Relativistic MHD

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§ 0. Prelude. Metric tensor, inertial frame, etc.

0.1. General stuff

$$ds^2 = g_{\nu\mu}\, dx^\nu dx^\mu \qquad (0.1)$$

– spacetime metric form.

$(\nu, \mu = 0, 1, 2, 3)$

$\{x^\nu\}$ – spacetime coordinates

- The coordinate basis of vectors:

$$\vec{e}_\nu = \frac{\partial}{\partial x^\nu} \qquad (0.2)$$

$$g_{\nu\mu} = (\vec{e}_\nu \cdot \vec{e}_\mu) \qquad (0.3)$$

- Normalized basis:

$$\hat{\vec{e}}_\nu = \frac{1}{\sqrt{|g_{\nu\nu}|}}\, \vec{e}_\nu \qquad (0.4)$$

$$\Rightarrow \quad |\hat{\vec{e}}_\nu|^2 = \pm 1 \qquad (0.5)$$

- Components of vectors

Contravariant, $\{A^\nu\}$: $\vec{A} = A^\nu \vec{e}_\nu \quad (0.6)$

(Notice the use of Einstein's summation rule.)

As a rule we will ~~not~~ use normalized basis in what follows

Covariant, $\{A_\nu\}$: $\quad A_\nu = g_{\nu\mu} A^\mu \quad (0.7)$

0.2. 3+1 splitting of inertial frames

(In curved spacetime only locally inertial frames)

The ~~special~~ spacial coordinate grid is fixed in space (static grid)! No deformations, rotation, etc.

$$\boxed{ds^2 = -c^2 dt^2 + dl^2} \qquad (0.8)$$

$$\boxed{dl^2 = \gamma_{ij} dx^i dx^j} \qquad (0.9)$$

γ_{ij} – metric tensor of space

$\{x^i\}$, $i = 1,2,3$ – the spacial coordinates

$$\gamma_{ij} = (\underline{e}_i \cdot \underline{e}_j) \qquad (0.10)$$

$\underline{e}_i = \frac{\partial}{\partial x^i}$ – basis vectors of spacial coordinates

$$\boxed{x^0 = ct} \qquad (0.11)$$

– time coordinate of spacetime

Thus, $\quad ds^2 = -dx^{0\,2} + \gamma_{ij} dx^i dx^j$

and

$$g_{\nu\mu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & (\gamma_{ij}) & \\ 0 & & & \end{pmatrix} \qquad (0.12)$$

– metric tensor of an inertial frame

$$g^{\nu\mu} = \begin{pmatrix} -1 & 0 \; 0 \; 0 \\ 0 & \\ 0 & (\gamma^{ij}) \\ 0 & \end{pmatrix} \qquad (0.13)$$

where (γ^{ij}) is inverse to (γ_{ij}),
that is $\gamma^{ik}\gamma_{kj} = \delta^i_j$.

$$g \equiv \det(g_{\nu\mu}) = -\gamma \qquad (0.14)$$

$$\gamma \equiv \det(\gamma_{ij}) \qquad (0.15)$$

Note that

$$\frac{\partial \gamma_{ij}}{\partial t} = 0 \,, \quad \frac{\partial g}{\partial t} = 0 \qquad (0.16)$$

– ~~[illegible]~~ static spacial grid ?

- <u>Components of vectors and tensors</u>

$$A^\nu = (\alpha, a^1, a^2, a^3) \equiv (\alpha, a^i) \qquad (0.17)$$

where α is a spacial scalar (3-scalar)
$\underline{a}$ is a spacial vector (3-vector)

$$A_\nu = (-\alpha, a_i), \qquad (0.18)$$

where $a_i = \gamma_{ij} a^j$.

For any second rank tensor

$$\Pi^{\nu\mu} = \begin{pmatrix} a & b^j \\ c^i & \Pi^{ij} \end{pmatrix} \qquad (0.19)$$

a – 3-scalar , $\vec{b}, \vec{c}$ – 3-vectors , Π^{ij} – 3-tensor

$$\Pi_{\nu\mu} = \begin{pmatrix} a & -b_i \\ -c_j & \Pi_{ij} \end{pmatrix} \qquad (0.20)$$

where $\Pi_{ij} = \gamma_{ia}\gamma_{jb}\Pi^{ab}$ are the covariant components of Π.

E.T.C.

