and non-Abelian Chern-Simons theory descriptions are given as generalizations of effective CS theory for QHE in the low energy regime.
- Low energy dynamics of strings-D-branes configurations are effectively captured by QHE on S^2 and S^4. Bernevig et.al. (2001), Fabinger (2002).

- Topological Insulators(TIs):
  1. A-class TIs: $\mathcal{T} = 0$, $\mathcal{C} = 0$, $\mathcal{S} = \mathcal{T}\mathcal{C} = 0$ and $\mathbb{Z}$ TI in even dimensions. These can be realized as QHE on even spheres S^{2k}. (Hasebe(1) , 2014).
  2. $AIII$-class TIs: $\mathcal{T} = 0$, $\mathcal{C} = 0$, $\mathcal{S} = 1$ and $\mathbb{Z}$ TI in odd dimensions. QHE on S^3 can be seen as a realization of $AIII$-class TI in 3-dimensions. (Hasebe(2),2014).

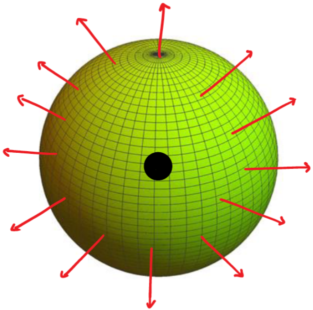
Landau Problem on S^2

- There are motivations from condensed matter physics.
  - Rotational Invariance, leading to invariance under magnetic translations, explaining the degeneracy of LL.
  - Compact geometry leads to finite number of degrees of freedom, leading to finite degeneracy of states at a given LL.
  - When $R \to \infty$, results on the plane are recovered.
- A Dirac monopole placed at the center of S^2:

$$\vec{B} = \frac{g}{r^3}\vec{r},$$

- Dirac quantization condition is:

$$eg = \frac{1}{2} l \hbar , \quad l \in \mathbb{Z} .$$



- The Hamiltonian for a charged particle on a sphere of radius R under the influence of the monopole field is

$$H = \frac{1}{2mR^2}\mathbf{\Lambda}\cdot\mathbf{\Lambda} = \frac{\omega_c}{I\hbar}\mathbf{\Lambda}\cdot\mathbf{\Lambda}\,, \quad \omega_c = \frac{eB}{m}\,.$$

- We have

$$\mathbf{\Lambda} = \boldsymbol{r}\times(-i\hbar\boldsymbol{\nabla} + e\boldsymbol{A})\,, \quad \boldsymbol{\nabla}\times\boldsymbol{A} = B\hat{\boldsymbol{r}}\,, \quad \mathbf{\Lambda}\cdot\hat{\boldsymbol{r}} = \hat{\boldsymbol{r}}\cdot\mathbf{\Lambda} = 0\,,$$

$$[\Lambda_\alpha\,,\Lambda_\beta] = i\hbar\varepsilon_{\alpha\beta\gamma}(\Lambda_\gamma - \frac{I\hbar}{2}\hat{r}_\gamma)$$

- Rotations are generated by

$$\boldsymbol{J} = \mathbf{\Lambda} + \frac{I\hbar}{2}\hat{\boldsymbol{r}}\,, \quad [J_\alpha\,,J_\beta] = i\hbar\varepsilon_{\alpha\beta\gamma}J_\gamma\,, \quad \boldsymbol{J}\cdot\hat{\boldsymbol{r}} = \hat{\boldsymbol{r}}\cdot\boldsymbol{J} = \frac{I\hbar}{2}\,,$$

- We have $[\boldsymbol{J}\,,H] = 0$.
- Eigenvalues of $\boldsymbol{J}^2$ should be $\hbar^2 j(j+1)$ where $j = q + \frac{I}{2}$.

