Scalar Fields in

Non-Commutative Spaces

I. **Quantum physics in a non-commutative (NC) space**

II. **The NC $\lambda\phi^4$ model: formulation**
Phase diagram in space-time dimension $d = 3$ and $d = 2$

III. **Can translation symmetry break spontaneously in $d = 2$?**

W.B., Frank Hofheinz, Héctor Mejía-Díaz, Jun Nishimura, Marco Panero,

JHEP 0406:042, JHEP 1410:56

I. Quantum physics in a non-commutative space

Since coordinates and momenta do not commute, we might "quantise further" and introduce also **NC space coordinates**.
Simplest case: constant NC "tensor" Θ in $d = 2$:

$$[\hat{x}_\mu, \hat{x}_\nu] = \mathrm{i}\,\Theta_{\mu\nu} = \mathrm{i}\,\theta\,\epsilon_{\mu\nu} \qquad \begin{array}{ccc} \hat{x}_\mu & : & \text{Hermitian operators} \\ \theta & : & \text{NC parameter} \end{array}$$

- Pre-history: { Heisenberg, Peierls, Pauli, Oppenheimer } (private commun.)
  Snyder '47, Yang '47

- 80's: mathematicians formulate field theory on NC spaces (Connes . . .)

  Application *as a formalism* in solid state physics
  (Q-Hall effect, *e.g.* Girvin et al.) Key relation: $\theta \propto 1/B$

- 1996-8: String theory at low energy $\longleftrightarrow$ NC field theory

  (Connes/Douglas/Schwarz, Seiberg/Witten, Sheikh-Jabbari . . .)

  $\Rightarrow$ Boom for more than a decade, > 3000 papers

- Qualitative difference to standard field theory:

  Non-Locality of range $\sim \sqrt{\|\Theta\|}$

  $\Rightarrow$ Conceptual problems, but hope for link to **quantum gravity** ?

Gedankenexperiment

Measure some event with accuracy $\Delta x_1, \Delta x_2, \Delta x_3, \Delta t$;
requires energy accumulation (Heisenberg) $\rightarrow$ gravitational field

Extreme case: event horizon > uncertainty $\Rightarrow$ event invisible

Avoiding this implies: (Doplicher/Fredenhagen/Roberts, '95)

$$\begin{aligned} \Delta x_1 \Delta x_2 + \Delta x_1 \Delta x_3 + \Delta x_2 \Delta x_3 &\geq l^2_{\text{Planck}} \\ (\Delta x_1 + \Delta x_2 + \Delta x_3)\, \Delta t &\geq l^2_{\text{Planck}} \quad (l_{\text{Planck}} = 1/\sqrt{G} \simeq 10^{-35}\,\text{m}) \end{aligned}$$

NC space "natural" framework for conciliation of quantum theory with gravity (?)

However: time is often kept commutative
(saves unitarity and reflection positivity, alleviates problems with causality)

This is modern motivation; how about Snyder etc.? Hope to remove/weaken UV divergences, avoid/simplify problems in field theory

WRONG ! Due to Θ, renormalisation is much harder :

- UV divergences in *planar diagrams* remain (Filk '96)
- rest "mixes" with a new type of IR divergences

Intuitive picture (Szabo):

$$\left.\begin{array}{lcc} \Delta x_j & \sim & 1/\Delta p_j \\ \Delta x_j & \sim & \Theta_{jk}/\Delta x_k \sim \Theta_{jk}\Delta p_k \end{array}\right\} \quad \Delta p_j \to 0 \;\Leftrightarrow\; \Delta p_k \to \infty$$

UV/IR mixing; in quantum theory, Θ has long-range impact

Renormalisation in perturbation theory beyond one loop mysterious.
Here: fully **non-perturbative** approach !

Lattice structure

NC plane, $[\hat{x}_i, \hat{x}_j] = \mathrm{i}\,\theta\,\epsilon_{ij}, \quad \theta = \text{const.} \qquad (i,\, j = 1, 2)$

A (fuzzy) lattice structure is imposed by the operator identity

$$\exp\left(\mathrm{i}\,\frac{2\pi}{a}\hat{x}_i\right) = \hat{1\!\!1}$$

Momentum components are commutative and periodic over Brillouin zone:

$$\begin{aligned} e^{\mathrm{i}\,k_i\hat{x}_i} &= e^{\mathrm{i}\,(k_i+\frac{2\pi}{a})\hat{x}_i} \\ \hat{1\!\!1} &= e^{\mathrm{i}\,(k_i+\frac{2\pi}{a})\hat{x}_i} e^{-\mathrm{i}\,k_j\hat{x}_j} = \cdots = \hat{1\!\!1}\exp\left(\frac{\mathrm{i}\,\pi}{a}\theta(k_2-k_1)\right) \;\Rightarrow\; \frac{\theta}{2a}k_i \in \mathbb{Z} \end{aligned}$$

$\Rightarrow$ Lattice is automatically periodic

Assume periodicity over $N \times N$ → momenta $k_n = \frac{2\pi}{aN} n \quad (n_i \in \mathbb{Z})$

$$\boxed{\theta = \frac{1}{\pi} N a^2}$$

- continuum limit: $a \to 0$
- infinite volume limit: $Na \to \infty$

The Double Scaling Limit

$$\boxed{a \to 0,\ N \to \infty \quad \text{at} \quad Na^2 = \text{const.}}$$

combines both at $\theta = \text{const.}$: continuous NC plane of infinite extent.

Simultaneous limit in the spirit of UV/IR mixing.

II. The NC $\lambda\phi^4$ model

Formulation for NC field theory in terms of ordinary coordinates x_μ, if all fields are multiplied by $\star$-products :

$$\phi(x) \star \psi(x) := \phi(x) \exp\left(\frac{\mathrm{i}}{2} \overleftarrow{\partial}_\mu \Theta_{\mu\nu} \overrightarrow{\partial}_\nu\right) \psi(x)$$

based on plane wave decomposition, $e^{\mathrm{i}\, p_\mu \hat{x}_\mu} e^{\mathrm{i}\, q_\nu \hat{x}_\nu} = e^{\mathrm{i}\, (p+q)_\mu \hat{x}_\mu - \frac{\mathrm{i}}{2} p_\mu \Theta_{\mu\nu} q_\nu}$

Euclidean action:

$$S[\phi] = \int d^d x \left[\frac{1}{2} \partial_\mu \phi \, \partial_\mu \phi + \frac{m^2}{2} \phi^2 + \frac{\lambda}{4} \phi \star \phi \star \phi \star \phi \right]$$

Bilinear terms under $\int$: $\star$-product $\equiv$ standard product (since $\Theta_{\mu\nu} = -\Theta_{\nu\mu}$)

$\Rightarrow$ λ determines extent of NC effects.