- <u>Example of proof calculations</u>

$$\Pi_{0j} = g_{0\nu}\, g_{j\mu}\, \Pi^{\nu\mu} = g_{00}\, g_{j\mu}\, \Pi^{0\mu} =$$

$$= g_{00}\, g_{jk}\, \Pi^{0k} = -\gamma_{jk} b^k = -b_j \qquad \blacksquare$$

§1 Particle Dynamics

$$p^\nu = m_0 u^\nu \qquad (1.1)$$

– energy-momentum vector (4-momentum).

m_0 – rest mass of a particle

$$u^\nu = \frac{dx^\nu}{d\tau} \qquad (1.2) \qquad \text{- 4-velocity}$$

τ – proper time of the particle

- Energy-momentum conservation law:

$$\sum_{k=1}^{N} p^\nu_{(k)} = \text{const} \qquad (1.3)$$

(N particles)

- Inertial frame 3+1 splitting

$$u^\nu = (\Gamma c, \Gamma v^i) \qquad (1.4)$$

$$\Gamma = \frac{dt}{d\tau} \quad \text{- Lorentz factor} \quad (1.5)$$

$$v^i = \frac{dx^i}{dt} \qquad (1.6)$$

$$p^\nu = (m_0 \Gamma c, m_0 \Gamma v^i) = \left(\frac{E}{c}, \underline{p}\right) \qquad (1.7)$$

where

$$E = mc^2 \quad \text{- energy} \qquad (1.8)$$

$$m = \Gamma m_0 \quad \text{- inertial mass} \qquad (1.9)$$

$$\underline{p} = m\underline{v} \quad \text{- 3-momentum} \qquad (1.10)$$

The minimum energy

$$E_0 = m_0 c^2 \quad \text{– rest mass-energy} \qquad (1.11)$$

$$K = E - E_0 = m_0(\Gamma - 1)c^2 \qquad (1.12)$$

– kinetic energy

$$m = m_0 + m_0(\Gamma - 1) = m_0 + \frac{K}{c^2} \qquad (1.13)$$

(i) The rest mass-energy contributes to the total energy!

(ii) The kinetic energy contributes to particle's inertial mass!

For a gas element these mean that

(i) Its energy includes the rest mass of its particles.

(ii) Its inertial mass includes the gas thermal energy.

§2 Thermodynamics

It describes the properties of gas in the gas frame.

All these quantities are measured in the gas frame:

n – volume number density of particles

$\rho = m_0 n$ – volume rest-mass density

$P = nkT$ – gas pressure

T – temperature

$$\boxed{\mathcal{E} = \rho c^2 + \mathcal{E}_t} \quad (2.1)$$

– volume energy density.

$\mathcal{E}_t$ – thermal energy density

$$\boxed{w = \mathcal{E} + P} \quad (2.2)$$

– enthalpy

- Polytropic equation of state (EOS)

$$\boxed{\mathcal{E}_t = \frac{1}{\tilde{\gamma}-1} P} \quad (2.3)$$

$$\boxed{w = \rho c^2 + \frac{\tilde{\gamma}}{\tilde{\gamma}-1} P} \quad (2.4)$$

$1 < \tilde{\gamma} < 2$ – ratio of specific heats

§3 Hydrodynamics

3.1 The continuity equation

(The rest mass conservation)

Not always satisfied.

- Nonrelativistic case

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla_i \rho v^i = 0} \quad (3.1)$$

∇_i – covariant derivative

$$\nabla_i a^j = \frac{\partial a^j}{\partial x^i} - \Gamma^j_{ik} a^k \qquad (3.2)$$

where Γ^j_{ik} – Cristoffel's symbols

For divergence this can be simplified!

$$\boxed{\nabla_i a^i = \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^i}\left(\sqrt{\gamma}\, a^i\right)} \qquad (3.3)$$

where $\gamma = \det(\gamma_{ij})$.

- Example (Cartesian coordinates)

$$\gamma_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \Rightarrow \gamma = 1$$

$$\Rightarrow \nabla_i a^i = \frac{\partial a^i}{\partial x^i} = \frac{\partial a^1}{\partial x^1} + \frac{\partial a^2}{\partial x^2} + \frac{\partial a^3}{\partial x^3} \qquad (3.4)$$

▨

• Relativistic version

This should involve ρ, u^ν and ∇_ν, the covariant derivative of spacetime, in order to have a 4-tensor equation. The simplest one is

$$\boxed{\nabla_\nu (\rho u^\nu) = 0} \qquad (3.5)$$

For the 4-divergence we have

$$\boxed{\nabla_\nu A^\nu = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu} (\sqrt{-g} A^\nu)} \qquad (3.6)$$

where $\quad g = \det(g_{\nu\mu})$.