- Spectrum of the Hamiltonian is then ($B = \frac{I\hbar}{2eR^2}$)

$$E = \frac{e\hbar B}{2m}(2q+1) + \frac{\hbar^2}{2mR^2} q(q+1)\,,$$

- The lowest Landau level has the energy $E = \frac{e\hbar B}{2m}$.
- Each Landau level has finite degeneracy $2j+1 = I + 1 + 2q$.
- On $SU(2)$, we have the Wigner functions $\mathcal{D}^{(j)}_{L_3,R_3}(g)$. Functions on S^2 may be viewed as the subset of functions on $SU(2)$ invariant under the $U(1)$ subgroup.
- Suppose that there there is no B-field. What is the Hamiltonian for an electron on the sphere?

$$H = \frac{L^2}{2mR^2}\,, \quad E = \frac{\ell(\ell+1)}{2mR^2}\,, \quad \psi_{\ell m} = D^{\ell}_{m0}(g) = \sqrt{\frac{4\pi}{2\ell+1}}\,Y^*_{\ell m}(\theta\,,\phi)\,.$$

- Eigenvalue equation for H is solved by the wave functions:

$$\mathcal{D}^{j}_{L_3\,\frac{I}{2}}(g)\,, \quad R_3 = \frac{I}{2}\,, \quad j = q + \frac{I}{2}$$

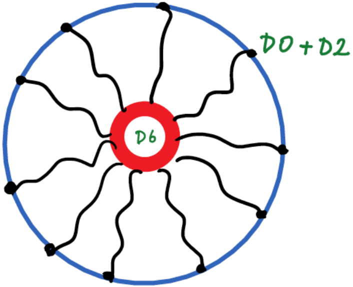
- Density correlation function between a pair of particles takes the form

$$\Omega(1,2) = |\Psi^{\mathbf{1}}|^2 |\Psi^{\mathbf{2}}|^2 - |\Psi^{*\mathbf{1}}_{\Lambda} \Psi^{\mathbf{2}}_{\Lambda}|^2,$$
$$= 1 - e^{-2B\left|\vec{X}^{\mathbf{1}} - \vec{X}^{\mathbf{2}}\right|^2},$$

- $\Omega(1,2)$ approaches 1 at separations $\gg \ell_B$. Here $\ell_B = \sqrt{\frac{1}{B}}$ is the magnetic length. Restoring e and $\hbar$ it is $\ell_B = \sqrt{\frac{\hbar}{eB}}$.
- Probability of finding two particles at the same location goes to zero.

A String Theory Perspective

- Very briefly:
  - Wrap a $D2$-brane on S^2 and dissolve N, $D0$-branes on it.
  - Take K flat $D6$-branes $\perp$ $D2$-brane and move them to the center of $D2$-brane.
  - K fundamental strings stretch between $D2$ and $D6$-branes.
  - Charged string-ends may be viewed as K- charged particles and in the low energy limit N is interpreted as the number of magnetic flux quantum with $\nu = \frac{K}{N}$ being the filling factor.
  - Background magnetic field may be viewed as density of $D0$-branes on the $D2$-brane. Thus, at low energies Quantum Hall system on S^2 appears to emerge.



Landau Problem on S^{2k-1}: Basic Setup and Geometry

- To specify the coordinates of S^{2k-1} we may embed it in $\mathbb{R}^{2k}$.
$S^{2k-1} \equiv \left\langle \vec{X} = (X_1, X_2, \cdots, X_{2k}) \in \mathbb{R}^{2k} \middle| \vec{X}\cdot\vec{X} = R^2 \right\rangle$.
- It is a coset space: $S^{2k-1} \equiv \frac{SO(2k)}{SO(2k-1)}$.
- We will need the gamma matrices, Γ_a, $a = (1, 2, \ldots, 2k)$ in $2k$-dimensions. These are $2^k \times 2^k$ matrices with $\{\Gamma_a\,, \Gamma_b\} = 2\delta_{ab}$. Iteratively:

$$\Gamma^\mu = \begin{pmatrix} 0 & -i\gamma_\mu \\ i\gamma_\mu & 0 \end{pmatrix}, \quad \Gamma^{2k} = \begin{pmatrix} 0 & 1_{2^{k-1}} \\ 1_{2^{k-1}} & 0 \end{pmatrix}, \quad \Gamma^{2k+1} = \begin{pmatrix} -1_{2^{k-1}} & 0 \\ 0 & 1_{2^{k-1}} \end{pmatrix},$$

Where we have $\{\gamma_\mu\,, \gamma_\nu\} = 2\delta_{\mu\nu}$ in $(2k-1)$-dimensions.