Perturbation theory:

1-loop diagrams: $\int d^d k \, \frac{1}{k^2+m^2}$, $\int d^d k \, \frac{\exp(\mathrm{i}\, k_\mu \Theta_{\mu\nu} p_\nu)}{k^2+m^2}$

planar non-planar

leading divergence
in $d = 4$, $|k| < \Lambda$: $\propto \Lambda^2$ $\propto [1/\Lambda^2 + p_\mu (\Theta^2)_{\mu\nu} p_\nu]^{-1}$

(Minwalla/Van Raamsdonk/Seiberg, '00)

$\|\Theta\| > 0$: removes non-planar UV divergence, unless $p \to 0$

Limit $\Theta \to 0$ is *not smooth*, beware of expansion in small $\|\Theta\|$

$\|\Theta\| \to \infty$ is commutative, but different from $\Theta = 0$

First consider $d=3$

$\phi(\vec{x},t)$, NC plane + commutative Euclidean time t

Action on a $N^2 \times T$ lattice can be mapped onto a matrix model with twisted boundary conditions (Ambjørn/Makeenko/Nishimura/Szabo, '00)

$$S[\bar\phi] \;=\; \mathrm{Tr}\sum_{t=1}^{T}\Big[\,\frac{1}{2}\sum_{i=1}^{2}\Big(\Gamma_i\bar\phi(t)\Gamma_i^\dagger-\bar\phi(t)\Big)^2+\frac{1}{2}\Big(\bar\phi(t+1)-\bar\phi(t)\Big)^2 + \frac{m^2}{2}\bar\phi^2(t)+\frac{\lambda}{4}\bar\phi^4(t)\,\Big]$$

$\bar\phi(t)$: Hermitian $N\times N$ matrices, at $t=1,\dots,T$

- Time direction: ordinary (discrete) kinetic term

- NC plane: unitary "twist eaters" Γ_i provide shift by one lattice unit, if

$$\Gamma_i\Gamma_j = Z_{ji}\Gamma_j\Gamma_i \qquad (\,'\text{t Hooft-Weyl algebra}).$$

We use $Z_{21} = Z_{12}^* = e^{2\pi \mathrm{i}\, k/N}$ with $k = (N+1)/2$, <u>N odd</u>

Solution for twist eaters: shift- and clock-operator

$$\Gamma_1 = \begin{pmatrix} 0 & 1 & & & \\ & . & . & & \\ & & . & . & \\ & & & . & 1 \\ 1 & & & & 0 \end{pmatrix} , \quad \Gamma_2 = \begin{pmatrix} 1 & & & & \\ & Z_{21} & & & \\ & & Z_{21}^2 & & \\ & & & . & \\ & & & & . \end{pmatrix}$$

Gubser/Sondhi, '01: 1-loop calculation in Hartree-Fock approximation
(exact for O($N \to \infty$))

$\Rightarrow$ **Conjectured phase diagram** (in $d = 3, 4$) :

- small θ : Ising type transition: disorder $\leftrightarrow$ uniform order
- larger θ : disorder $\leftrightarrow$ striped order (new!)

(order at $m^2 \ll -||\Theta||^{-1}$ $\sim$ very low temperature)

- Chen/Wu, '02: RG study in $d = 4 - \varepsilon$: striped phase for $\theta > 12/\sqrt{\varepsilon}$

- Castorina/Zappalà, '02: approach with $S_{\rm eff}$
  supports Gubser/Sondhi conjecture

- W.B./F. Hofheinz/J. Nishimura '04: **numerical study**

Simulations reveal phase diagram in $m^2 - \lambda$ plane
(large $\lambda \longrightarrow$ strong NC effects)

$N = T = 15,\ 25,\ 35,\ 45,$ phase transitions stable for $N \geq 25$

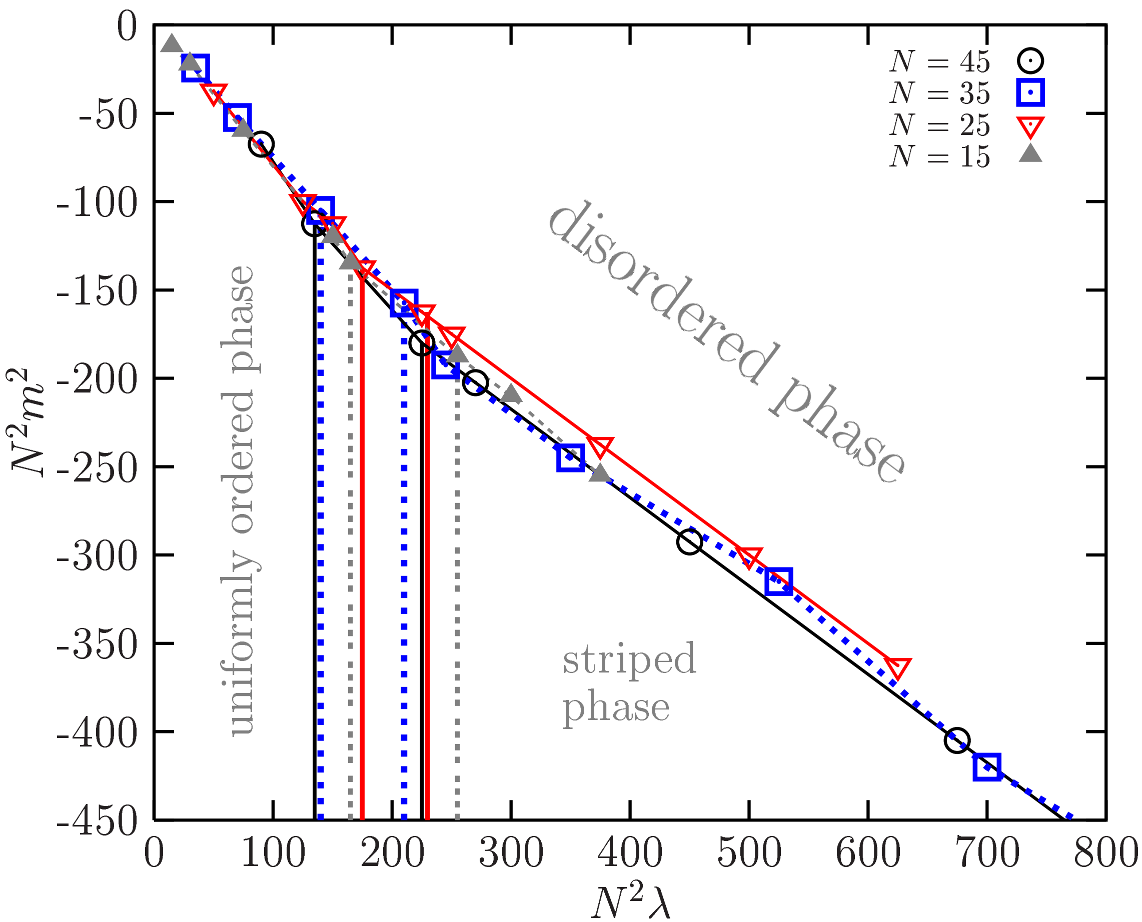
Ordered regime splits indeed into

- uniform phase: small λ
- striped phase: larger λ

Evaluation relies on **momentum dependent order parameter**

$$M(k) = \frac{1}{NT} \max_{k=|\vec{p}|N/2\pi} \Big| \sum_t \tilde{\phi}(\vec{p}, t) \Big|$$

(rotation to capture pattern of each configurations)
$M(0)$ uniform order, $M(k > 0)$ detects stripes with width $\propto 1/k$
(For decreasing m^2, M indicates type of order. Transition best seen from $\langle M^2 \rangle_C$.)