• 3+1 splitting of inertial frames

$$\nabla_\nu A^\nu = \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^\nu} (\sqrt{\gamma} A^\nu) = \frac{1}{\sqrt{\gamma}} \frac{1}{c} \frac{\partial}{\partial t} (\sqrt{\gamma} A^0) + \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^i} (\sqrt{\gamma} A^i) =$$

$$= \frac{1}{c} \frac{\partial}{\partial t} (A^0) + \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^i} (\sqrt{\gamma} A^i)$$

Thus. (3.5) reads

$$\frac{1}{c} \frac{\partial}{\partial t} (\rho u^0) + \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^i} (\sqrt{\gamma} \rho u^i) = 0$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t} (\rho \Gamma) + \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^i} (\sqrt{\gamma} \rho \Gamma v^i) = 0} \qquad (3.7)$$

This can also be written in vector form as

$$\frac{\partial}{\partial t} (\rho \Gamma) + \underline{\nabla} (\rho \Gamma \underline{v}) = 0 \, .$$

3.2 Energy-momentum equation

$$\boxed{\nabla_\nu T^{\nu\mu} = 0 \qquad \text{or} \qquad \nabla_\nu T^{\nu}{}_{\mu} = 0} \quad (3.8)$$

where $T^{\nu\mu} = T^{\mu\nu}$ is the stress-energy-momentum tensor

- 3+1 splitting at inertial frames:

T^{00} - energy density
cT^{0i} – 3-vector of energy flux
T^{0i}/c – 3-vector of momentum density
T^{ij} – 3-stress tensor (flux of momentum)

3.3 Stress-energy-momentum of perfect gas

In the fluid frame

$$T^{00} = \varepsilon \ , \ T^{0i} = T^{i0} = 0$$

$$T^{ij} = P\gamma^{ij} \quad \text{- pure pressure.}$$

Thus.

$$T^{\nu\mu} = \begin{pmatrix} \varepsilon & 0\ 0\ 0 \\ 0 & \\ 0 & (P\gamma^{ij}) \\ 0 & \end{pmatrix} \qquad (3.9)$$

In general $T^{\nu\mu}$ is given by the tensor equation

$$\boxed{T^{\nu\mu} = \frac{\omega}{c^2} u^\nu u^\mu + P g^{\nu\mu}} \qquad (3.10)$$

In order to verify (3.10) it is just sufficient to show that (3.10) reduces to (3.9) in the fluid frame, where

$$u^\nu = (c, \underline{0}) \qquad g^{\nu\mu} = \begin{pmatrix} -1 & 0\,0\,0 \\ \begin{matrix}0\\0\\0\end{matrix} & (\gamma^{ij}) \end{pmatrix} .$$

Indeed

$$T^{00} = \frac{w}{c^2} u^0 u^0 + p g^{00} = \frac{\varepsilon + p}{c^2} c^2 - p = \varepsilon$$

$$T^{0i} = \frac{w}{c^2} u^0 u^i + p g^{0i} = 0 + 0 = 0$$

$$T^{ij} = \frac{w}{c^2} u^i u^j + p g^{ij} = 0 + p\gamma^{ij} = p\gamma^{ij}$$

⊠

3.4 3+1 splitting of inertial frames

For the divergence of symmetric tensor one has

$$\boxed{\nabla_\nu T^\nu{}_\mu = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu}\left(\sqrt{-g}\, T^\nu{}_\mu\right) - \frac{1}{2} \frac{\partial g_{\alpha\beta}}{\partial x^\mu} T^{\alpha\beta}} \quad (3.10)$$

For inertial frames of special relativity

$$\nabla_\nu T^\nu{}_0 = \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^\nu}\left(\sqrt{\gamma}\, T^\nu{}_0\right) - \frac{1}{2c} \frac{\partial g_{\alpha\beta}}{\partial t} T^{\alpha\beta}$$

Since $\partial g_{\alpha\beta}/\partial t = 0$ (static grid)

$$\nabla_\nu T^\nu{}_0 = \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^\nu}\left(\sqrt{\gamma}\, T^\nu{}_0\right) = \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^0}\left(\sqrt{\gamma}\, T^0{}_0\right) + \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^i}\left(\sqrt{\gamma}\, T^i{}_0\right)$$

$$= \frac{1}{c} \frac{\partial}{\partial t}\left(T^0{}_0\right) + \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^i}\left(\sqrt{\gamma}\, T^i{}_0\right).$$

