Synchrotron Radiation

The synchrotron radiation, the emission of very relativistic and ultrarelativistic electrons gyrating in a magnetic field, is the process which dominates much of high energy astrophysics. It was originally observed in early betatron experiments in which electrons were first accelerated to ultrarelativistic energies. This process is responsible for the radio emission from the Galaxy, from supernova remnants and extragalactic radio sources. It is also responsible for the non-thermal optical and X-ray emission observed in the Crab Nebula and possibly for the optical and X-ray continuum emission of quasars.

The word *non-thermal* is used frequently in high energy astrophysics to describe the emission of high energy particles. This an unfortunate terminology since all emission mechanisms are 'thermal' in some sense. The word is conventionally taken to mean 'continuum radiation from particles, the energy spectrum of which is not Maxwellian'. In practice, continuum emission is often referred to as 'non-thermal' if it cannot be described by the spectrum of thermal bremsstrahlung or black-body radiation.

Motion of an Electron in a Uniform, Static Magnetic field

We begin by writing down the equation of motion for a particle of rest mass m_0, charge ze and Lorentz factor $\gamma = (1 - v^2/c^2)^{-1/2}$ in a uniform static magnetic field $\boldsymbol{B}$.

$$\frac{\mathrm{d}}{\mathrm{d}t}(\gamma m_0 \boldsymbol{v}) = ze(\boldsymbol{v} \times \boldsymbol{B}) \tag{1}$$

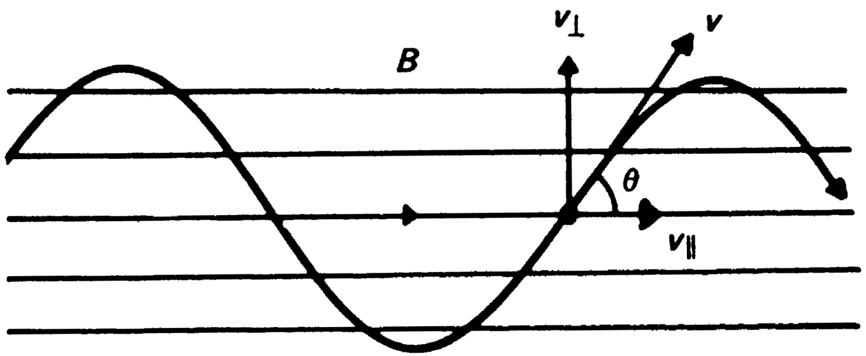
We recall that the left-hand side of this equation can be expanded as follows:

$$m_0 \frac{\mathrm{d}}{\mathrm{d}t}(\gamma \boldsymbol{v}) = m_0 \gamma \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} + m_0 \gamma^3 \boldsymbol{v} \frac{(\boldsymbol{v} \cdot \boldsymbol{a})}{c^2} \tag{2}$$

because the Lorentz factor γ should be written $\gamma = (1 - \boldsymbol{v} \cdot \boldsymbol{v}/c^2)^{-1/2}$. In a magnetic field, the three-acceleration $\boldsymbol{a} = \mathrm{d}\boldsymbol{v}/\mathrm{d}t$ is always perpendicular to $\boldsymbol{v}$ and consequently $\boldsymbol{v} \cdot \boldsymbol{a} = 0$. As a result,

$$\gamma m_0 \, \mathrm{d}\boldsymbol{v}/\mathrm{d}t = ze(\boldsymbol{v} \times \boldsymbol{B}) \tag{3}$$

Motion of an Electron in a Uniform, Static Magnetic field



We now split $\boldsymbol{v}$ into components parallel and perpendicular to the uniform magnetic field, $v_\parallel$ and $v_\perp$ respectively. The pitch angle θ of the particle's path is given by $\tan\theta = v_\perp / v_\parallel$, that is, θ is the angle between the vectors $\boldsymbol{v}$ and $\boldsymbol{B}$. Since $v_\parallel \times \boldsymbol{B} = 0$, $v_\parallel =$ constant. The acceleration is perpendicular to the magnetic field direction and to $v_\parallel$.

$$\gamma m_0 \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = zev_\perp B(\boldsymbol{i}_\perp \times \boldsymbol{i}_{\boldsymbol{B}}) = zevB(\boldsymbol{i}_{\boldsymbol{v}} \times \boldsymbol{i}_{\boldsymbol{B}}) \quad (4)$$

where $\boldsymbol{i}_{\boldsymbol{v}}$ and $\boldsymbol{i}_{\boldsymbol{B}}$ are unit vectors in the directions of $\boldsymbol{v}$ and $\boldsymbol{B}$ respectively.

Gyrofrequencies

Thus, the motion of the particle consists of a constant velocity along the magnetic field direction and circular motion with radius r about it. This means that the particle moves in a *spiral path* with *constant pitch angle* θ. The radius r is known as the *gyroradius* of the particle. The angular frequency of the particle in its orbit ω_{g} is known as the *angular cyclotron frequency* or *angular gyrofrequency* and is given by

$$\omega_{\mathrm{g}} = v_{\perp}/r = zeB/\gamma m_0 \tag{5}$$

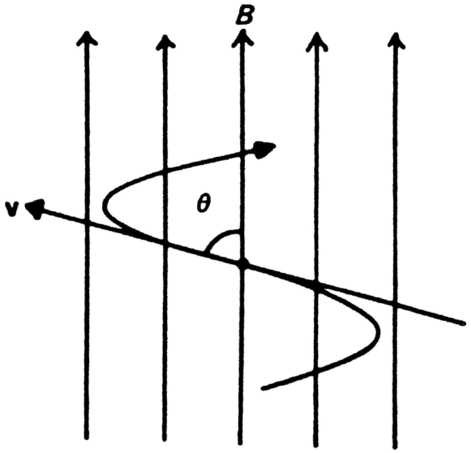
The corresponding *gyrofrequency* ν_{g}, that is, the number of times per second that the particle rotates about the magnetic field direction, is

$$\nu_{\mathrm{g}} = \omega_g/2\pi = zeB/2\pi\gamma m_0 \tag{6}$$

In the case of a non-relativistic particle, $\gamma = 1$ and hence $\nu_{\mathrm{g}} = zeB/2\pi m_0$.

A useful figure to remember is the non-relativistic gyrofrequency of an electron $\nu_{\mathrm{g}} = eB/2\pi m_{\mathrm{e}} = 28$ GHz T^{-1} where the magnetic field strength is measured in tesla; alternatively, $\nu_{\mathrm{g}} = 2.8$ MHz G^{-1} for those not yet converted from gauss (G) to teslas (T).

The Total Energy Loss Rate

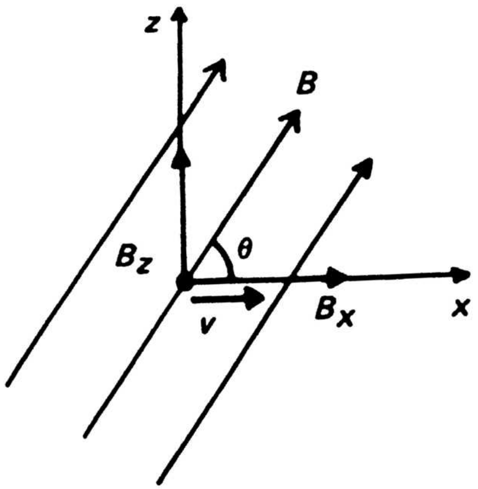


Most of the essential tools needed in this analysis have already been derived. First of all, use the expression for the acceleration of the electron in its orbit and insert this into the expression for the radiation rate of a relativistic electron. The acceleration is always perpendicular to the velocity vector of the particle and to $\boldsymbol{B}$ and hence $a_{\parallel} = 0$. Therefore, the total radiation loss rate of the electron is

$$-\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right) = \frac{\gamma^4 e^2}{6\pi\varepsilon_0 c^3}|a_{\perp}|^2 = \frac{\gamma^4 e^2}{6\pi\varepsilon_0 c^3}\frac{e^2 v^2 B^2 \sin^2\theta}{\gamma^2 m_{\mathrm{e}}^2} \quad (7)$$

$$= \frac{e^4 B^2}{6\pi\varepsilon_0 c m_{\mathrm{e}}^2}\frac{v^2}{c^2}\gamma^2 \sin^2\theta \quad (8)$$

The Total Energy Loss Rate - Another Approach



Let us start from the fact that in the instantaneous rest frame of the electron, the acceleration of the particle is small and so we can use the non-relativistic expression for the radiation rate. The instantaneous direction of motion of the electron in the laboratory frame, the frame in which $\boldsymbol{B}$ is fixed, is taken as the positive x-axis. We now transform the field quantities into the instantaneous rest frame of the electron. In S', the force on the electron is

$$\boldsymbol{f}' = m_e \dot{\boldsymbol{v}}' = e(\boldsymbol{E}' + \boldsymbol{v}' \times \boldsymbol{B}') = e\boldsymbol{E}' \qquad (9)$$

since the particle is instantaneously at rest in S'. Therefore, in transforming $\boldsymbol{B}$ into S', we need only consider the transformed components for the electric field $\boldsymbol{E}'$.