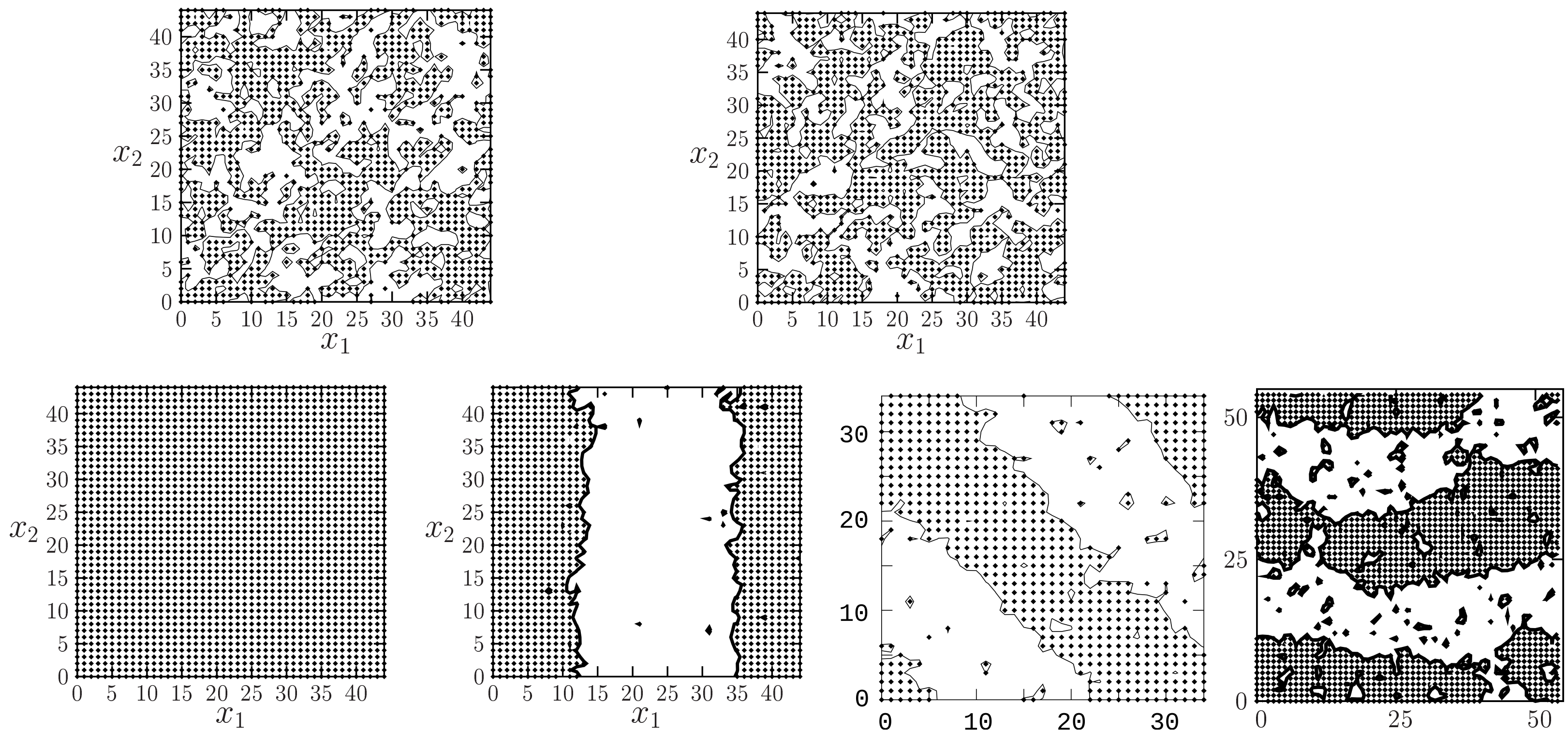


Thermal cycle: • phase transitions order-disorder of 2^{nd} order (in both cases)

• transition uniform-striped : 1^{st} order (cycle des not close)

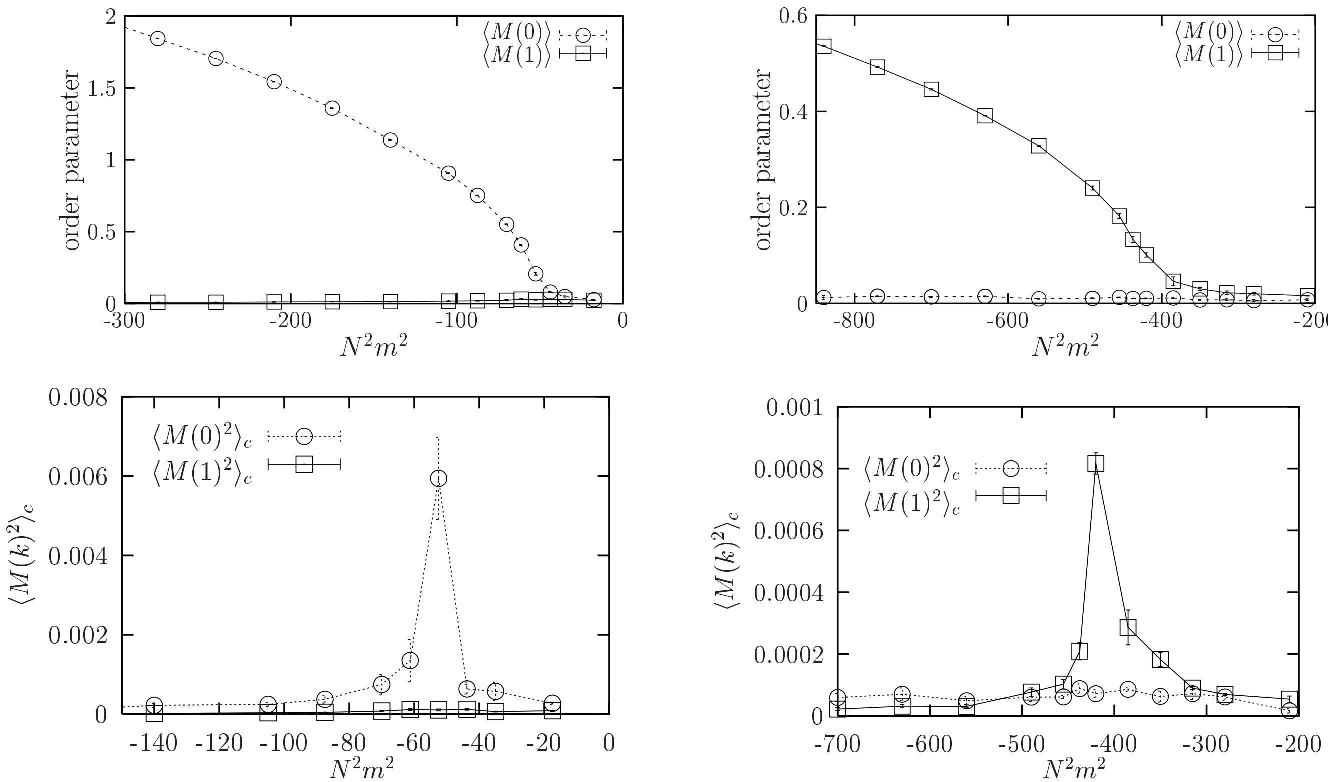
Mapping back to the NC plane visualises the pattern.