Thus, the energy equation is

$$\boxed{\frac{\partial}{\partial t}\left(T^0{}_0\right) + \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^i}\left(\sqrt{\gamma}\, c T^i{}_0\right) = 0} \quad (3.11)$$

Similarly we have the momentum equation:

$$\boxed{\frac{\partial}{\partial t}\left(T^{0}_{\;j}/c\right) + \frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x^{i}}\left(\sqrt{\gamma}\,T^{i}_{\;j}\right) - \frac{1}{2}\frac{\partial g_{ab}}{\partial x^{j}}T^{ab} = 0} \quad (3.12)$$

- If $\{x^i\}$ are Cartesian coordinates then

$\gamma = 1$, $\frac{\partial g_{ab}}{\partial x^{j}} = 0$

$T^{0}_{\;j} = T^{0j}$, $T^{i}_{\;j} = T^{ij}$, $T^{0}_{\;0} = -T^{00}$, $T^{i}_{\;0} = -T^{i0}$

and (3.11), (3.12) reduce to

$$\frac{\partial}{\partial t}T^{00} + \frac{\partial}{\partial x^{i}}\left(cT^{i0}\right) = 0 \qquad (3.13)$$

$$\frac{\partial}{\partial t}\left(T^{0j}/c\right) + \frac{\partial}{\partial x^{i}}\left(T^{ij}\right) = 0 \qquad (3.14)$$

Substituting $T^{\nu\mu}$ from (3.10) one obtains

$$\boxed{\frac{\partial}{\partial t}\left(\omega\Gamma^{2} - p\right) + \frac{\partial}{\partial x^{i}}\left(\omega\Gamma^{2}v^{i}\right) = 0} \qquad (3.15)$$

$$\boxed{\frac{\partial}{\partial t}\left(\frac{\omega\Gamma^{2}}{c^{2}}v^{i}\right) + \frac{\partial}{\partial x^{i}}\left(\frac{\omega\Gamma^{2}}{c^{2}}v^{i}v^{j} + p\,\delta^{ij}\right) = 0} \qquad (3.16)$$

where δ^{ij} is the Kronecker's delta.

§4 Electromagnetism

4.1 Levi-Civita tensor of space

Levi-Civita symbol:

$$\epsilon^{ijk} \equiv \epsilon_{ijk} = \begin{cases} 1 & \text{if } i,j,k \text{ is even permutation of } 1,2 \\ -1 & \text{if } \text{---''---} \text{ odd } \text{---''---} \\ 0 & \text{otherwise} \end{cases} \quad (4.1)$$

Levi-Civita tensor of space

$$e_{ijk} = \begin{cases} \sqrt{\gamma}\,\epsilon_{ijk} & \text{for right-handed coordinates} \\ -\sqrt{\gamma}\,\epsilon_{ijk} & \text{for left-handed coordinates} \end{cases} \quad (4.2)$$

$$e^{ijk} = \begin{cases} \frac{1}{\sqrt{\gamma}}\,\epsilon^{ijk} & \text{for right-handed} \\ -\frac{1}{\sqrt{\gamma}}\,\epsilon^{ijk} & \text{for left-handed} \end{cases} \quad (4.3)$$

Usage

1) $\underline{E} = \underline{B} \times \underline{V} \iff \begin{cases} E_i = e_{ijk} B^j V^k \\ E^i = e^{ijk} B_j V_k \end{cases} \quad (4.4)$

2) $\underline{B} = \underline{\nabla} \times \underline{A} \iff \begin{cases} B^i = e^{ijk} \nabla_j A_k \\ B_i = e_{ijk} \nabla^j A^k \end{cases} \quad (4.5)$

3) Dual vectors and anti-symmetric tensors:

$$\boxed{\begin{array}{ll} \text{If} & F^{ij} = \epsilon^{ijk} B_k \\ \text{then} & B_k = \frac{1}{2} \epsilon_{kij} F^{ij} \end{array}} \qquad (4.6)$$

Notice that

$$F^{ij} = -F^{ji}$$

and has only 3 independent components, just as B^k. Hence both contain the same amount of information.

4.2 The Maxwell tensor

$\underline{B}$ and $\underline{E}$ are 3-vectors. Hence 6 components.

For a covariant formulation of electrodynamics we need a spacetime tensor.

(i) 1-rank tensor (4-vector) → 4 components → too few
(ii) 2-rank tensor → 16 components → too many
(iii) 2-rank symmetric tensor → 10 components → still too many
(iv) 2-rank anti-symmetric tensor → 6 components → exact match!