$$\begin{aligned} E'_x &= E_x \\ E'_y &= \gamma(E_y - vB_z) \\ E'_z &= \gamma(E_z + vB_y) \end{aligned} \quad \text{and so} \quad \begin{aligned} E'_x &= 0 \\ E'_y &= -v\gamma B_z = -v\gamma B \sin\theta \\ E'_z &= 0 \end{aligned}$$

Therefore

$$\dot{\boldsymbol{v}}' = -\frac{e\gamma v B \sin\theta}{m_{\mathrm{e}}} \tag{10}$$

Consequently, in the rest frame of the electron, the loss rate by radiation is

$$-\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)' = \frac{e^2|\dot{\boldsymbol{v}}'|^2}{6\pi\varepsilon_0 c^3} = \frac{e^4\gamma^2 B^2 v^2 \sin^2\theta}{6\pi\varepsilon_0 c^3 m_{\mathrm{e}}^2} \tag{11}$$

Since $(\mathrm{d}E/\mathrm{d}t)$ is a Lorentz invariant, we recover the formula (8). Recalling that $c^2 = (\mu_0\varepsilon_0)^{-1}$, let us rewrite this in the following way

$$-\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right) = 2\left(\frac{e^4}{6\pi\varepsilon_0^2 c^4 m_{\mathrm{e}}^2}\right)\left(\frac{v}{c}\right)^2 c\,\frac{B^2}{2\mu_0}\,\gamma^2\ \sin^2\theta \tag{12}$$

The quantity in the first set of round brackets on the right-hand side of (8) is the Thomson cross-section σ_{T}. Therefore

$$\boxed{-\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right) = 2\sigma_{\mathrm{T}} c U_{\mathrm{mag}}\left(\frac{v}{c}\right)^2 \gamma^2 \sin^2\theta} \tag{13}$$

where $U_{\mathrm{mag}} = B^2/2\mu_0$ is the energy density of the magnetic field.

The Average Energy Loss Rate

In the ultrarelativistic limit, $v \rightarrow c$, we may approximate this result by

$$-\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right) = 2\sigma_{\mathrm{T}} c U_{\mathrm{mag}} \gamma^2 \sin^2\theta \quad (14)$$

These results apply for electrons of a specific pitch angle θ. Particles of a particular energy E, or Lorentz factor γ, are often expected to have an isotropic distribution of pitch angles and therefore we can work out their average energy loss rate by averaging over such a distribution of pitch angles $p(\theta)\,\mathrm{d}\theta = \frac{1}{2}\sin\theta\,\mathrm{d}\theta$

$$\begin{aligned} -\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right) &= 2\sigma_{\mathrm{T}} c U_{\mathrm{mag}} \gamma^2 \left(\frac{v}{c}\right)^2 \frac{1}{2}\int_0^{\pi} \sin^3\theta d\theta \\ &= \tfrac{4}{3}\sigma_{\mathrm{T}} c U_{\mathrm{mag}} \left(\frac{v}{c}\right)^2 \gamma^2 \quad (15) \end{aligned}$$

There is a deeper sense in which (15) is the average loss rate for a particle of energy E. During its lifetime, it is likely that the high energy particle is randomly scattered in pitch angle and then (15) is the correct expression for its average energy loss rate.

Non-relativistic gyroradiation and cyclotron radiation

Consider first the simplest case of non-relativistic gyroradiation, in which case $v \ll c$ and hence $\gamma = 1$. Then, the expression for the loss rate of the electron is

$$-\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right) = 2\sigma_T c U_{\mathrm{mag}} \left(\frac{v}{c}\right)^2 \sin^2\theta = \frac{2\sigma_T}{c} U_{\mathrm{mag}} v_\perp^2 \qquad (16)$$

and the radiation is emitted at the gyrofrequency of the electron $\nu_{\mathrm{g}} = eB/2\pi m_{\mathrm{e}}$.

In the non-relativistic case, there are no beaming effects and the polarisation observed by the distant observer can be derived from the rules given above. When the magnetic field is perpendicular to the line of sight, *linearly polarised radiation* is observed because the acceleration vector is observed to perform simple harmonic motion in a plane perpendicular to the magnetic field by the distant observer. The electric field strength varies sinusoidally at the gyrofrequency. When the magnetic field is parallel to the line of sight, the acceleration vector is continually changing direction as the electron moves in a circular orbit about the magnetic field lines and therefore the radiation is observed to be 100% *circularly polarised*. Between these cases, the radiation is observed to be *elliptically polarised*, the ratio of axes of the polarisation ellipse being $\cos\theta$.

Mildly Relativistic Cyclotron Radiation

Even for slowly moving electrons, $v \ll c$, not all the radiation is emitted at the gyrofrequency because there are small aberration effects which slightly distort the polar diagram from a $\cos\theta$ law. From the symmetry of these aberrations, the observed polar diagram of the radiation can be decomposed by Fourier analysis into a sum of equivalent dipoles radiating at harmonics of the relativistic gyrofrequency ν_{r} where $\nu_{\mathrm{r}} = \nu_{\mathrm{g}}/\gamma$. These harmonics have frequencies

$$\nu_l = l\nu_{\mathrm{r}} / \left(1 - \frac{v_{\parallel}}{c}\cos\theta\right) \tag{17}$$

where l takes integral values, $l = 1, 2, 3,$ .. and the fundamental gyrofrequency has $l = 1$. The factor $[1 - (v_{\parallel}/c)\cos\theta]$ in the denominator takes account of the Doppler shift of the radiation of the electron due to its translational velocity along the field lines, projected onto the line of sight to the observer, $v_{\parallel}$. In the limit $lv/c \ll 1$, it can be shown that the total power emitted in a given harmonic for the case $v_{\parallel} = 0$ is

$$-\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_l = \frac{2\pi e^2 \nu_{\mathrm{g}}^2}{\varepsilon_0 c}\frac{(l+1)l^{2l+1}}{(2l+1)!}\left(\frac{v}{c}\right)^2 \tag{18}$$

Mildly Relativistic Cyclotron Radiation

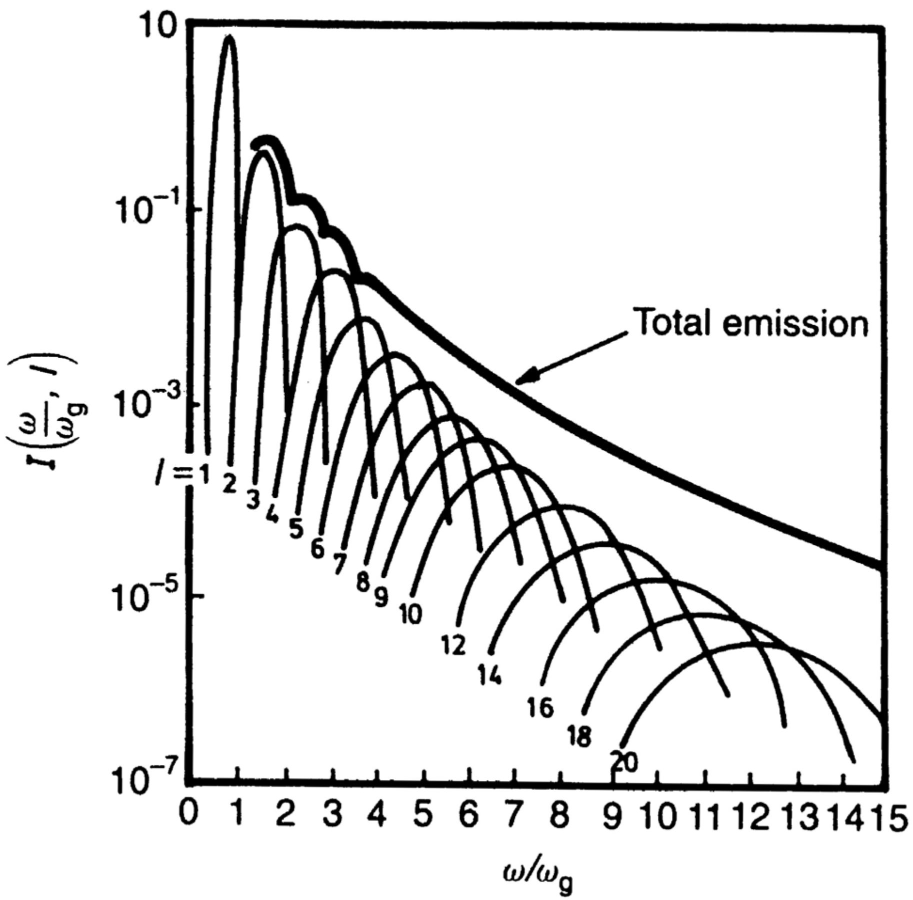
Hence, to order of magnitude,

$$\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{l+1} \Big/ \left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{l} \approx \left(\frac{v}{c}\right)^2 \tag{19}$$

Thus, the energy radiated in high harmonics is small when the particle is non-relativistic.