- $\Sigma_{\mu\nu} := -\frac{i}{4}[\gamma_\mu\,, \gamma_\nu]$, generate the spinor IRR of $SO(2k-1)$ of dimension 2^{k-1}.
- We also have

$$\Xi_{ab} := -\frac{i}{4}[\Gamma_a\,, \Gamma_b] = \begin{pmatrix} \Xi^+_{ab} & 0 \\ 0 & \Xi^-_{ab} \end{pmatrix}$$

Fundamental spinor IRRs of $SO(2k)$ are $\Xi^\pm_{ab} = (\Xi^\pm_{\mu\nu}\,, \Xi^\pm_{2k\mu}) = (\Sigma_{\mu\nu}, \mp\frac{1}{2}\gamma_\mu)$, generating $SO(2k)$. Each of dimension 2^{k-1}.

Gauge Field Background

- We introduce the 2^k-component spinor:

$$\Psi = \frac{1}{\sqrt{2R(R+x_{2k})}}((R+x_{2k})\mathbb{I}_{2^k}+x_\mu\Gamma^\mu)\phi\,, \quad \Psi^\dagger\Psi = 1\,, \quad \phi = \frac{1}{\sqrt{2}}\begin{pmatrix}\tilde{\phi}\\ \tilde{\phi}\end{pmatrix}$$

- Coordinates of S^{2k-1} : $\frac{X_a}{R} = \Psi^\dagger\Gamma_a\Psi$
- $SO(2k-1)$ gauge field, (i.e. the spin connection) on S^{2k-1} is given as

$$A_\mu = \Psi^\dagger\partial_\mu\Psi \implies A_\mu = -\frac{1}{R(R+X_{2k})}\Sigma_{\mu\nu}X_\nu\,, \quad A_{2k} = 0\,.$$

- Corresponding field strength is $F_{ab} = -i[D_a, D_b]$. Here $D_a = \partial_a + iA_a$ are the covariant derivatives. It takes the form

$$F_{\mu\nu} = \frac{1}{R^2}(X_\nu A_\mu - X_\mu A_\nu + \Sigma_{\mu\nu})\,, \quad F_{2k\mu} = -\frac{R+X_{2k}}{R^2}A_\mu\,.$$

- $R^4\sum_{a<b}F_{ab}^2 = \sum_{\mu<\nu}\Sigma_{\mu\nu}^2 = C^2_{SO(2k-1)}(\tfrac{1}{2}) = \tfrac{1}{2}\left(k-\tfrac{1}{2}\right)(k-1)\mathbb{I}_{2^{k-1}}\,.$

Gauge Field Background Continued

- A natural choice for a constant gauge field background is the I-fold symmetric tensor product[1]

$$\left(\frac{I}{2}\right) := \underbrace{\left(\frac{I}{2}, \ldots, \frac{I}{2}\right)}_{(k-1)\text{ terms}} = \bigotimes_{Sym}^{I} \left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$$

- Thus we have:

$$R^4 \sum_{a<b} F_{ab}^2 = \sum_{\mu<\nu} \Sigma_{\mu\nu}^2 = C^2_{SO(2k-1)}\left(\frac{I}{2}\right) = \frac{I}{2}\left(\frac{I}{2} + (k-1)\right)(k-1)\mathbb{I}_d$$

- d stands for the dimension of the representation $\left(\frac{I}{2}\right)$.

[1] *Note: For IRRs we are using the highest weight labels.*

Hamiltonian for Charged Particles

- For the Hamiltonian of charged particles on S^{2k-1} in the constant $SO(2k-1)$ gauge field background, we write

$$H = \frac{\hbar}{2MR^2} \sum_{a<b} \Lambda_{ab}^2 \,, \quad \Lambda_{ab} := -i(X_a D_b - X_b D_a)$$

- Total angular momentum is: Orbital angular momentum of the particles plus the angular momentum of the background gauge field.

$$\begin{aligned} L_{ab} &= \Lambda_{ab} + R^2 F_{ab} \,, \\ [L_{ab}, L_{cd}] &= i(\delta_{ac} L_{bd} + \delta_{bd} L_{ac} - \delta_{bc} L_{ad} - \delta_{ad} L_{bc}) \,. \end{aligned}$$