Hence introduce an anti-symmetric tensor

$$F^{\nu\mu} = -F^{\mu\nu}, \qquad (4.7)$$

the Maxwell tensor, to describe the electromagnetic field.

The Lorentz 4-force acting on a charge q with 4-velocity u^ν has to be

$$\phi^\nu = \alpha q F^{\nu\mu} u_\mu \qquad (4.8)$$

where α is a space-time scalar (constant)

~~Fr~~ <u>3+1 splitting of inertial frames</u>

We have to split $F^{\mu\nu}$ into $\underline{E}$ and $\underline{B}$ of inertial frame in such a way that (4.8) give the Lorentz 3-force

$$\underline{f} = q(\underline{E} + \tfrac{1}{c}\underline{v}\times\underline{B}) \qquad (4.9)$$

If we put

$$\boxed{E_i = F_{i0} \quad (E^i = -F^{i0})} \qquad (4.10)$$

and

$$\boxed{\begin{array}{l} B_i = \frac{1}{2} e_{ijk} F^{jk} \\ F^{jk} = e^{jks} B_s \end{array}} \qquad (4.11)$$

then we recover (4.9) from (4.8) if $\alpha = \frac{1}{c}$. Thus.

$$\boxed{\phi^\nu = \frac{q}{c} F^{\nu\mu} u_\mu} \qquad (4.12).$$

<u>Exercise</u> Derive (4.9) from (4.11) (4.12).

$$\phi_0 = \frac{dP_0}{d\tau} \Rightarrow \phi_i = \frac{dP_i}{d\tau} = \Gamma \frac{dP_i}{dt} = \Gamma f_i \quad (*)$$

$$(4.12) \Rightarrow \phi_i = \frac{q}{c}\left(F_{i0}u^0 + F_{ij}u^j\right) =$$

$$= \frac{qu^0}{c}\left(F_{i0} + F_{ij}\frac{u^j}{u^0}\right) = \frac{qu^0}{c}\left(E_i + \frac{1}{c}e_{ijk}B^k V^j\right) =$$

$$= q\Gamma\left(E_i + \frac{1}{c}e_{ijk}V^j B^k\right).$$

$$\Rightarrow f_i = q\left(E_i + \frac{1}{c}e_{ijk}V^j B^k\right)$$

or $\underline{f} = q\left(\underline{E} + \frac{1}{c}\underline{V}\times\underline{B}\right)$ ☒

According to (4.10) and (4.11) the 3+1 splitting is

$$F^{\nu\mu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & -B_1 & B_2 & 0 \end{pmatrix} \quad (4.13)$$

4.3 The Faraday tensor

4-dimensional Levi-Civita symbol:

$$\epsilon^{\alpha\beta\gamma\delta} = \epsilon_{\alpha\beta\gamma\delta} = \begin{cases} 1 & \text{if } \alpha,\beta,\gamma,\delta \text{ is an even permutation of } 0,1,\ldots \\ -1 & \text{———} \text{"} \text{——— odd ———} \text{"} \text{———} \\ 0 & \text{otherwise} \end{cases} \qquad (4.14)$$

Levi-Civita tensor of space-time

$$\boldsymbol{\epsilon}_{\alpha\beta\delta\gamma} = \begin{cases} \sqrt{-g}\,\epsilon_{\alpha\beta\delta\gamma} & \text{for right-handed coordinates} \\ -\sqrt{-g}\,\epsilon_{\alpha\beta\delta\gamma} & \text{for left-handed} \end{cases} \qquad (4.15)$$

From this it follows that

$$\boldsymbol{\epsilon}^{\alpha\beta\delta\gamma} = \begin{cases} -\frac{1}{\sqrt{-g}}\,\epsilon^{\alpha\beta\delta\gamma} & \text{for right-handed systems} \\ \frac{1}{\sqrt{-g}}\,\epsilon^{\alpha\beta\delta\gamma} & \text{for left-handed} \end{cases} \qquad (4.16)$$

The Faraday tensor

$$\boxed{{}^*F^{\mu\nu} = \frac{1}{2}\,\boldsymbol{\epsilon}^{\mu\nu\alpha\beta} F_{\alpha\beta}} \qquad (4.17)$$

From this we have

$$\boxed{{}^*F^{\mu\nu} = -{}^*F^{\nu\mu}} \qquad (4.18)$$

${}^*F^{\nu\mu}$ is also anti-symmetric and hence contains the same amount of information as $F^{\nu\mu}$. Hence, it can also describe the electromagnetic field.