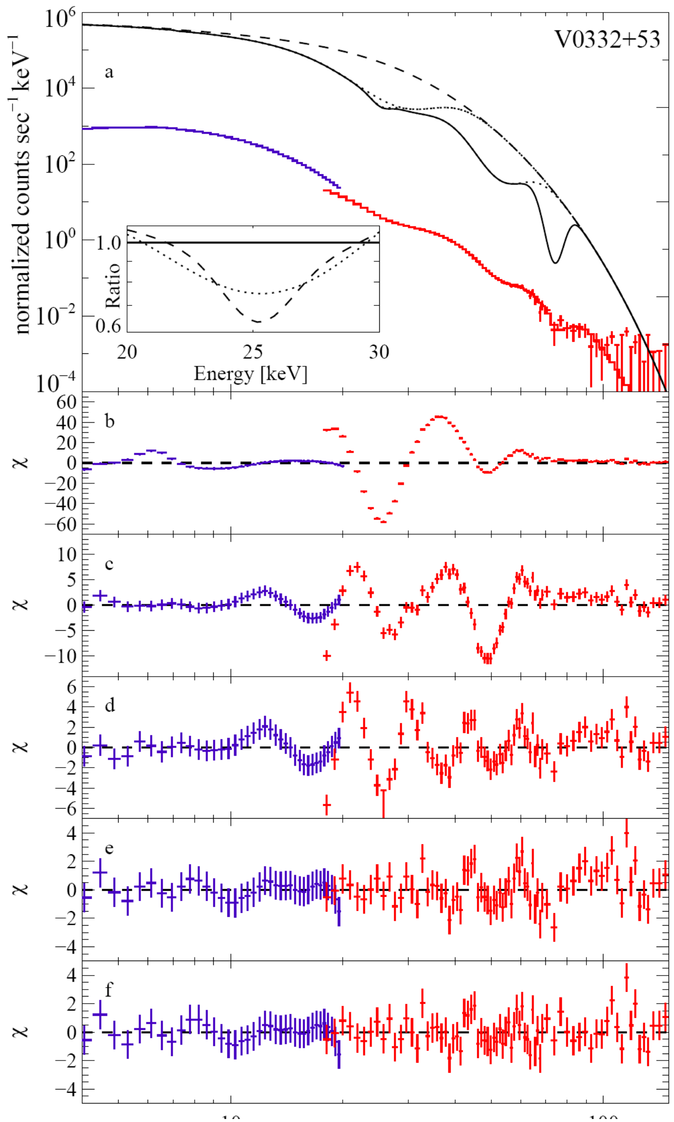
When the particle becomes significantly relativistic, $v/c \geq 0.1$, the energy radiated in the higher harmonics becomes important. The Doppler corrections to the observed frequency of the emitted radiation become significant and a wide spread of emitted frequencies is associated with the different pitch angles of an electron of energy $E = \gamma mc^2$. The result is broadening of the width of the emission line of a given harmonic and, for high harmonics, the lines are so broadened that the emission spectrum becomes continuous rather than consisting of a series of harmonics at well defined frequencies.

Mildly Relativistic Cyclotron Radiation



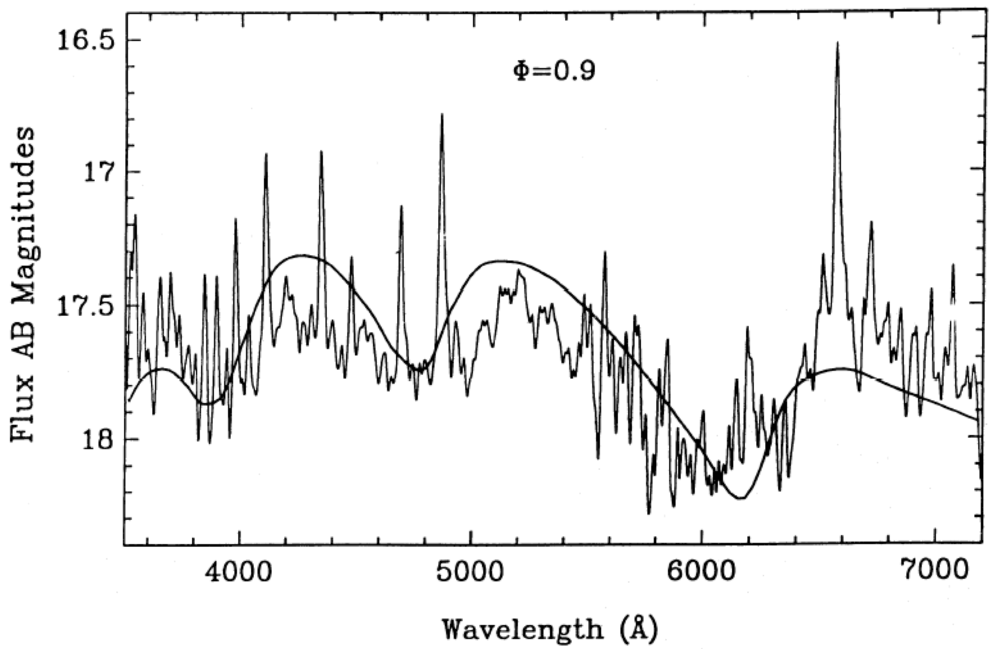
The results of calculations for a relativistic plasma having $kT_e/m_ec^2 = 0.1$, corresponding to $\gamma = 1.1$ and $v/c \approx 0.4$, are shown in the diagram. The spectra of the first twenty harmonics are shown as well as the total emission spectrum found by summing the spectra of the individual harmonics. One way of looking at synchrotron radiation is to consider it as the relativistic limit of the process illustrated in the diagram in which all the harmonics are washed out and a smooth continuum spectrum is observed.

Mildly Relativistic Cyclotron Absorption



A remarkable example of the application of these formulae in absorption occurred in the transient X-ray pulsar V0332+53 observed by the INTEGRAL and RXTE satellites. The source exhibited a very powerful outburst in Dec, 2004 – Feb, 2005 in a wide (3-100 keV) energy band. A cyclotron resonance scattering line at an energy of 26 keV was detected together with its two higher harmonics at 50 and 73 keV, respectively (Tsygankov, Lutovinov, Churazov and Sunyaev 2005).

Cyclotron Emission in AM Herculis Binaries



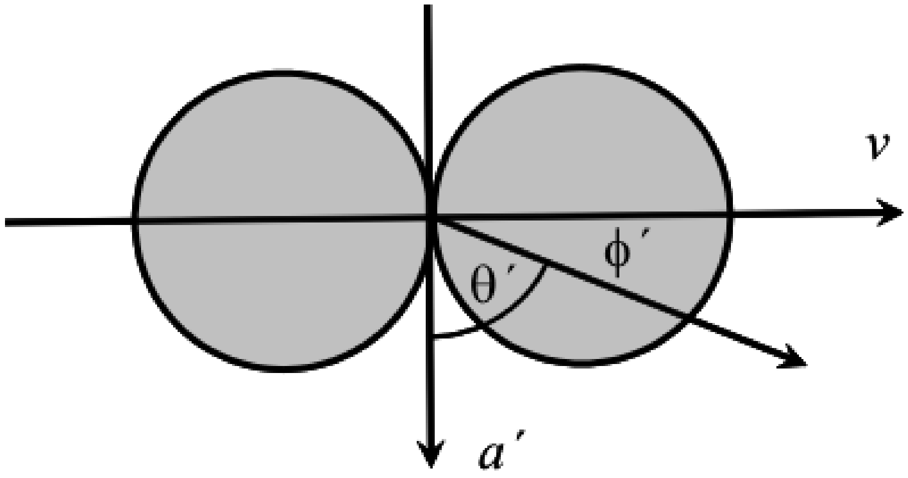
The harmonics of cyclotron radiation are circularly polarised and so it is possible to learn a great deal about the strength of the magnetic field strength and its orientation with respect to the line of sight. Circularly polarised optical emission has been discovered in the eclipsing magnetic binary stars known as *AM Herculis binaries*, in which a red dwarf star orbits a white dwarf. Circular polarisation percentages as large as 40% is observed. Accretion of matter from the surface of the red dwarf onto the magnetic poles of the white dwarf heats the matter to temperatures in excess of 10^7 K. In the X-ray source EXO 033319-2554.2, the magnetic flux density turns is 5600 T.

Spectrum of Synchrotron Radiation
Physical Arguments

One of the basic features of the radiation of relativistic particles in general is the fact that the radiation is *beamed* in the direction of motion of the particle. Part of this effect is associated with the relativistic aberration formulae between the frame of reference of the particle and the observer's frame of reference. There are, however, subtleties about what is observed by the distant observer because, in addition to aberration, we have to consider the time development of what is seen by the distant observer.

Let us consider first the simple case of a particle gyrating about the magnetic field at a pitch angle of 90°. The electron is accelerated towards its guiding centre, that is, radially inwards, and in its instantaneous rest frame it emits the usual dipole pattern with respect to the acceleration vector.