Typical snapshots in the 4 sectors of the phase diagram at $N = 45$:



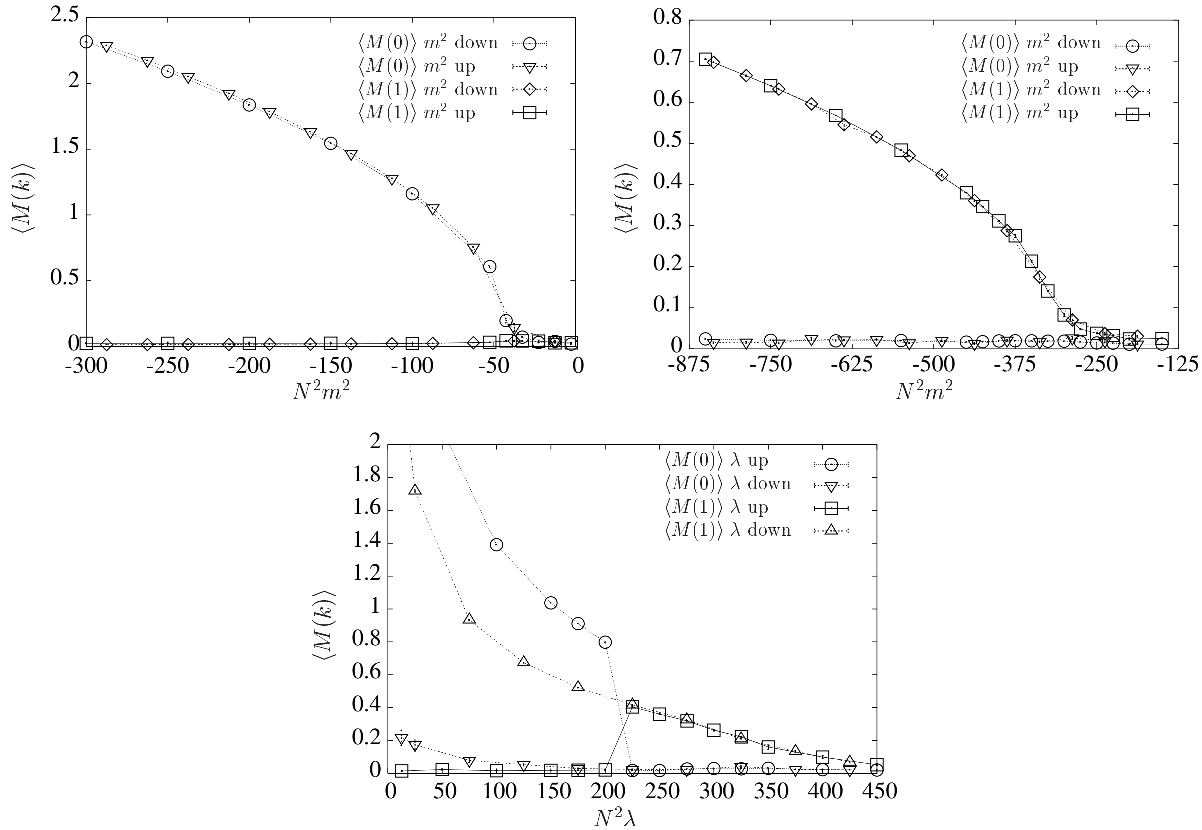
(dotted/blank areas correspond to the signs of ϕ)

But: does striped phase persist in the Double Scaling Limit?



Above: momentum dependent order parameter $\langle M(k)\rangle$, $k = 0, 1$ $(N = 35)$
in the uniform phase (left, $\lambda = 2$) and striped phase (right, $\lambda = 20$).

The transition is best detected by the connected 2-point function (below).

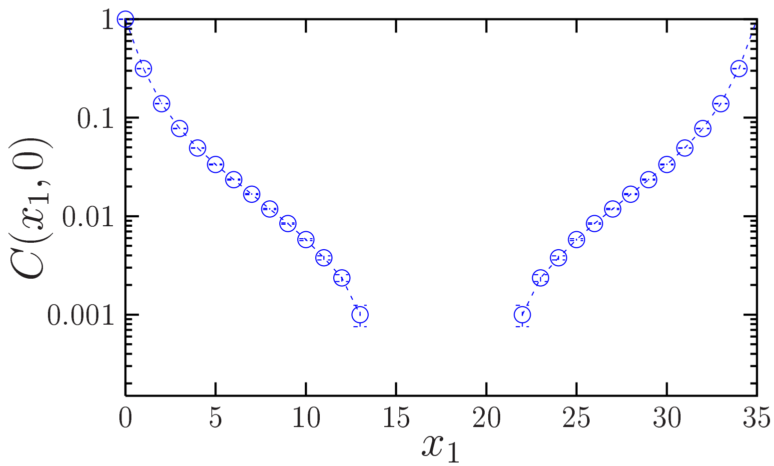


Hysteresis behaviour for the phase transitions:

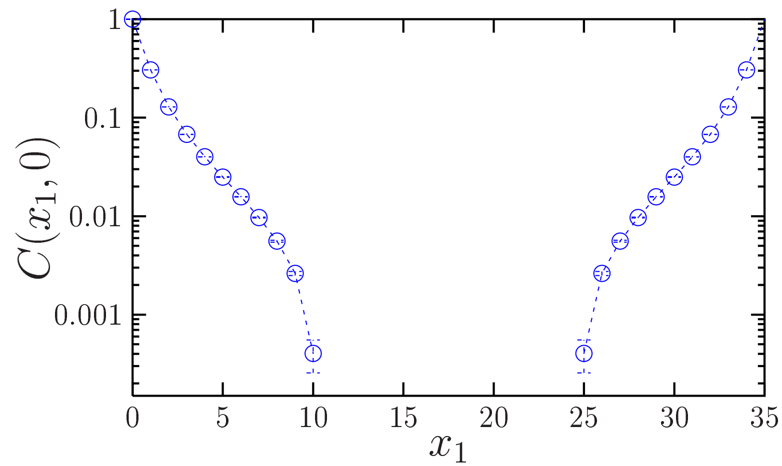
left: disorder $\leftrightarrow$ uniform, right: disorder $\leftrightarrow$ striped } 2nd order

below: uniform $\leftrightarrow$ striped } 1st order

Correlation function $C(\vec{x}) = \langle \phi(\vec{0},t)\phi(\vec{x},t) \rangle$ in the *disordered phase, close to ordering* (b.c. harmless) Examples at $N = 35$:



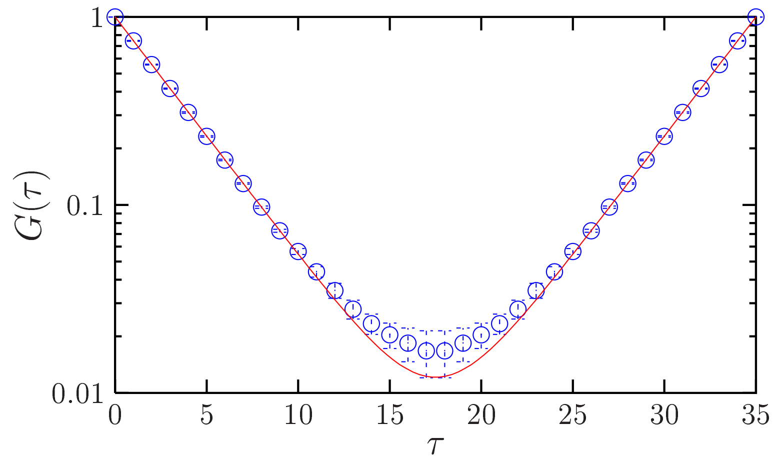
$N^2\lambda = 70\ ,\ N^2m^2 = -17.5$
close to uniform order



$N^2\lambda = 3500\ ,\ N^2m^2 = -140$
close to striped order

Unusual decay in spatial directions: fast, but not exponential, NC distorted

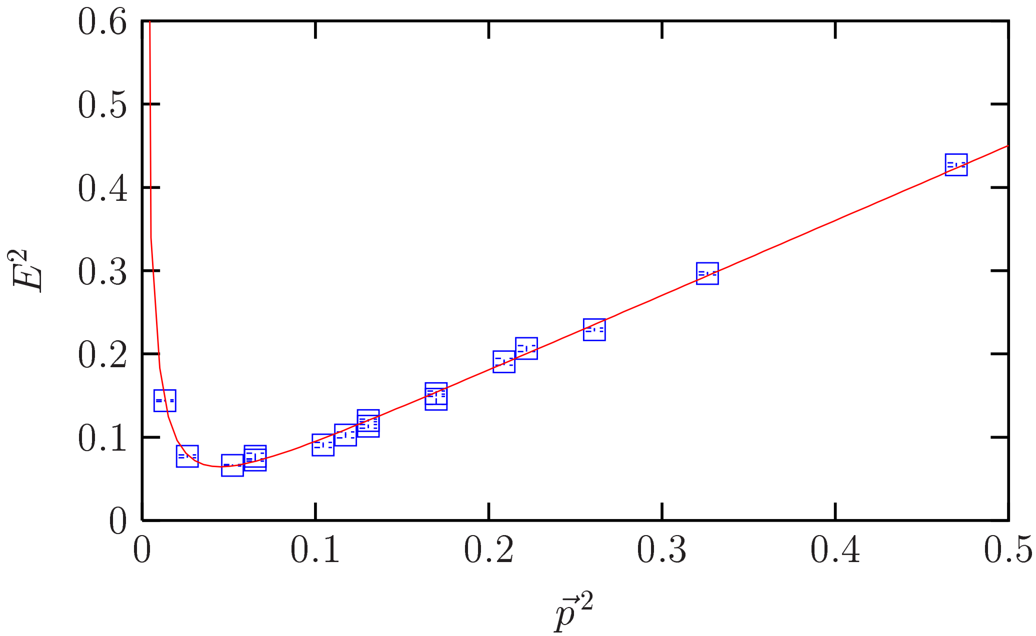
Temporal correlation function: $G(\tau) = \langle \tilde{\phi}(\vec{p} = \vec{0}, t) \tilde{\phi}(\vec{p} = \vec{0}, t + \tau) \rangle$
in the disordered phase, close to the stripe formation



Data at $N^2 \lambda = 35$, $N^2 m^2 = -140$ vs. cosh fit
In the (commutative) time direction the decay **is** exponential.

Evaluate $G(\tau) = \langle \tilde{\phi}(\vec{p}, t)\tilde{\phi}(\vec{p}, t + \tau)\rangle$ also at $\vec{p} \neq \vec{0} \Rightarrow$ energy $E(\vec{p})$.

Deformed dispersion relation at $N = 55$



Non-zero energy minimum

For decreasing m^2 condensation $\rightarrow$ stable stripe pattern

Different patterns (multi-stripe, checkerboard, etc.) in close competition

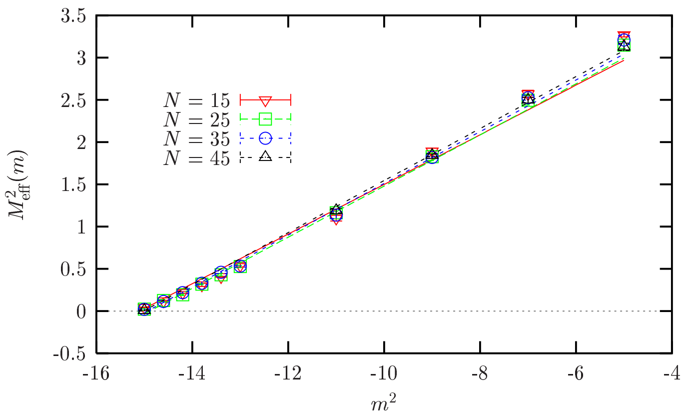
Continuum limit

So far lattice units; we need a *dimensional* quantity to introduce "physical units"

Linear extrapolation introduces effective mass $E^2 = M_{\rm eff}^2 + \vec{p}^{\,2}$.

We observe linear dependence on $m^2 > m_{\rm c}^2$:

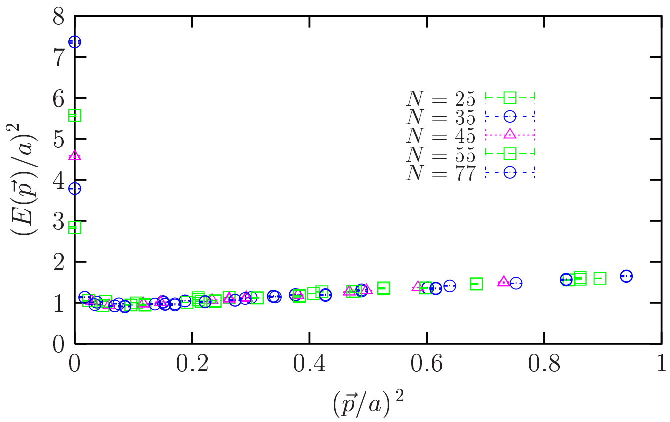
$$M_{\rm eff}^2|_{\lambda={\rm const.}} \propto m^2 - m_{\rm c}^2 \ , \qquad \text{e.g. for } \lambda = 50 \ ; \ m_c^2 = -15.01(8)$$

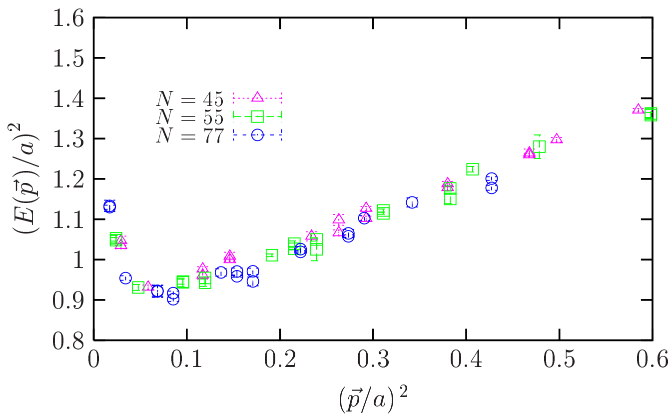


Double scaling: keep $M_{\rm eff}/a$ fixed, $\theta \propto Na^2 = {\rm const.}$ is implemented as

$$N \to \infty, \ m^2 \searrow m_c^2 \quad \text{with} \quad N(m^2 - m_c^2) = {\rm const.}$$