- L_{ab} generates the $SO(2k)$ group.
- Hamiltonian commutes with the total angular momentum: $[H \,, L_{ab}] = 0$.
- Hamiltonian takes the form (Using $\Lambda_{ab} F_{ab} = F_{ab} \Lambda_{ab} = 0$)

$$H = \frac{\hbar}{2MR^2} \left(\sum_{a<b} L_{ab}^2 - \sum_{\mu<\nu} \Sigma_{\mu\nu}^2 \right) .$$

Energy Spectrum

- What is the generic form of $SO(2k)$ IRR carried by L_{ab}?
- Its branching under $SO(2k-1)$ should include the $\left(\frac{I}{2}, \ldots, \frac{I}{2}\right)$ IRR of $SO(2k-1)$.
- From branching rules we find: $(n + \frac{I}{2}, \frac{I}{2}, \cdots, \frac{I}{2}, s)$, $n \geq 0 \in \mathbb{Z}$ and $|s| \leq \frac{I}{2}$. Indeed we have:

$$\left(n + \frac{I}{2}, \frac{I}{2}, \cdots, \frac{I}{2}, s\right) = \bigoplus_{\mu_1 = \frac{I}{2}}^{n + \frac{I}{2}} \bigoplus_{\mu_2 = s}^{\frac{I}{2}} \left(\mu_1, \frac{I}{2}, \cdots, \frac{I}{2}, \mu_2\right) .$$

- For the energy spectrum we find:

$$\begin{aligned} E &= \frac{\hbar}{2MR^2}\left(C^2_{SO(2k)} - C^2_{SO(2k-1)}\right) \\ &= \frac{\hbar}{2MR^2}\left(n^2 + s^2 + n(I + 2k - 2) + \frac{I}{2}(k-1)\right) . \end{aligned}$$

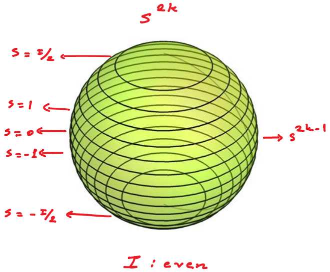
- For a fixed I, (n, s) are the quantum numbers labeling the Landau levels.

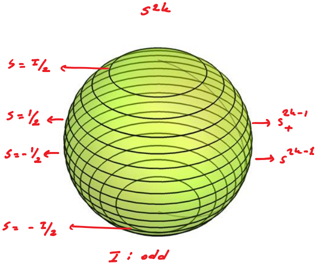
Degenaracies

- Degeneracy $d(n, \frac{I}{2}, s)$ of each LL is given by the dimension of the IRR $(n + \frac{I}{2}, \frac{I}{2}, \cdots, \frac{I}{2}, s)$.
- $n = 0$, energy levels split into sub-levels with $-\frac{I}{2} \leq s \leq \frac{I}{2}$. It is easily seen that

$$\sum_{|s| \leq \frac{I}{2}} d(n, \frac{I}{2}, s)_{SO(2k)} = d(0, \frac{I}{2})_{SO(2k+1)}$$

$d(0, \frac{I}{2})_{SO(2k+1)}$ being the degeneracy of LLL of QHE on S^{2k}.





Degeneracies & the LLL

- Degeneracy $d(n,s)$ of each LL is given by the dimension of the IRR $(n+\frac{I}{2},\frac{I}{2},\cdots,\frac{I}{2},s)$.
- LLL differ for odd and even values of I.

$$E_{LLL} = \begin{cases} \frac{\hbar}{2MR^2}\frac{I}{2}(k-1) & \text{for even } I\,,(n,s)=(0,0)\,, \\ \frac{\hbar}{2MR^2}\left(\frac{I}{2}(k-1)+\frac{1}{4}\right) & \text{for odd } I\,,(n,s)=(0,\pm\frac{1}{2}) \end{cases}$$

- For large I, degeneracy at the LLL for all values of I is :

$$d_{LLL} \approx I^{\frac{1}{2}(k-1)(k+2)}\,.$$

- In the limit $I,R \longrightarrow \infty$ with finite $\ell_M = \frac{R}{\sqrt{I}}$.

$$E(n,s) \longrightarrow \frac{\hbar}{2M\ell_M^2}\left(n+\frac{1}{2}(k-1)\right)\,, \quad E_{LLL} = \frac{\hbar}{2M\ell_M^2}\frac{k-1}{2}\,,$$

- Note that the spacing between LL levels remains finite.