Beaming of the Emitted Radiation

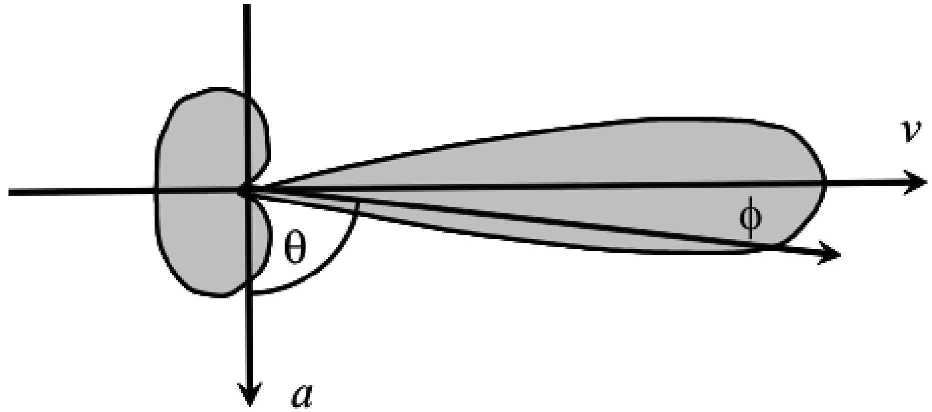


To centre of particle's orbit

We can therefore work out the radiation pattern in the laboratory frame of reference by applying the aberration formulae with the results illustrated schematically in the diagrams. The angular distribution of radiation with respect to the velocity vector in the frame S′ is $I_\nu \propto \sin^2\theta' = \cos^2\phi'$. We may think of this as being the probability distribution with which photons are emitted by the electron in its rest frame. The appropriate aberration formulae between the two frames are:

$$\sin\phi = \frac{1}{\gamma}\frac{\sin\phi'}{1 + (v/c)\cos\phi'} \quad ; \quad \cos\phi = \frac{\cos\phi' + v/c}{1 + (v/c)\cos\phi'} \qquad (20)$$

Beaming of the Emitted Radiation



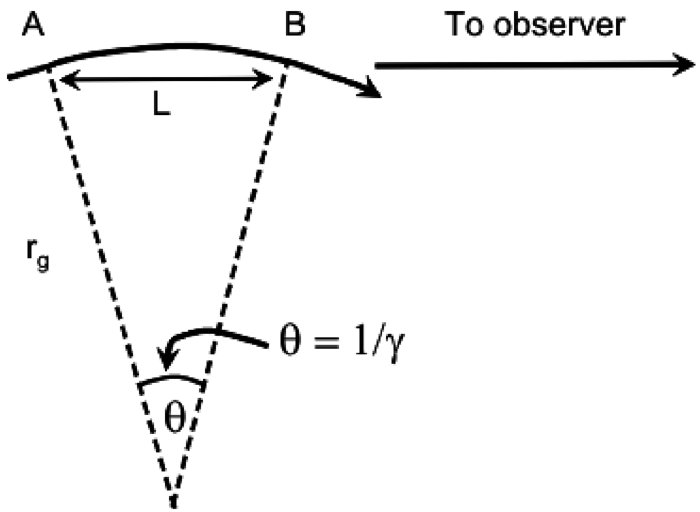
To centre of particle's orbit

Consider the angles $\phi' = \pm\pi/4$ in S', the angles at which the intensity of radiation falls to half its maximum value in the instantaneous rest frame. The corresponding angles ϕ in the laboratory frame of reference are

$$\sin\phi \approx \phi \approx 1/\gamma \tag{21}$$

The radiation emitted within $-\pi/4 < \phi' < \pi/4$ is beamed in the direction of motion of the electron within $-1/\gamma < \phi < 1/\gamma$. A large 'spike' of radiation is observed every time the electron's velocity vector lies within an angle of about $1/\gamma$ to the line of sight to the observer. The spectrum of the radiation is the Fourier transform of this pulse once the effects of time retardation and aberration are taken into account.

The Duration of the Observed Pulse



The observer sees significant radiation for only about $1/\gamma$ of the particle's orbit but the observed duration of the pulse is less than $1/\gamma$ times the period of the orbit because radiation emitted at the trailing edge of the pulse almost catches up with the radiation emitted at the leading edge. The observer is located at a distance R from the point A. The radiation from A reaches the observer at time R/c. The radiation is emitted from B at time L/v later and then travels a distance $(R - L)$ at the speed of light to the observer. The trailing edge of the pulse therefore arrives at a time $L/v + (R - L)/c$. The duration of the pulse as measured by the observer is therefore

$$\Delta t = \left[\frac{L}{v} + \frac{(R - L)}{c}\right] - \frac{R}{c} = \frac{L}{v}\left(1 - \frac{v}{c}\right) . \tag{22}$$

The Duration of the Observed Pulse

Notice that the observed duration of the pulse is much less than value L/v. Only if light propagated at an infinite velocity would the duration of the pulse be L/v. The factor $1-(v/c)$ is exactly the same factor which appears in the Liénard-Weichert potential and takes account of the fact that the source of radiation is not stationary but is moving towards the observer. We now rewrite the above expression noting that

$$\frac{L}{v} = \frac{r_{\mathrm{g}}\theta}{v} \approx \frac{1}{\gamma\omega_{\mathrm{r}}} = \frac{1}{\omega_{\mathrm{g}}} \tag{23}$$

where ω_{g} is the non-relativistic angular gyrofrequency and $\omega_r = \omega_{\mathrm{g}}/\gamma$. We can also rewrite $(1-v/c)$ as

$$\left(1-\frac{v}{c}\right) = \frac{[1-(v/c)]\,[1+(v/c)]}{[1+(v/c)]} = \frac{\left(1-v^2/c^2\right)}{1+(v/c)} \approx \frac{1}{2\gamma^2} \tag{24}$$

since $v \approx c$. Therefore, the observed duration of the pulse is roughly

$$\Delta t \approx \frac{1}{2\gamma^2\omega_{\mathrm{g}}} \tag{25}$$

The duration of the pulse in the laboratory frame of reference is roughly $1/\gamma^2$ times shorter than the non-relativistic gyroperiod $T_{\mathrm{g}} = 2\pi/\omega_{\mathrm{g}}$.

The Observed Frequency of Synchrotron Radiation

The maximum Fourier component of the spectral decomposition of the observed pulse of radiation is expected to correspond to a frequency $\nu \sim \Delta t^{-1}$, that is,

$$\nu \sim \Delta t^{-1} \sim \gamma^2 \nu_{\mathrm{g}} \tag{26}$$

where ν_{g} is the non-relativistic gyrofrequency.

In the above analysis, it has been assumed that the particle moves in a circle about the magnetic field lines, that is, the pitch angle θ is 90°. The same calculation can be performed for any pitch angle and then the result becomes

$$\nu \sim \gamma^2 \nu_{\mathrm{g}} \sin\theta \tag{27}$$

The reason for performing this simple exercise in detail is that the beaming of the radiation of ultrarelativistic particles is a very general property and does not depend upon the nature of the force causing the acceleration.

Curvature Radiation

Returning to an earlier part of the calculation, the observed frequency of the radiation can also be written

$$\boxed{\nu \approx \gamma^2 \nu_{\mathrm{g}} = \gamma^3 \nu_{\mathrm{r}} = \frac{\gamma^3 v}{2\pi r_{\mathrm{g}}}} \tag{28}$$

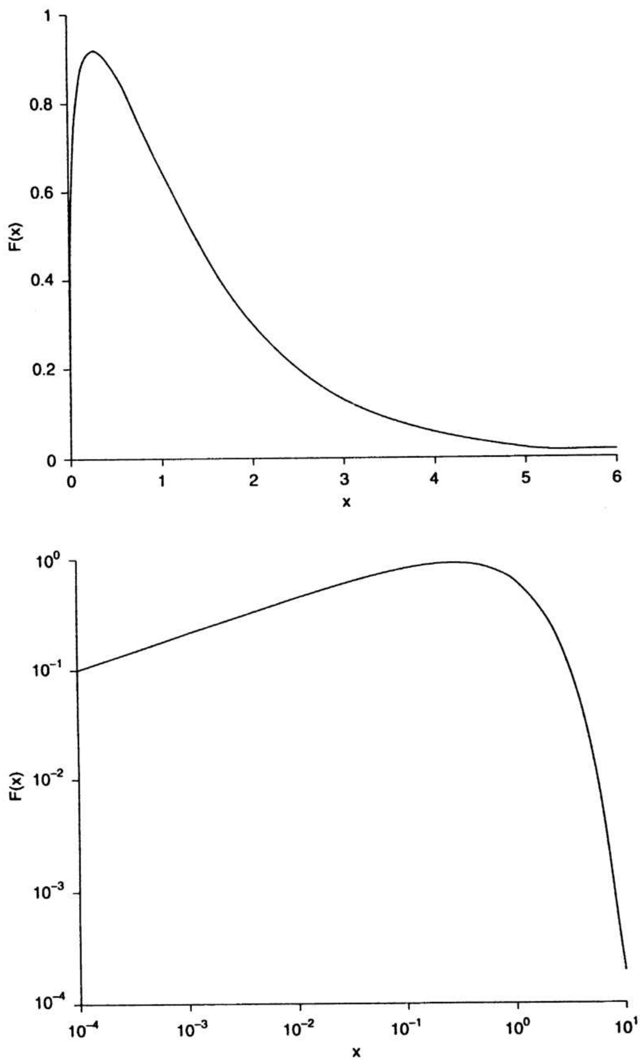
where ν_{r} is the relativistic gyrofrequency and r_{g} is the radius of curvature of the particle's orbit. Notice that, in general, we may interpret r_{g} as the instantaneous radius of curvature of the particle's orbit and v/r_{g} is the angular frequency associated with it. This is a useful result because it enables us to work out the frequency at which most of the radiation is emitted, provided we know the radius of curvature of the particle's orbit. The frequency of the observed radiation is roughly γ^3 times the angular frequency v/r where r is the *instantaneous radius of curvature* of the particle in its orbit. This result is important in the study of *curvature radiation* which has important applications in the emission of radiation from the magnetic poles of pulsars.