Results in dimensional units





The dispersion relation **stabilises in the double scaling limit** (up to rest energy E_0)

$a \to 0\,,\ V \to \infty\,,\ \theta = \text{const.} \ \Rightarrow$ looks **non-perturbatively renormalisable**

Finite minimum at $\vec{p}^{\,2}/a^2 \lesssim 0.1 \ \Rightarrow$ **finite stripe width dominates**

Demonstration for the striped phase in the continuum limit

Implies spontaneous breaking of translation and rotation symmetry

Corresponding model in $\mathrm{d} = 2$ (skip time coordinate)

Usually a continuous, global sym. cannot break spontaneously in $d \leq 2$

However, Mermin-Wagner Theorem assumes locality and IR regularity.

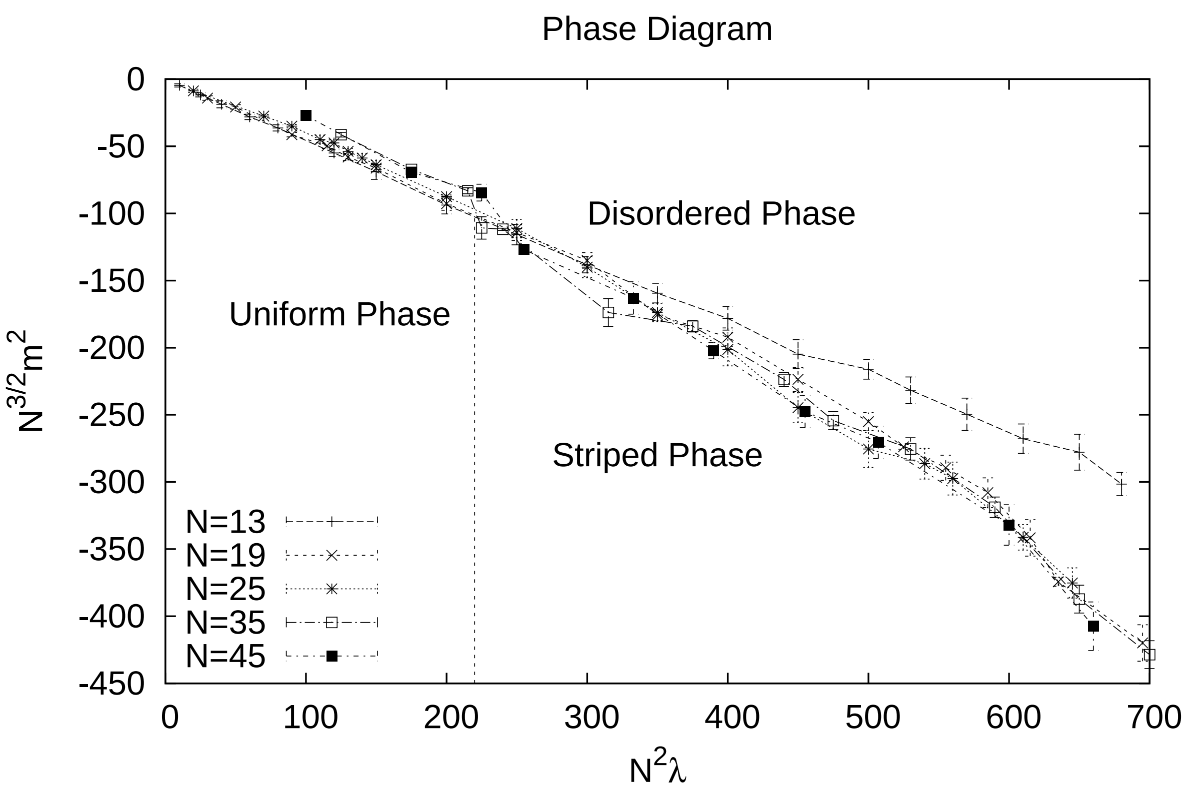
Still, Gubser/Sondhi '01 do not expect a striped phase (generalised M-W)

But: Castorina/Zappalà '07: analysis of $S_{\rm eff}$ seems to allow stripes.

Numerical:
Ambjørn/Catterall '02, W.B./Hofheinz/Nishimura '04 see stripes.

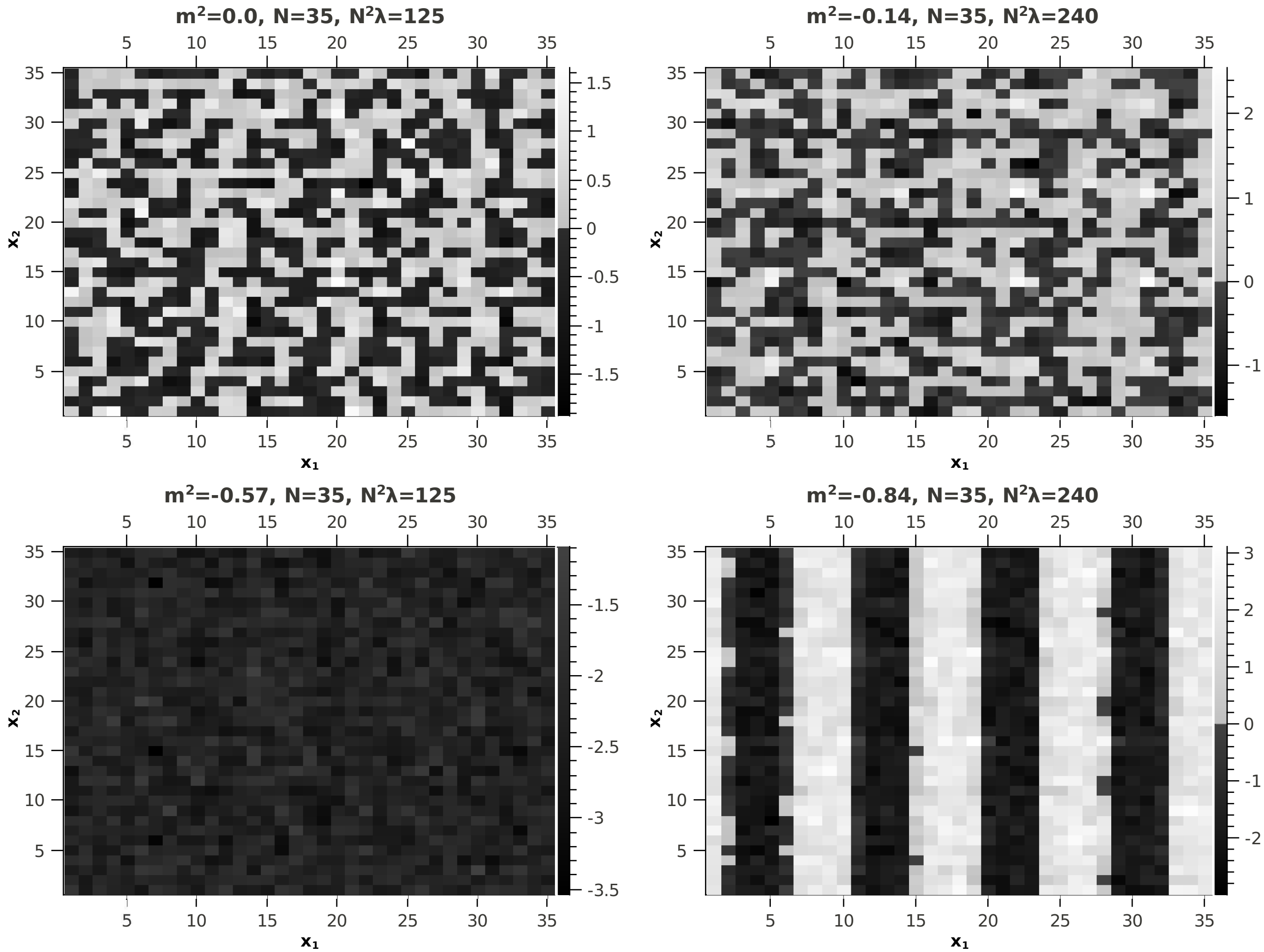
However: open question if it survives the Double Scaling Limit, or fate like confinement phase of lattice QED (no exp. decay of correlation functions!)