Wave Functions

- Wave functions corresponding to the LL: Wigner $\mathcal{D}$-functions of $SO(2k)$: $\mathcal{D}^{(n+\frac{I}{2},\frac{I}{2},\cdots,\frac{I}{2},s)}(g)_{[L][R]}$.
- For $I=1$, LLL wave functions corresponding to $s=\pm\frac{1}{2}$ are

$$
\begin{aligned}
\Psi^{\pm} &= K^{\pm}\tilde{\phi}, \quad \Psi = \begin{pmatrix} \Psi^{+} \\ \Psi^{-} \end{pmatrix} \\
K^{\pm} &= \frac{1}{2}\frac{1}{\sqrt{R(R+X_{2k})}}((R+X_{2k})\mathbb{I}_{2^{k-1}} \mp i x_{\mu}\gamma^{\mu})
\end{aligned}
$$

- LLL-functions may be written as the I-fold symmetric tensor product of $\Psi^{\pm}$:

$$
\Psi^{I} = \sum_{\alpha_1,\cdots\alpha_I} f_{\alpha_1\cdots\alpha_I}\Psi_{\alpha_1}\cdots\Psi_{\alpha_I},
$$

- For N particles the LLL wave function can be obtained via the Slater determinant

$$
\Psi^{I}_{N} = \sum_{\alpha_1,\cdots\alpha_I} \varepsilon_{\alpha_1\cdots\alpha_I}\Psi^{I}_{\alpha_1}(x_1)\cdots\Psi^{I}_{\alpha_I}(x_N),
$$

The Equatorial S^{2k-2}

We can see that constant gauge field backgrounds on even-spheres can be accessed by confining to the equatorial spheres S^{2k-2}.

- We first note that

$$(K^{\pm})^2 = \frac{1}{R}(X_{2k}\mathbb{I}_{2^{k-1}} \mp iX_\mu\gamma^\mu)\,.$$

- On the equatorial S^{2k-2} this gives

$$(K_0^{\pm})^2 := (K^{\pm})^2\Big|_{X_{2k}=0} = \mp i\frac{1}{R}X_\mu\gamma^\mu\,, \quad R \text{ is the radius of } S^{2k-2}$$

- An idempotent on S^{2k-2} is

$$Q = i(K_0^{\pm})^2\,, \quad Q^\dagger = Q\,, \quad Q^2 = \mathbb{I}_{2^{k-1}}\,,$$

- Rank-2^{k-2} projection operators are:

$$\mathcal{P}_\pm = \frac{\mathbb{I}_{2^{k-1}} \pm Q}{2}\,.$$

- If $\mathcal{A}$ denotes the algebra of functions on S^{2k-2}. A free $\mathcal{A}$-module is $\mathcal{A}^{2^{k-1}} = \mathcal{A} \otimes \mathbb{C}^{2^{k-1}}$.
- We may decompose the free $\mathcal{A}^{2^{k-1}}$-module as

$$\mathcal{A}^{2^{k-1}} = \mathcal{P}_+ \mathcal{A}^{2^{k-1}} \oplus \mathcal{P}_- \mathcal{A}^{2^{k-1}} \,,$$

  $\mathcal{P}_\pm \mathcal{A}^{2^{k-1}}$ are the projective modules, each of dimension 2^{k-2}.
- Connection two-forms associated with $\mathcal{P}_\pm$ are

$$\mathcal{F}_\pm = \mathcal{P}_\pm \, d\left(\mathcal{P}_\pm\right) d\left(\mathcal{P}_\pm\right).$$

- $(k-1)^{th}$ Chern numbers are:

$$c_{k-1}^{\pm} = \frac{1}{k!(2\pi)^k} \int_{S^{2k-2}} \operatorname{Tr}\left(\mathcal{F}_\pm\right)^{k-1}.$$

- $c_{k-1} = d_{LLL}^{S^{2k-2}}(k-1)$ for $I = 1$. Higher rank projective modules an be constructed to give $c_{k-1}(I)$ which matches exactly with the number of zero modes, i.e. the index of the gauged Dirac operator on S^{2k-2}.