Synchrotron radiation - improved version

There is no particularly simple way of deriving the spectral distribution of synchrotron radiation and I do not find the analysis particularly appealing - see *HEA3* for the gruelling details. The analysis proceeds by the following steps:

- Write down the expression for the energy emitted per unit bandwidth for an arbitrarily moving electron;
- Select a suitable set of coordinates in which to work out the field components radiated by the electron spiralling in a magnetic field;
- Then battle away at the algebra to obtain the spectral distribution of the field components.

The Results



The emitted spectrum of a single electron, averaged over the particle's orbit is

$$j(\omega) = j_\perp(\omega) + j_\parallel(\omega) = \frac{\sqrt{3}e^3 B \sin\theta}{8\pi^2\varepsilon_0 c m_\mathrm{e}} F(x) \quad (29)$$

where

$$x = \omega/\omega_\mathrm{c}, \quad \omega_\mathrm{c} = \frac{3}{2}\left(\frac{c}{v}\right)\gamma^3 \omega_\mathrm{r} \sin\theta \quad (30)$$

and

$$F(x) = x\int_x^\infty K_{5/3}(z)\,\mathrm{d}z. \quad (31)$$

ω_c is known as the critical angular frequency. $K_{5/3}(z)$ is a modified Bessel function of order $5/3$. The form of this spectrum in terms of angular frequency ω is shown in linear and logarithmic form in the diagrams. It has a broad maximum centred roughly at the frequency $\nu \approx \nu_\mathrm{c}$ with $\Delta\nu/\nu \sim 1$. The maximum of the emission spectrum has value $\nu_\mathrm{max} = 0.29\nu_\mathrm{c}$.

The Results (continued)

The high frequency emissivity of the electron is given by an expression of the form

$$j(\nu) \propto \nu^{1/2} \mathrm{e}^{-\nu/\nu_{\mathrm{c}}} \tag{32}$$

which is dominated by the exponential cut-off at frequencies $\nu \gg \nu_{\mathrm{c}}$. This simply means that there is very little power at frequencies $\nu > \nu_{\mathrm{c}}$ which can be understood on the basis of the physical arguments developed earlier – there is very little structure in the observed polar diagram of the radiation emitted by the electron at angles $\theta \ll \gamma^{-1}$. At low frequencies, $\nu \ll \nu_{\mathrm{c}}$, the spectrum is given by $j(\nu) \propto \nu^{1/3}$

The ratio of the powers emitted in the polarisations parallel and perpendicular to the magnetic field direction is

$$\frac{I_{\perp}}{I_{\parallel}} = 7. \tag{33}$$

To find the polarisation observed from a distribution of electrons at a particular observing frequency, however, we need to integrate over the energy spectrum of the emitting electrons.

The Synchrotron Radiation of a Power-law Distribution of Electron Energies

The emitted spectrum of electrons of energy E is quite sharply peaked near the critical frequency ν_{c} and is very much narrower than the breadth of the electron energy spectrum. Therefore, to a good approximation, it may be assumed that all the radiation of an electron of energy E is radiated at the critical frequency ν_{c} which we may approximate by

$$\nu \approx \nu_{\mathrm{c}} \approx \gamma^2 \nu_{\mathrm{g}} = \left(\frac{E}{m_{\mathrm{e}} c^2}\right)^2 \nu_{\mathrm{g}}; \qquad \nu_{\mathrm{g}} = \frac{eB}{2\pi m_{\mathrm{e}}}. \tag{34}$$

Therefore, the energy radiated in the frequency range ν to $\nu + \mathrm{d}\nu$ can be attributed to electrons with energies in the range E to $E + \mathrm{d}E$, which we assume to have power-law form $N(E) = \kappa E^{-p}$. We may therefore write

$$J(\nu)\,\mathrm{d}\nu = \left(-\frac{\mathrm{d}E}{\mathrm{d}t}\right) N(E)\,\mathrm{d}E. \tag{35}$$

(continued)

Now

$$E = \gamma m_{\mathrm{e}} c^2 = \left(\frac{\nu}{\nu_{\mathrm{g}}}\right)^{1/2} m_{\mathrm{e}} c^2, \qquad \mathrm{d}E = \frac{m_{\mathrm{e}} c^2}{2\nu_{\mathrm{g}}^{1/2}} \nu^{-1/2} \mathrm{d}\nu, \tag{36}$$

and

$$-\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right) = \tfrac{4}{3} \sigma_{\mathrm{T}} c \left(\frac{E}{m_{\mathrm{e}} c^2}\right)^2 \frac{B^2}{2\mu_0}. \tag{37}$$

Substituting these quantities into (35), the emissivity may be expressed in terms of κ, B, ν and fundamental constants.

$$J(\nu) = (\mathrm{constants})\ \kappa B^{(p+1)/2} \nu^{-(p-1)/2}. \tag{38}$$

If the electron energy spectrum has power law index p, the spectral index of the synchrotron emission of these electrons, defined by $J(\nu) \propto \nu^{-\alpha}$, is $\alpha = (p-1)/2$. The spectral shape is determined by the shape of the electron energy spectrum rather than by the shape of the emission spectrum of a single particle. The quadratic nature of the relation between emitted frequency and the energy of the electron accounts for the difference in slopes of the emission spectrum and the electron energy spectrum.

Why is Synchrotron Radiation Taken so Seriously?

- Comparison of the local flux of relativistic electrons measured at the top of the atmosphere with the predicted synchrotron emissivity of the interstellar medium are in reasonable agreement.

- A convincing case can be made that the relativistic electrons are accelerated in supernova remnants which are very strong radio sources with power-law intensity spectra and the radio emission is linearly polarised.

- The intense extragalactic radio sources have qualitatively the same properties of power-law radio spectra and polarised radiation, but with intrinsic luminosities which are up to 10^8 greater than that of our own Galaxy. The radio emission originates from enormous radio lobes rather than from the galaxy itself. The only reasonable explanation is that the emission is the synchrotron radiation of high energy electrons gyrating in magnetic fields within the radio lobes (see later).

- Direct evidence for relativistic particles in active galactic nuclei comes from the very high brightness temperatures observed in compact radio sources, that it, we need to analyse the process of synchrotron self-absorption.

Synchrotron Self-absorption

According to the principle of detailed balance, to every emission process there is a corresponding absorption process – in the case of synchrotron radiation, this is known as *synchrotron self-absorption*.

Suppose a source of synchrotron radiation has a power law spectrum, $S_\nu \propto \nu^{-\alpha}$, where the spectral index $\alpha = (p-1)/2$. Its *brightness temperature* is defined to be $T_\mathrm{b} = (\lambda^2/2k)(S_\nu/\Omega)$, and is proportional to $\nu^{-(2+\alpha)}$, where S_ν is its flux density and Ω is the solid angle it subtends at the observer at frequency ν. We recall that brightness temperature is the temperature of a black-body which would produce the observed surface brightness of the source at the frequency ν in the Rayleigh-Jeans limit, $h\nu \ll kT_\mathrm{e}$. Thus, at low enough frequencies, the brightness temperature of the source may approach the kinetic temperature of the radiating electrons. When this occurs, self-absorption becomes important since thermodynamically the source cannot emit radiation of brightness temperature greater than its kinetic temperature.

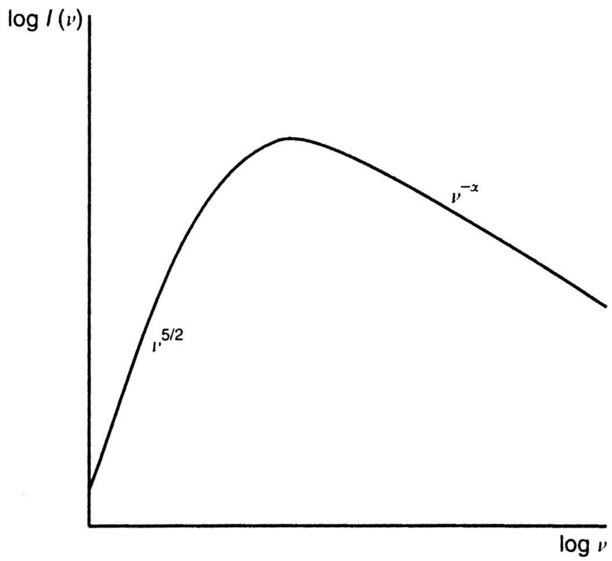
Synchrotron Self-absorption

The energy spectrum of the electrons $N(E)\,\mathrm{d}E = \kappa E^{-p}\,\mathrm{d}E$ is *not* a thermal equilibrium spectrum, which for relativistic particles would be a *relativistic Maxwellian distribution*. The concept of temperature can still be used, however, for particles of energy E. The spectrum of the radiation emitted by particles of energy E is peaked about the critical frequency $\nu \approx \nu_{\mathrm{c}} \approx \gamma^2 \nu_g$. Thus, the emission and absorption processes at frequency ν are associated with electrons of roughly the same energy. Second, the characteristic time-scale for the relativistic electron gas to relax to an equilibrium spectrum is very long indeed under typical cosmic conditions because the particle number densities are very small and all interaction times with matter are very long. Therefore, we can associate a temperature T_{e} with electrons of a given energy through the relativistic formula which relates particle energy to temperature

$$\gamma m_{\mathrm{e}} c^2 = 3kT_{\mathrm{e}}. \tag{39}$$

Recall that the ratio of specific heats γ_{SH} is $\frac{4}{3}$ for a relativistic gas. The internal thermal energy density of a gas is $u = NkT/(\gamma_{\mathrm{SH}} - 1)$, where N is the number density of particles. Setting $\gamma_{\mathrm{SH}} = \frac{5}{3}$, we obtain the classical result and, setting $\gamma_{\mathrm{SH}} = \frac{4}{3}$, we obtain the above result for the mean energy per particle.