Thesis by Héctor Mejía-Díaz (UNAM)
with W.B., M. Panero, JHEP 1410 (2014) 56

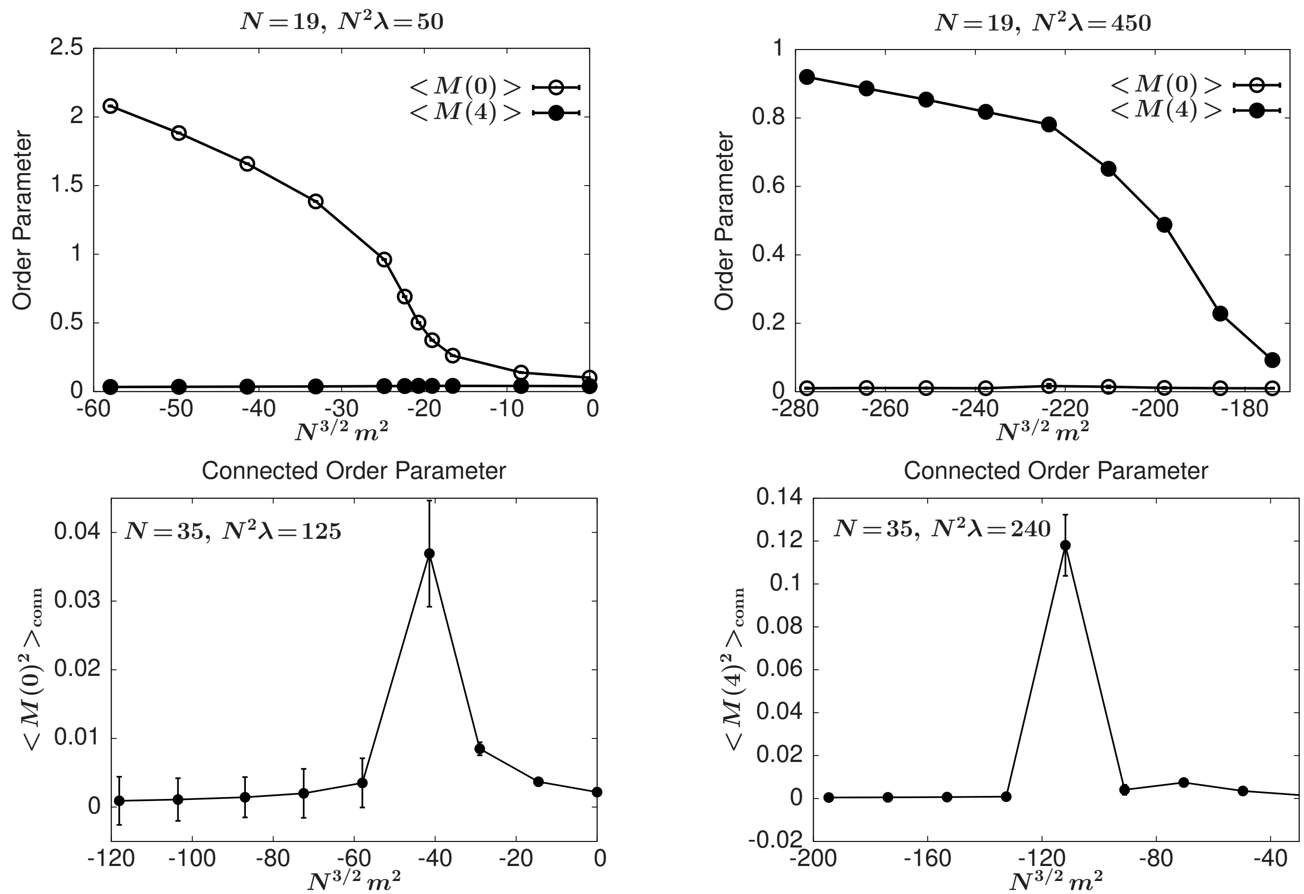


Phase diagram in $\mathbf{d = 2}$: requires different scaling of the axes $N^{3/2}m^2$ vs. $N^2\lambda$ (not predicted)

Stabilisation for $N \geq 19$



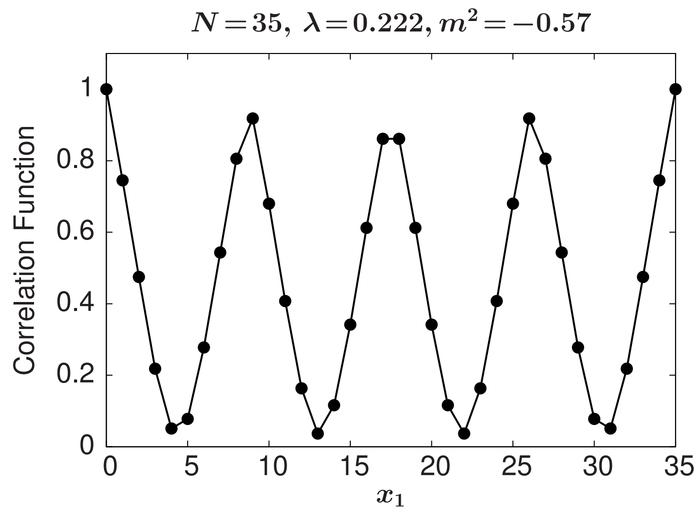
Snapshots: above: disordered, below: uniform/striped order



Identification of the phase transition order/disorder :

Above: uniform/striped order parameter jumps up for decreasing m^2

Below: peak of connected correlator $\langle M^2\rangle_{\rm C} = \langle M^2\rangle - \langle M\rangle^2$ localises critical value $m_{\rm c}^2$



Correlation $\langle \phi_{(0,0)} \phi_{(x,0)} \rangle$ near striped phase $(N^{3/2} m^2, N^2 \lambda) \simeq (-118, 272)$, pattern not condensed $\longrightarrow$ disordered.