Example 1: S^3

- Energy spectrum reads $E = \frac{\hbar}{2MR^2}(n^2 + 2n + In + \frac{I}{2} + s^2)$.
- Degeneracy of the Landau levels are

$$d(n,s) = (n + \frac{I}{2} + 1)^2 - s^2 \,,$$

- LLL has

$$E_{LLL} = \frac{\hbar}{2MR^2}\frac{I}{2} \,, \quad (\text{I even}) \,, \qquad E_{LLL} = \frac{\hbar}{2MR^2}\left(\frac{I}{2} + \frac{1}{4}\right) \,, \quad (\text{I odd})$$

- With the degeneracies

$$d(n=0, s=0) = (\frac{I}{2} + 1)^2 \,, \quad d(n=0, s=\pm\frac{1}{2}) = \frac{1}{4}(I+1)(I+3) \,,$$

- On the equatorial S^2: $Q = \boldsymbol{\sigma} \cdot \hat{\boldsymbol{X}}$ and $\mathcal{P}_\pm = \frac{\mathbb{I}_2 \pm \boldsymbol{\sigma} \cdot \hat{\boldsymbol{X}}}{2}$
- This yields the usual abelian Dirac monopole field

$$B_\mu = \frac{1}{2}\varepsilon_{\mu\nu\rho}F_{\nu\rho} = \frac{I}{2}\frac{X_i}{R^3} \,.$$

- $c_1(I) = I$ gives the zero modes of the Dirac operator on S^2 in the Dirac monopole background.

Example 2: S^5

- Energy spectrum is : $E = \frac{\hbar}{2MR^2}(n^2 + 4n + In + I + s^2)$.
- For the LLL we find

$$E_{LLL} = \frac{\hbar}{2MR^2} I \,, \quad (\text{I even})\,, \qquad E_{LLL} = \frac{\hbar}{2MR^2}\left(I + \frac{1}{4}\right)\,, \quad (\text{I odd})$$

$$d(n=0, s=0) = \frac{1}{3\cdot 2^6}(I+2)^2(I+3)(I+4)^2\,, \quad (\text{I even})\,,$$
$$d(n=0, s=\pm\frac{1}{2}) = \frac{1}{3\cdot 2^6}(I+1)(I+3)^3(I+5)\,, \quad (\text{I odd})\,.$$

- On the equatorial S^4: $Q = \frac{\gamma_\mu X_\mu}{R}$ and $\mathcal{P}_\pm = \frac{\mathbb{I}_4 \pm Q}{2}$
- Curvature and connection take the form $i = (1\,, \cdots\,, 4)$

$$F_{ij} = \frac{1}{R^2}(X_j A_i - X_i A_j + \Sigma^+_{ij})\,, \quad F_{5i} = -\frac{R + X_5}{R^2} A_i\,,$$
$$A_i = -\frac{1}{R(R+X_5)}\Sigma^+_{ij} X_j\,, \quad A_5 = 0\,, \quad \Sigma^+_{ij} = -i\frac{1}{4}[\sigma_i\,, \sigma_j]$$

- Number of zero modes of the Dirac operator on S^4 with this $SU(2)$ background is $c_2(I) = \frac{1}{6}I(I+1)(I+2)$.

Gauged Dirac Operator on S^{2k-1}

- Dirac operator on S^{2k-1} in the background of F_{ab} has the form

$$\mathcal{D}_{(1,2)} = \frac{1}{2}(\mathbb{I} \mp \Gamma_{2k+1}) \sum_{a<b} \left(-\Xi_{ab}(L_{ab} - R^2 F_{ab}) + k - \frac{1}{2} \right) .$$

- Here Ξ_{ab} carries $\left(\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2}\right) \oplus \left(\frac{1}{2}, \frac{1}{2}, \cdots, -\frac{1}{2}\right)$ representation of $SO(2k)$.
- On a symmetric coset space, say $K \equiv G/H$, with holonomy group H taken as the gauge group (& identifying the gauge connection with the spin connection), t is possible to write

$$(i\mathcal{D}_{Gauged})^2 = C^2(G) - C^2(H) + \frac{\mathcal{R}}{8} ,$$

- This fits perfectly to our problem!: Therefore, we can write:

$$(i\mathcal{D}_{(1,2)})^2 = C^2_{SO(2k)}(n + J, J, \cdots, J, \pm\tilde{s}) - C^2_{SO(2k-1)}\left(\frac{I}{2}, \frac{I}{2}, \cdots, \frac{I}{2}\right) + \frac{1}{4}(2k^2 - 3k + 1) .$$

- $\mathcal{R} = 2(2k^2 - 3k + 1)$ is the Ricci scalar of S^{2k-1}.