From (4.17) it follows that

$$\boxed{F^{\mu\nu} = -\frac{1}{2} e^{\mu\nu\alpha\beta}\, {}^*F_{\alpha\beta}}\,. \qquad (4.19)$$

• <u>3+1 splitting of inertial frames</u>

$$\boxed{B_i = {}^*F_{0i}} \qquad (4.20)$$

$$\boxed{E^i = \frac{1}{2} e^{ijk}\, {}^*F_{jk}} \qquad (4.21)$$

4.4 Maxwell equations

$$\boxed{\nabla_\beta\, {}^*F^{\alpha\beta} = 0 \quad \text{– Faraday group}} \qquad (4.22)$$

$$\boxed{\nabla_\beta F^{\alpha\beta} = \frac{4\pi}{c} I^\alpha \quad \text{– Maxwell group}} \qquad (4.23)$$

where I^α is the electric current 4-vector

• <u>3+1 splitting of inertial frames</u>

$$I^\alpha = (cq, \underline{j}) \qquad (4.24)$$

q - electric charge density
$\underline{j}$ - electric current density

$$(4.22) \Rightarrow \begin{cases} \nabla \cdot \underline{B} = 0 & (4.25) \\ \frac{1}{c}\frac{\partial B}{\partial t} + \nabla \times \underline{E} = 0 & (4.26) \end{cases}$$

$$(4.23) \Rightarrow \begin{cases} \nabla \cdot \underline{E} = 4\pi q & (4.27) \\ -\frac{1}{c}\frac{\partial E}{\partial t} + \nabla \times \underline{B} = \frac{4\pi}{c}\underline{j} & (4.28) \end{cases}$$

These are obtained using the result

$$\boxed{\nabla_\beta F^{\alpha\beta} = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\beta}\left(\sqrt{-g}\, F^{\alpha\beta}\right)} \qquad (4.29)$$

for any anti-symmetric tensor.

- **Exercise** (Derive (4.25) and (4.26))

(4.22) reads

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\beta}\left(\sqrt{-g}\,{}^*F^{\alpha\beta}\right) = 0$$

$$\Rightarrow \frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x^\beta}\left(\sqrt{\gamma}\,{}^*F^{\alpha\beta}\right) = 0 \Rightarrow \frac{\partial}{\partial x^\beta}\left(\sqrt{\gamma}\,{}^*F^{\alpha\beta}\right) = 0$$

$$\Rightarrow \frac{1}{c}\frac{\partial}{\partial t}\left(\sqrt{\gamma}\,{}^*F^{\alpha 0}\right) + \frac{\partial}{\partial x^i}\left(\sqrt{\gamma}\,{}^*F^{\alpha i}\right) = 0$$

$$\Rightarrow \frac{1}{c}\frac{\partial}{\partial t}\left({}^*F^{\alpha 0}\right) + \frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x^i}\left(\sqrt{\gamma}\,{}^*F^{\alpha i}\right) = 0$$

$\underline{\alpha = 0}$

$$\frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x^i}\left(\sqrt{\gamma}\,{}^*F^{0 i}\right) = 0$$

$$\Rightarrow \frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x^i}\left(\sqrt{\gamma}\, B^i\right) = 0 \Rightarrow \nabla \cdot \underline{B} = 0 \quad //$$

$\alpha = j$

$$\frac{1}{c}\frac{\partial}{\partial t}{}^*F^{j0} + \frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x^i}\left(\sqrt{\gamma}\,{}^*F^{ji}\right) = 0 \qquad (*)$$

$${}^*F^{j0} = -{}^*F^{0j} = B^j$$
$${}^*F^{ji} = e^{jik}E_k = \frac{1}{\sqrt{\gamma}}\epsilon^{jik}E_k$$

Thus $(*)$ reads

$$\frac{1}{c}\frac{\partial}{\partial t}(B^j) + \frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x^i}\left(\epsilon^{jik}E_k\right) = 0$$

$$\Rightarrow \frac{1}{c}\frac{\partial}{\partial t}(B^j) + \frac{1}{\sqrt{\gamma}}\epsilon^{jik}\frac{\partial}{\partial x^i}(E_k) = 0$$

$$\Rightarrow \frac{1}{c}\frac{\partial}{\partial t}(B^j) + e^{jik}\frac{\partial}{\partial x^i}E_k = 0$$

$$\Rightarrow \frac{1}{c}\frac{\partial \underline{B}}{\partial t} + \underline{\nabla}\times\underline{E} = 0 \qquad //$$