Synchrotron Self-absorption



The important point is that the *effective temperature* of the particles now becomes a function of their energies. Since $\gamma \approx (\nu/\nu_{\rm g})^{1/2}$,

$$T_{\rm e} \approx (m_{\rm e}c^2/3k)(\nu/\nu_{\rm g})^{1/2}. \qquad (40)$$

For a self-absorbed source, the brightness temperature of the radiation must be equal to the kinetic temperature of the emitting particles, $T_{\rm b} = T_{\rm e}$, and therefore, in the Rayleigh-Jeans limit,

$$S_\nu = \frac{2kT_{\rm e}}{\lambda^2}\Omega = \frac{2m_{\rm e}}{3\nu_{\rm g}^{1/2}}\Omega\,\nu^{5/2} \propto \frac{\theta^2\nu^{5/2}}{B^{1/2}}, \qquad (41)$$

where Ω is the solid angle subtended by the source, $\Omega \approx \theta^2$. Spectra of roughly this form are found at radio, centimetre and millimetre wavelengths from the nuclei of active galaxies and quasars.

Synchrotron Self Absorption

It is a straightforward, but long, calculation to work out the absorption coefficient $\chi(\nu)$ for synchrotron self-absorption (see *HEA3*). The result for a randomly oriented magnetic field is,

$$\chi_\nu = \frac{\sqrt{3\pi} e^3 \kappa B^{(p+2)/2} c}{64\pi^2 \epsilon_0 m_\mathrm{e}} \left(\frac{3e}{2\pi m_\mathrm{e}^3 c^4} \right)^{p/2} \frac{\Gamma\left(\frac{3p+22}{12}\right) \Gamma\left(\frac{3p+2}{12}\right) \Gamma\left(\frac{p+6}{4}\right)}{\Gamma\left(\frac{p+8}{4}\right)} \nu^{-(p+4)/2}, \tag{42}$$

where the Γs are gamma-functions.

To work out the emission spectrum from, say, a slab of thickness l, we write down the transfer equation

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}x} = -\chi_\nu I_\nu + \frac{J(\nu)}{4\pi}. \tag{43}$$

The solution is

$$I_\nu = \frac{J(\nu)}{4\pi \chi_\nu} [1 - e^{-\chi_\nu l}]. \tag{44}$$

Synchrotron Self-absorption

If the source is optically thin, $\chi(\nu)l \ll 1$, we obtain $I_\nu = J(\nu)l/4\pi$.

If the source is optically thick, $\chi(\nu)l \gg 1$, we find

$$I_\nu = \frac{J(\nu)}{4\pi\chi_\nu}. \tag{45}$$

The quantity $J(\nu)/4\pi\chi_\nu$ is often referred to as the *source function*. Substituting for the absorption coefficient $\chi(\nu)$ from (170) and for J_ν from (166), we find

$$I_\nu = (\text{constant})\,\frac{m_\mathrm{e}\nu^{5/2}}{B^{1/2}}\,. \tag{46}$$

This is the same dependence as was found from the above physical arguments.

VLBI observations show that the angular sizes of many of the synchrotron self-absorbed sources have angular sizes $\theta \approx 10^{-3}$ arcsec. For 1 Jy radio sources, the corresponding brightness temperatures are

$$T_\mathrm{b} = \frac{\lambda^2}{2k_\mathrm{B}}\frac{S_\nu}{\Omega} \sim 10^{11}\,\mathrm{K}. \tag{47}$$

This is direct evidence for relativistic electrons within the source regions.

Distortions of Injection Spectra of the Electrons

In the optically thin regime of sources of synchrotron radiation, spectral breaks or cut-offs are often observed. In addition, different regions within individual sources may display spectral index variations. Both of these phenomena can be attributed to the effects of ageing of the spectrum of the electrons within the source regions and so provide useful information about time-scales. The lifetimes τ of the electrons in the source regions are

$$\tau = \frac{E}{(\mathrm{d}E/\mathrm{d}t)} = \frac{m_{\mathrm{e}}c^2}{\frac{4}{3}\sigma_{\mathrm{T}} c U_{\mathrm{mag}} \gamma}. \tag{48}$$

For typical extended powerful radio sources, $\gamma \sim 10^3$ and $B \sim 10^{-9}$ T and so the lifetimes of the electrons are expected to be $\tau \leq 10^7 - 10^8$ years. In the case of X-ray sources, for example, the diffuse X-ray emission from the Crab Nebula and the jet of M87, the energies of the electrons are very much greater, the inferred magnetic field strengths are greater and so the relativistic electrons have correspondingly shorter lifetimes. Since the lifetimes of the electrons are shorter than the light travel time across the sources, the electrons must be continuously accelerated within these sources.

The Diffusion Loss Equation

To obtain a quantitative description of the resulting distortions of synchrotron radiation spectra, it is convenient to introduce the *diffusion-loss equation* for the electrons (*HEA3*). If we write the loss rate of the electrons as

$$-\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right) = b(E), \tag{49}$$

the diffusion-loss equation is

$$\frac{\partial N(E)}{\partial t} = D\nabla^2 N(E) + \frac{\partial}{\partial E}[b(E)N(E)] + Q(E,t), \tag{50}$$

where D is a scalar diffusion coefficient and $Q(E)$ is a source term which describes the rate of injection of electrons and their injection spectra into the source region. We can obtain some useful results by inspection of a few special steady-state solutions.

Steady-state Electron Spectrum with Losses

Suppose, first of all, that there is an infinite, uniform distribution of sources, each injecting high energy electrons with an injection spectrum $Q(E) = \kappa E^{-p}$. Then, diffusion is not important and the diffusion-loss equation reduces to

$$\frac{\mathrm{d}}{\mathrm{d}E}[b(E)N(E)] = -Q(E) \qquad ; \qquad \int \mathrm{d}[b(E)N(E)] = -\int Q(E)\,\mathrm{d}E. \tag{51}$$

We assume $N(E) \rightarrow 0$ as $E \rightarrow \infty$ and hence integrating we find

$$N(E) = \frac{\kappa E^{-(p-1)}}{(p-1)b(E)}. \tag{52}$$

We now write down $b(E)$ for high energy electrons under interstellar conditions

$$b(E) = -\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right) = A_1\left(\ln\frac{E}{m_\mathrm{e}c^2} + 19.8\right) + A_2E + A_3E^2. \tag{53}$$

The first term on the right-hand side, containing the constant A_1, describes *ionisation losses* and depends only weakly upon energy; the second term, containing A_2, represents *bremsstrahlung* and *adiabatic losses* and the last term, containing A_3, describes *inverse Compton* and *synchrotron losses*.

Steady-state Electron Spectrum with Losses

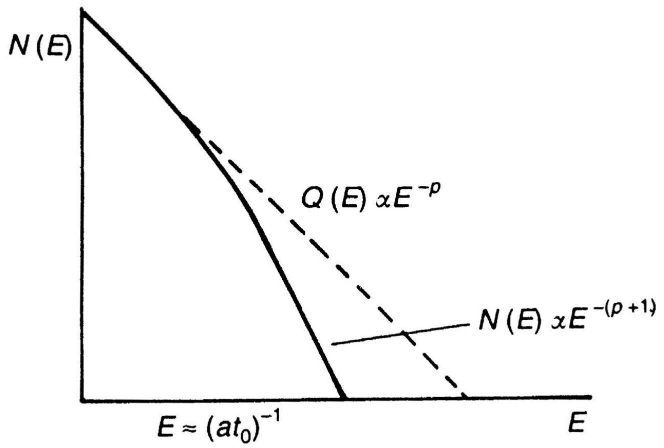
This analysis enables us to understand the effect of continuous energy losses upon the initial spectrum of the high energy electrons. Thus, from (52),

- if ionisation losses dominate, $N(E) \propto E^{-(p-1)}$, that is, the energy spectrum is flatter by one power of E;
- if bremsstrahlung or adiabatic losses dominate, $N(E) \propto E^{-p}$, that is, the spectrum is unchanged;
- if inverse Compton or synchrotron losses dominate, $N(E) \propto E^{-(p+1)}$, that is, the spectrum is steeper by one power of E.