Spectrum of $\mathcal{D}^2_{(1,2)}$ *and Zero Modes*

- We have $|\tilde{s}| \leq J$ and

$$J = \begin{cases} \frac{I}{2} + \frac{1}{2} & \text{for spin up \&} \quad I \geq 0 \\ \frac{I}{2} - \frac{1}{2} & \text{for spin down \&} \quad I \geq 1 \end{cases}$$

- Spectrum of $(i\mathcal{D}_{(1,2)})^2$ reads

$$\begin{aligned} \mathcal{E}_\uparrow &= n(n+2k-1) + I(n+k-1) + k(k-1) + \tilde{s}^2 \,, \quad (I \geq 0) \,, \\ \mathcal{E}_\downarrow &= n(n+I+2k-3) + \tilde{s}^2 \,, \quad (I \geq 1) \end{aligned}$$

- For even I, LLL is : $\mathcal{E}_\downarrow^{LLL}(n=0\,, \tilde{s} = \pm\frac{1}{2}) = \frac{1}{4}$.
- For odd I, LLL is : $\mathcal{E}_\downarrow^{LLL}(n=0\,, \tilde{s} = 0) = 0$. These are the zero modes of the Dirac operator.
- Note that spectrum of $(i\mathcal{D}_1)^2$ and $(i\mathcal{D}_2)^2$ are the same. This can be seen by taking $\tilde{s} \to -\tilde{s}$ in $\mathcal{E}_\uparrow$ and $\mathcal{E}_\downarrow$.

- For S^3, we find the LLL degeneracies:

$$\frac{I(I+2)}{4} \qquad \text{for even } I$$
$$\frac{(I+1)^2}{4} \qquad \text{for odd } I \text{, (zero modes)}$$

- For S^5, LLL degeneracies are:

$$\frac{1}{3\cdot 2^6} I(I+2)^3(I+4) \qquad \text{for even } I$$
$$\frac{1}{3\cdot 2^6} (I+1)^2(I+2)(I+3)^2 \qquad \text{for odd } I \text{, (zero modes)}$$

- No index theorem in odd dimensions to relate the zero modes to a topological number.
- For $I = 0$ and $\tilde{s} = \pm\frac{1}{2}$ we recover the spectrum for vanishing gauge background:

$$\mathcal{E}_\uparrow = (n + k - \frac{1}{2})^2 \,.$$

$$E_\uparrow = \sqrt{\mathcal{E}_\uparrow}\,, \quad E_\downarrow = -\sqrt{\mathcal{E}_\uparrow}$$

with $n \to n-1$ and $\tilde{s} \to -\tilde{s}$ in $E_\downarrow$.

Concluding Remarks and Outlook

- We have solved the Landau problem and the Dirac Landau problem for charged particles on S^{2k-1} in the background of $SO(2k-1)$ gauge field. Obtained the energy spectrum and wave functions.
- It is possible to show that there is exact correspondence between the direct sum of Hilbert spaces of LLLs with I ranging from 0 to $I_{max} = 2K$ or $I_{max} = 2K + 1$ correspond respectively to the Hilbert spaces of the fuzzy $\mathbb{C}P^3$ or that of winding number ± 1 line bundle over $\mathbb{C}P^3$ at level K.
  This correspondence also means that the quantum number $s = \pm\frac{1}{2}$ for the LLL over S^5 is actually related to the winding number $\kappa = \pm 1$ of the monopole bundles over $\mathbb{C}P_F^3$ via $s = \frac{\kappa}{2}$, which permits us to give, in a sense, a topological meaning to the ± 1 values of $2s$.