$\blacksquare$

4.5 Stress-energy-momentum tensor

$$\boxed{T^{\mu\nu} = \frac{1}{4\pi}\left(F^{\mu\gamma}F^{\nu}{}_{\gamma} - \frac{1}{4}\left(F_{\alpha\beta}F^{\alpha\beta}\right)g^{\mu\nu}\right)} \qquad (4.30)$$

- 3+1 splitting of inertial frames

$$T^{00} = \frac{1}{8\pi}(E^2+B^2), \quad T^{0i} = \frac{1}{4\pi}e^{ijk}E_jB_k$$

$$T^{ij} = \frac{1}{4\pi}\left[-(E^iE^j + B^iB^j) + \frac{1}{2}\gamma^{ij}(E^2+B^2)\right] \qquad (4.31)$$

4.6 Ohm's Law

The simplest 3+1 form in the fluid frame is

$$\boxed{\underline{j} = \sigma \underline{E}} \qquad (4.32)$$

where σ is the conductivity.

The corresponding covariant form is

$$\boxed{I_\nu = \frac{\sigma}{c} F_{\nu\mu} u^\mu + q_0 u_\nu} \qquad (4.33)$$

where

$$q_0 = -\frac{I_\nu u^\nu}{c^2} \qquad (4.34)$$

is the electric charge density in the fluid frame.

- Exercise (Derive (4.32) from (4.33))

In the fluid frame $u^\nu = (c, \underline{0})$, $u_\nu = (-c, \underline{0})$,
$I_\nu = (-c q_0, \underline{j})$, $F_{\nu\mu} u^\mu = F_{\nu 0}\, c = (0, \underline{E} c)$

Thus (4.33) reads

$$(-cq, \underline{j}) = \frac{\sigma}{c}(0, \underline{E} c) + q_0(-c, \underline{0})$$

$$\Rightarrow (-cq, \underline{j}) = (-c q_0, \sigma \underline{E})$$

$$\Rightarrow \begin{cases} q = q_0 \\ \underline{j} = \sigma \underline{E} \end{cases}$$

- **The perfect conductivity limit** ($\sigma \to \infty$)

I^ν remains finite only if $F^{\nu\mu} u_\mu \to 0$
Thus, in this limit

$$\boxed{F^{\nu\mu} u_\mu = 0} \qquad (4.35)$$

In the fluid frame (4.35) reads

$$F^{\nu 0} = 0 \Rightarrow F^{i0} = 0 \Rightarrow \boxed{\underline{E} = 0} \qquad (4.36)$$

General 3+1 splitting of inertial frames

$$F^{\nu\mu} u_\mu = 0 \begin{cases} F^{0\mu} u_\mu = 0 \\ F^{i\mu} u_\mu = 0 \end{cases}$$

$$F^{0\mu} u_\mu = 0 \Rightarrow F^{0i} u_i = 0 \Rightarrow E^i v_i = 0 \Rightarrow$$

$$\Rightarrow \boxed{(\underline{E} \cdot \underline{v}) = 0} \qquad (4.37)$$

$$F^{i\mu} u_\mu = F^{i0} u_0 + F^{ij} u_j = 0$$

$$\Rightarrow E^i c\Gamma + e^{ijk} B_k \Gamma v_j = 0$$

$$\Rightarrow E^i + \frac{1}{c} e^{ijk} v_j B_k = 0$$

$$\Rightarrow \boxed{\underline{E} + \frac{1}{c} \underline{v} \times \underline{B} = 0} \qquad (4.38)$$

— the same result as in non-relativistic MHD

§5 Relativistic MHD

5.1 Resistive RMHD

- The energy-momentum equation

$$\boxed{\nabla_\mu \left(T^{\nu\mu}_{(m)} + T^{\nu\mu}_{(e)} \right) = 0} \qquad (5.1)$$

$T^{\nu\mu}_{(m)}$ – tensor of matter (eq. 3.10)

$T^{\nu\mu}_{(e)}$ – tensor of electromagnetic field (eq 4.30)

Using the Maxwell equations

$$\nabla_\mu \left(T^{\nu\mu}_{(m)} \right) = -\nabla_\mu T^{\nu\mu}_{(e)} = \frac{1}{c} F^{\mu\gamma} I_\gamma \qquad (5.2)$$