Steady-state Electron Spectrum with Losses

These are also the equilibrium spectra expected whenever the continuous injection of electrons takes place over a time-scale longer than the lifetimes of the individual electrons involved. For example, if we inject electrons continuously with a spectrum E^{-p} into a source component for a time t and synchrotron radiation is the only important loss process, an electron of energy E_{S} loses all its energy in a time τ such that $-(\mathrm{d}E/\mathrm{d}t)\tau = E_{\mathrm{S}}$. For lower energies than E_{S}, the electrons do not lose a significant fraction of their energy and therefore the spectrum is the same as the injection spectrum, $N(E) \propto E^{-p}$. For energies greater than E_{S}, the particles have lifetimes less than t and we only observe those produced during the previous synchrotron lifetime τ_{S} of the particles of energy E, that is, $\tau_{\mathrm{S}} \propto 1/E$. Therefore the spectrum of the electrons is one power of E steeper, $N(E) \propto E^{-(p+1)}$, in agreement with the analysis proceeding from the steady-state solution of the diffusion-loss equation.

Steady-state Electron Spectrum with Losses



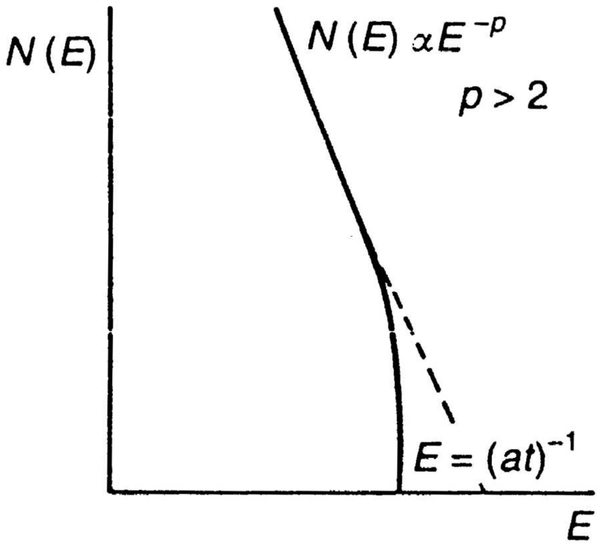
There are two useful analytic solutions for the electron energy distribution under continuous energy losses due to synchrotron radiation and inverse Compton scattering. In the first case, it is assumed that there is continuous injection of electrons with a power-law energy spectrum $Q(E) = \kappa E^{-p}$ for a time t_0. If we write the loss rate of the electrons in the form $b(E) = aE^2$, the energy spectrum after time t_0 has the form

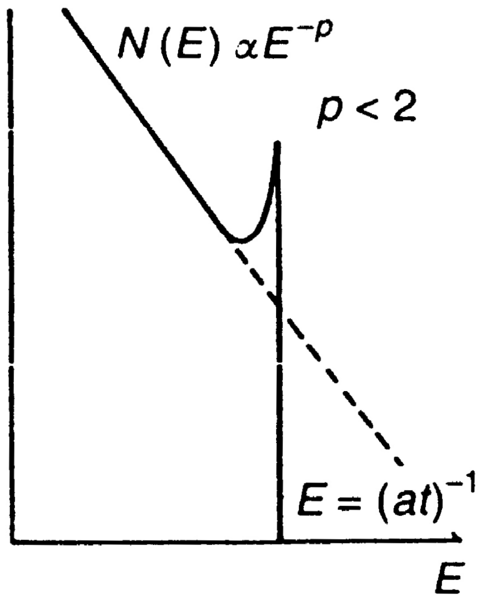
$$N(E) = \frac{\kappa E^{-(p+1)}}{a(p-1)}[1-(1-aEt)^{p-1}] \quad \text{if} \quad aEt_0 \leq 1; \tag{54}$$

$$N(E) = \frac{\kappa E^{-(p+1)}}{a(p-1)} \text{ if } aEt_0 > 1. \tag{55}$$

This form of spectrum agrees with the physical arguments given in the last paragraph.

No Injection of Electrons





A second useful case is that of the injection of electrons with a power-law energy spectrum at $t = 0$ with no subsequent injection of electrons. We can then write $Q(E) = \kappa E^{-p}\delta(t)$, where $\delta(t)$ is the Dirac delta function. It is straightforward to show that the solution of the diffusion-loss equation, ignoring the diffusion term, is

$$N(E) = \kappa E^{-p}(1 - aEt)^{p-2}. \qquad (56)$$

Thus, after time t, there are no electrons with energies greater than $(at)^{-1}$. Notice that, if $p > 2$, the spectrum steepens smoothly to zero at $E = (at)^{-1}$; if $p < 2$, there is a cusp in the energy spectrum at $E = (at)^{-1}$. The number of electrons, however, remains finite and constant.

Example of Use of Diffusion-Loss Equation

Let us look briefly at the *adiabatic loss problem* which pervades much of the astrophysics of clouds of relativistic plasma. The concern is that, if high energy particles are accelerated in a supernova explosion, for example, they lose all their energy adiabatically the expansion.

The relativistic gas exerts a pressure on its surroundings and consequently suffers adiabatic losses. For a relativistic particle, the energy decreases with increasing radius as $E \propto r^{-1}$. This also applies to the total relativistic particle energy and thus, if the total energy is W_0 at radius r_0, when the remnant expands to radius r, the internal energy of the relativistic gas is only $(r_0/r)W_0$.

If the expansion were purely adiabatic and the magnetic field strength decreased as $B \propto r^{-2}$, as is expected if magnetic flux freezing is applicable, the radio luminosity should decrease rapidly as the remnant expands.

The Adiabatic-Loss Problem

The diffusion-loss equation reduces to

$$\frac{\mathrm{d}N(E)}{\mathrm{d}t} = \frac{\partial}{\partial E}[b(E)N(E)]$$

where $b(E) = (1/r)(\mathrm{d}r/\mathrm{d}t)E$. Note that $N(E)$ now refers to all the particles in the remnant rather than the number per unit volume. Therefore, during the expansion, we can write $N(E) = V\kappa(r)E^{-p}$ since the spectral index does not change under adiabatic losses. If $N(E)$ were the number per unit volume, we would have to add the term $-N\nabla \cdot \mathbf{v}$ to the right hand side. Therefore, assuming $N(E) = \kappa E^{-p}$,

$$\frac{\mathrm{d}N(E)}{\mathrm{d}t} = \frac{\partial}{\partial E}\left(\frac{1}{r}\frac{\mathrm{d}r}{\mathrm{d}t}\kappa E^{-(p-1)}\right)$$

$$\frac{\mathrm{d}N(E)}{N(E)} = -(p-1)\frac{\mathrm{d}r}{r}$$

The Adiabatic-Loss Problem

$$\frac{N(E,r)}{N(E,r_0)} = \left(\frac{r}{r_0}\right)^{-(p-1)}$$, that is, $$\frac{\kappa(r)}{\kappa(r_0)} = \left(\frac{r}{r_0}\right)^{-(p-1)}$$

We can now work out how the synchrotron radio luminosity varies with radius because

$$I_\nu = A(\alpha)\kappa(R)B^{(1+\alpha)}\nu^{-\alpha}$$

and hence

$$I_\nu(r) \propto r^{-(p-1)}r^{-2(1+\alpha)} .$$

$p = 2\alpha + 1$ and therefore

$$I_\nu(r) \propto r^{-2(2\alpha+1)} = r^{-2p} .$$

Thus, if sources expand adiabatically, the synchrotron emissivity decreases very rapidly with radius. There needs to be a way of recovering the kinetic energy of the expanding source.

The Energetics of Sources of Synchrotron Radiation

An important calculation involving sources of synchrotron radiation is the estimation of the minimum energy requirements in relativistic electrons and magnetic fields to account for the observed synchrotron emission. Suppose a source has luminosity L_ν at frequency ν and its volume is V. The spectrum of the radiation is of power-law form, $L_\nu \propto \nu^{-\alpha}$. The following arguments can be applied to the synchrotron radiation emitted by the source at any frequency, be it radio, optical or X-ray wavelengths. The luminosity can be related to the energy spectrum of the ultrarelativistic electrons and the magnetic field B present in the source through the expression (38) for synchrotron radiation

$$L_\nu = A(\alpha) V \kappa B^{1+\alpha} \nu^{-\alpha}, \tag{57}$$

where the electron energy spectrum per unit volume is $N(E)\,\mathrm{d}E = \kappa E^{-p}\,\mathrm{d}E$, $p = 2\alpha + 1$ and $A(\alpha)$ is a constant which depends only weakly on the spectral index α. Writing the energy density in relativistic electrons as ε_e, the total energy present in the source is

$$W_\mathrm{total} = V\varepsilon_\mathrm{e} + V\frac{B^2}{2\mu_0} = V\int \kappa E N(E)\,\mathrm{d}E + V\frac{B^2}{2\mu_0}. \tag{58}$$

The Minimum Energy Requirements

The luminosity of the source L_ν determines only the product $V\kappa B^{1+\alpha}$. If V is assumed to be known, the luminosity may either be produced by a large flux of relativistic electrons in a weak magnetic field, or *vice versa*. There is no way of deciding which combination of ε_e and B is appropriate from observations of L_ν. Between the extremes of dominant magnetic field and dominant particle energy, there is a minimum total energy requirement.