- We have noticed a peculiar relation between the Landau problem on S^{2k-1} and that on the equatorial S^{2k-2}, which allowed us to give the background $SO(2k-2)$ gauge fields over S^{2k-2} by constructing the relevant projective modules.
- LL on S^{2k-1} with $n = 0$ and $|s| \leq \frac{I}{2}$ can be visualized as embedded in the LLL of S^{2k} where s is thought of as a latitude parameter with discrete values. This picture can be described in terms of higher dimensional fuzzy spheres (Hasebe,2016).

HAPPY BIRTHDAY BAL!

A Curious Connection with $\mathbb{C}P_F^3$:

- An exact correspondence between the direct sum of Hilbert spaces of LLLs with I ranging from 0 to $I_{max} = 2K$ or $I_{max} = 2K + 1$ correspond respectively to the Hilbert spaces of the fuzzy $\mathbb{C}P^3$ or that of winding number ± 1 line bundle over $\mathbb{C}P^3$ at level K.
- Recall that the isometry group $SU(4)$ for $\mathbb{C}P^3$ is isomorphic to that of S^5, which is $Spin(6) \approx SO(6)$.
- Fuzzy $\mathbb{C}P^3$ at level K is given in term of the matrix algebra $Mat(d_K)$, where $d_K = \frac{1}{6}(K+3)(K+2)(K+1)$. It covers all the IRRs of $SU(4)$ which emerge from the tensor product

$$\left(\frac{K}{2}, \frac{K}{2}, \frac{K}{2}\right) \otimes \left(\frac{K}{2}, \frac{K}{2}, -\frac{K}{2}\right) = \bigoplus_{k=0}^{K} (k, k, 0)$$

- Expansion of an element of $Mat(d_K)$ in terms of $SU(4)$ harmonics carries the IRRs of $SU(4)$ appearing in the direct sum decomposition given in the r.h.s.
- Just observe, that each summand in the latter is equal to the $SU(4) \approx SO(6)$ IRR carried by the LLL for $I = 2k$.
- So, for even I, $(I = 2k)$, the direct sum of all the LLL Hilbert spaces with $0 \le k \le K$ spans the matrix algebra $Mat(d_K)$ of $\mathbb{C}P_F^3$.

- Sections of complex line bundles with winding number 1 over $\mathbb{C}P_F^3$ are described via the tensor product decomposition

$$\left(\frac{K+1}{2},\frac{K+1}{2},\frac{K+1}{2}\right)\otimes\left(\frac{K}{2},\frac{K}{2},-\frac{K}{2}\right)=\bigoplus_{k=0}^{K}\left(k+\frac{1}{2},k+\frac{1}{2},\frac{1}{2}\right)$$

- Elements in this nontrivial line bundle are $d_{K+1}\times d_K$ rectangular matrices forming a right module $\mathcal{A}^{(1)}(\mathbb{C}P_F^3)$ under the action of $Mat(d_K)$.
- We observe that each summand corresponds to an $SO(6)$ IRR carried by the LLL for $I=2k+1$ and $s=\frac{1}{2}$.
- So the direct sum of all the LLL Hilbert spaces with $0\leq k\leq K$ spans $\mathcal{A}^{(1)}(\mathbb{C}P_F^3)$ over $\mathbb{C}P_F^3$.
- It is easy to check that the total number of states in this direct sum of LLLs is precisely $d_{K+1}d_K$:

$$\sum_{k=0}^{K}\frac{1}{12}(k+4)(k+3)(k+2)^2(k+1)=d_{K+1}d_K$$

- A similar correspondence also follows for $\mathcal{A}^{-1}(\mathbb{C}P_F^3)$.

A Few Facts on QHE on S^4

- Landau problem for charged particles on S^4 formulated and solved by Hu and Zhang (2000).
- Particles are under influence of a background $SU(2)$ gauge field. This is provided by a Yang monopole.
- Multiparticle problem: In LLL, with filling factor $\nu = 1$, finite spatial density occurs iff the charges particles carry infinitely large IRR's of $SU(2)$.
- In $2D$ edge excitations give spin zero particles (massless chiral bosons), in $4D$ edge excittions have higher spin particles like photons and gravitons. However, other massless higher-spin states also occur.
- Effective Abelian and non-Abelian Chern-Simons theory descriptions in $6 + 1$ and $4 + 1$, respectively are also given as generalizations of effective CS theory for QHE in the low energy regime.