Here
$$\phi^\mu = \frac{1}{c} F^{\mu\gamma} I_\gamma \qquad (5.3)$$
is the MHD version of the Lorentz 4-force. (Compare with eq. 4.12)

- The continuity equation

$$\boxed{\nabla_\alpha \rho u^\alpha = 0} \qquad (5.4)$$

- Maxwell's equations

$$\boxed{\nabla_\beta {}^*F^{\alpha\beta} = 0} \qquad (5.5)$$

$$\boxed{\nabla_\beta F^{\alpha\beta} = I^\beta} \qquad (5.6)$$

- Ohm's law

$$\boxed{I_\nu = \frac{\sigma}{c} F_{\nu\mu} u^\mu + q_0 u_\nu} \quad (5.7)$$

$$q_0 = -\frac{1}{c^2}(I_\nu u^\nu)$$

5.2 Ideal RMHD

$$\sigma = \infty \implies \boxed{F_{\nu\mu} u^\mu = 0} \quad (5.8)$$

This equation replaces the Maxwell equation (5.6)

5.3 Ideal RMHD in the Lichnerowicz formulation

Introduce 4-vectors of electric

$$\boxed{e_\nu = \frac{1}{c} F_{\nu\mu} u^\mu} \quad (5.9)$$

and magnetic fields

$$\boxed{b_\nu = -\frac{1}{c} {}^*F_{\nu\mu} u^\mu} \quad (5.10).$$

From this we have

$$e_\nu u^\nu = 0 \quad (5.11)$$

$$b_\nu u^\nu = 0 \quad (5.12)$$

- <u>3+1 splitting of e_ν and b_ν in the fluid frame</u>

$$u^\mu = (c, \underline{0}) \Rightarrow e_\nu = F_{\nu 0} = (0, \underline{E}) \quad (5.13)$$

$$b_\nu = -\frac{1}{c}\, {}^*F_{\nu 0}\, c = (0, \underline{B}) \quad (5.14)$$

- <u>$F_{\nu\mu}$ and ${}^*F_{\nu\mu}$ in terms of b_ν, e_ν and u_ν</u>

$$F_{\nu\mu} = \frac{1}{c}\left[u_\nu e_\mu - e_\nu u_\mu + e_{\nu\mu\alpha\beta} u^\alpha b^\beta\right] \quad (5.15)$$

$${}^*F_{\nu\mu} = \frac{1}{c}\left[b_\nu u_\mu - u_\nu b_\mu + e_{\nu\mu\alpha\beta} u^\alpha e^\beta\right] \quad (5.16)$$

- ~~RMHD equations~~
<u>Ideal RMHD equations</u>

Since in the fluid frame $\underline{E} = 0$ we have $e_\nu = 0$. See also (5.9) and (5.8).

Then

$$F_{\nu\mu} = \frac{1}{c} e_{\nu\mu\alpha\beta} u^\alpha b^\beta \quad (5.17)$$

$${}^*F_{\nu\mu} = \frac{1}{c}\left[b_\nu u_\mu - u_\nu b_\mu\right] \quad (5.18)$$

Combining (5.17) with (4.30) one obtains

$$T^{\nu\mu}_{(e)} = \frac{1}{4\pi}\left[\frac{b^2}{c^2} u^\nu u^\mu - b^\nu b^\mu + \frac{b^2}{2} g^{\nu\mu}\right] \quad (5.19)$$

Given (5.18) the Faraday equation becomes

$$\boxed{\nabla_\mu \left(b^\nu u^\mu - b^\mu u^\nu \right) = 0} \qquad (5.20)$$

Given (5.19) and (3.10) the energy-momentum equation becomes

$$\boxed{\nabla_\mu \left[\left(w + \frac{b^2}{4\pi} \right) \frac{u^\nu u^\mu}{c^2} - \frac{b^\nu b^\mu}{4\pi} + \left(p + \frac{b^2}{4\pi} \right) g^{\nu\mu} \right] = 0} \qquad (5.21)$$

The continuity equation is as before

$$\boxed{\nabla_\alpha \left(\rho u^\alpha \right) = 0} \qquad (5.23)$$

These PDEs are complemented with

$w = w(p, \rho)$ - EOS

$b_\nu u^\nu = 0$

$u_\nu u^\nu = -c^2$

Notice that the magnetic field contributes not only to stress-tensor but also to fluid inertia via the term $b^2/4\pi$ in

$$\left(w + \frac{b^2}{4\pi} \right) \frac{u^\nu u^\mu}{c^2} \quad \text{in (5.21)}$$