Concept: approach $m^2 \searrow m_\mathrm{c}^2$ for increasing N such that the correlator down to the first dip stabilises.

Thus $\Delta m^2 := m^2 - m_\mathrm{c}^2$ defines a scale, which translates — with a suitable exponent — into the desired Double Scaling Limit: $a^2 \propto (\Delta m^2)^\sigma$

Question: does proximity to striped phase persist in this limit?

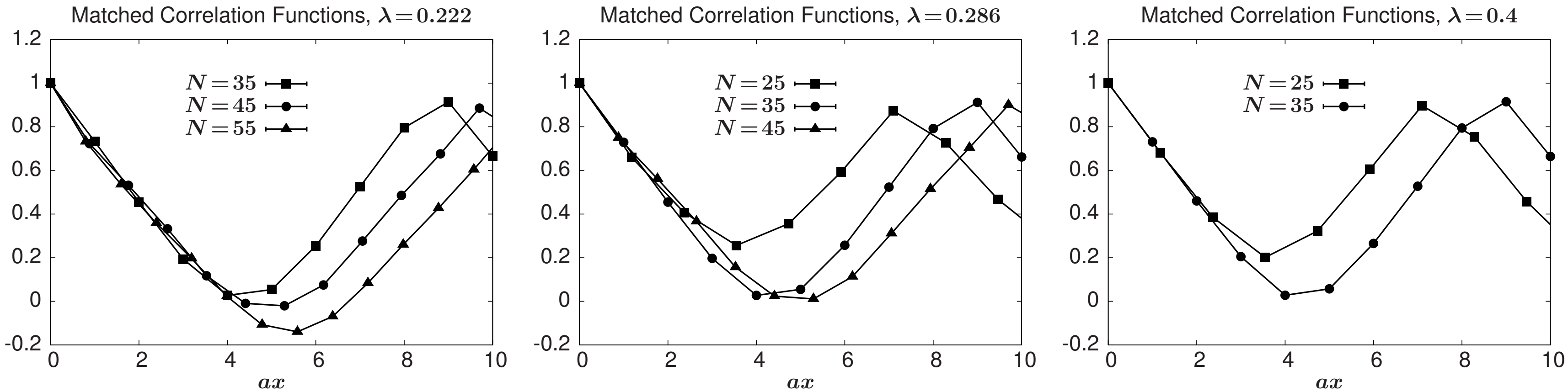
Ansatz: define $a = 1$ at $N = 35$: $N a^2 = \mathrm{const.} \;\Rightarrow\; a\,x = \sqrt{\frac{35}{N}}\,x$
Adjust dimension, like dim'less temperature $\tau = (T - T_{\rm c})/T_{\rm c}$,

$$a^2 = \frac{(\Delta m^2)^{\sigma}}{(m_{\rm c}^2)^{1-\sigma}}$$

Take two sizes N_1, N_2 with Δm_1^2, Δm_2^2, at fixed λ ($\rightarrow$ the dim'less term $\lambda\theta$ remains const.), same correlation decay. Extract exponent

$$\sigma = \frac{\ln(m_{1,{\rm c}}^2/m_{2,{\rm c}}^2)}{\ln(\Delta m_{1,{\rm c}}^2/\Delta m_{2,{\rm c}}^2) + \ln(m_{1,{\rm c}}^2/m_{2,{\rm c}}^2)}$$

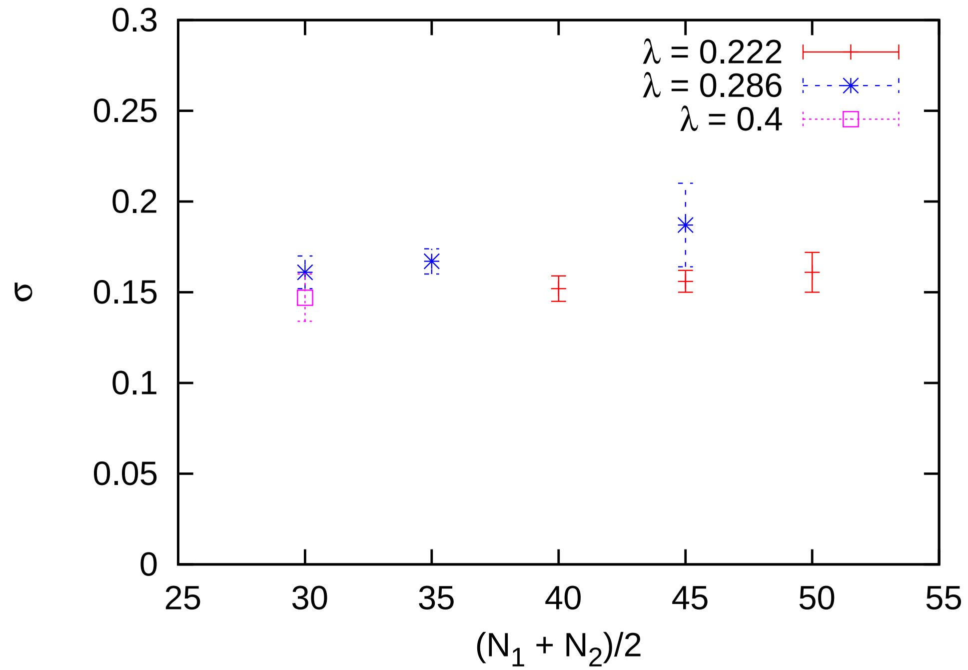
σ will stabilise for sufficiently large N_i and small $\Delta m_{i,{\rm c}}^2$,
if we stay near the striped phase.



λ	N_1	N_2	σ
0.222	35	45	0.152 (7)
	35	55	0.156 (6)
	45	55	0.161 (11)
0.286	25	35	0.161 (9)
	25	45	0.167 (7)
	35	45	0.178 (23)
0.4	25	35	0.147 (13)

(Feasibility of the simulation restricts the accessible values of $N^2\lambda$;
too large $\rightarrow$ landscape of deep semi-stable minima)

Stabilisation of σ is manifest:



$$\sigma = 0.16(1)$$

Striped phase persists in the Double Scaling Limit,
translation symmetry does break spontaneously.

IV. Conclusions

We studied the 3d and 2d $\lambda\phi^4$ model with a NC plane.

Lattice version can be mapped on a **Hermitian matrix model.**

This enables MC simulations (standard Metropolis algorithm).

$m^2 \ll 0$ **enforces order** :

λ resp. θ **small**: **uniform** order ; λ resp. θ **large**: **striped** order

Striped phase survives the Double Scaling Limit
($a \to 0$ *and* $L = Na \to \infty$, at $\theta = \text{const.}$)

SSB of translation invariance **even in $\mathbf{d = 2}$**

Mermin-Wagner Theorem invalidated
by IR divergence and non-locality

HAPPY BIRTHDAY, BAL !!!