We need to account for the energy which might be present in the form of relativistic protons. In our own Galaxy, there seems to be about 100 times as much energy in relativistic protons as there is in electrons, whereas in the Crab Nebula, the energy in relativistic protons cannot be much greater than the energy in the electrons from dynamical arguments. It is therefore assumed that the protons have energy β times that of the electrons, that is,

$$\varepsilon_\mathrm{protons} = \beta\varepsilon_\mathrm{e}\,, \qquad \varepsilon_\mathrm{total} = (1+\beta)\varepsilon_\mathrm{e} = \eta\varepsilon_\mathrm{e}. \tag{59}$$

We therefore write

$$W_\mathrm{total} = \eta V \int_{E_\mathrm{min}}^{E_\mathrm{max}} \kappa E N(E)\,\mathrm{d}E + V\frac{B^2}{2\mu_0}. \tag{60}$$

The Minimum Energy Requirements

The energy requirements as expressed in (60) depend upon the unknown quantities κ and B, but they are related through (57) for the observed luminosity of the source L_ν. We also require the relation between the frequency of emission of an ultrarelativistic electron of energy $E = \gamma m_e c^2 \gg m_e c^2$ in a magnetic field of strength B. We use the result that the maximum intensity of synchrotron radiation occurs at a frequency

$$\nu = \nu_{\mathrm{max}} = 0.29\nu_{\mathrm{c}} = 0.29\,\tfrac{3}{2}\gamma^2\nu_{\mathrm{g}} = CE^2B\,, \tag{61}$$

where ν_{g} is the non-relativistic gyrofrequency and $C = 1.22 \times 10^{10}/(m_e c^2)^2$. Therefore, the relevant range of electron energies in the integral (60) is related to the range of observable frequencies through

$$E_{\mathrm{max}} = \left(\frac{\nu_{\mathrm{max}}}{CB}\right)^{1/2} \qquad E_{\mathrm{min}} = \left(\frac{\nu_{\mathrm{min}}}{CB}\right)^{1/2}. \tag{62}$$

ν_{max} and ν_{min} are the maximum and minimum frequencies for which the spectrum is known or the range of frequencies relevant to the problem at hand.

The Minimum Energy Requirements

Then

$$W_{\text{particles}} = \eta V \int_{E_{\text{min}}}^{E_{\text{max}}} E \kappa E^{-p} \, \mathrm{d}E = \frac{\eta V \kappa}{(p-2)} (CB)^{(p-2)/2} \left[\nu_{\text{min}}^{(2-p)/2} - \nu_{\text{max}}^{(2-p)/2} \right].$$

Substituting for κ in terms of L_ν and B from (57),

$$W_{\text{particles}} = \frac{\eta V}{(p-2)} \left[\frac{L_\nu}{A(\nu) V B^{1+\alpha} \nu^{-\alpha}} \right] (CB)^{(p-2)/2} [\nu_{\text{min}}^{(2-p)/2} - \nu_{\text{max}}^{(2-p)/2}]. \quad (63)$$

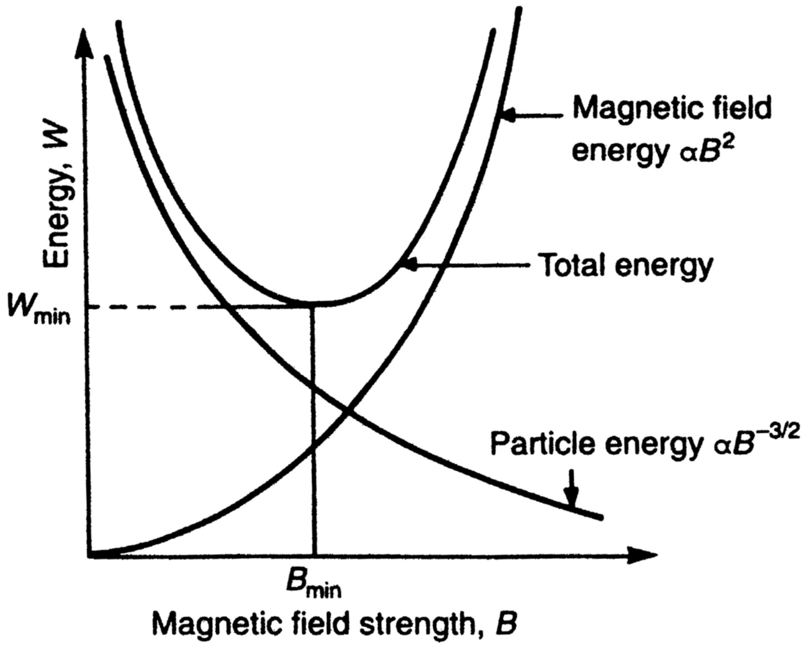
Preserving only the essential dependences,

$$W_{\text{particles}} = G(\alpha) \eta L_\nu B^{-3/2}, \quad (64)$$

where $G(\alpha)$ is a constant which depends weakly on $\alpha,\ \nu_{\text{max}}$ and ν_{min} if $\alpha \approx 1$. Therefore

$$W_{\text{total}} = G(\alpha) \eta L_\nu B^{-3/2} + V \frac{B^2}{2\mu_0}. \quad (65)$$

Minimum Energy Requirements



The diagram shows the variation of the energies in particles and magnetic field as a function of B. There is a minimum total energy,

$$B_{\text{min}} = \left[\frac{3\mu_0}{2} \frac{G(\alpha)\eta L_\nu}{V} \right]^{2/7} . \qquad (66)$$

This magnetic field strength B_{min} corresponds to approximate equality of the energies in the relativistic particles and magnetic field. Substituting B_{min} into (64), we find

$$W_{\text{mag}} = V \frac{B_{\text{min}}^2}{2\mu_0} = \frac{3}{4} W_{\text{particles}} \qquad (67)$$

Thus, the condition for minimum energy requirements corresponds closely to the condition that there are equal energies in the relativistic particles and the magnetic field.

Equipartition of Energy?

This condition is often referred to as *equipartition*. The minimum total energy is

$$W_{\mathrm{total}}(\mathrm{min}) = \frac{7}{6\mu_0} V^{3/7} \left[\frac{3\mu_0}{2} G(\alpha) \eta L_\nu\right]^{4/7} . \tag{68}$$

This is the minimum total energy needed to account for the observed luminosity of the source. These results are frequently used in the study of the synchrotron radiation from radio, optical and X-ray sources but their limitations should be appreciated.

- There is *no physical justification* for the source components being close to equipartition. It has been conjectured that the magnetic field in the source components may be stretched and tangled by motions in the plasma and so there might be rough equipartition between the magnetic energy density and the energy density in turbulent motions. The turbulent motions might also be responsible for accelerating the high energy particles and these particles might come into equipartition with the turbulent energy density if the acceleration mechanism were very efficient. In this way, it is possible that there might be a physical justification for the source components being close to equipartition, but this is no more than a conjecture.

Equipartition of Energy?

- The amount of energy present in the source is sensitive to the value of η, that is, the amount of energy present in the form of relativistic protons and nuclei.

- The total amount of energy in relativistic particles is dependent upon the limits assumed to the energy spectrum of the particles. If $\alpha = 1$, we need only consider the dependence upon ν_{min}, $W_{\text{min}} \propto \nu_{\text{min}}^{-0.5}$. However, there might be large fluxes of low energy relativistic electrons present in the source components with a quite different energy spectrum and we would have no way of knowing that they are present from the radio observations.

- The energy requirements depend upon the volume of the source. The calculation has assumed that the particles and magnetic field fill the source volume uniformly. The emitting regions might occupy only a small fraction of the apparent volume of the source, for example, if the synchrotron emission originated in filaments or subcomponents within the overall volume V. Then, the volume which should be used in the expressions (66) and (68) should be smaller than V. Often, a *filling factor* f is used to describe the fraction of the volume occupied by radio emitting material. The energy requirements are reduced if f is small.

Equipartition of Energy?

- On the other hand, we can obtain a firm lower limit to the *energy density* within the source components since

$$U_{\rm min} = \frac{W_{\rm total}({\rm min})}{V} = \frac{7}{6\mu_0} V^{-4/7} \left[\frac{3\mu_0}{2} G(\alpha) \eta L_\nu\right]^{4/7} . \qquad (69)$$

  For dynamical purposes, the energy density is more important than the total energy since it is directly related to the pressure within the source components $p = (\gamma - 1)U$ where γ is the ratio of specific heats. In the case of an ultrarelativistic gas, $\gamma = 4/3$ and so $p = \frac{1}{3}U$ as usual.

Therefore, the values of the magnetic field strength and minimum energy which come out of these arguments should be considered only order of magnitude estimates. If the source components depart radically from the equipartition values, the energy requirements are increased and this can pose problems for some of the most luminous sources.

Equipartition of Energy?

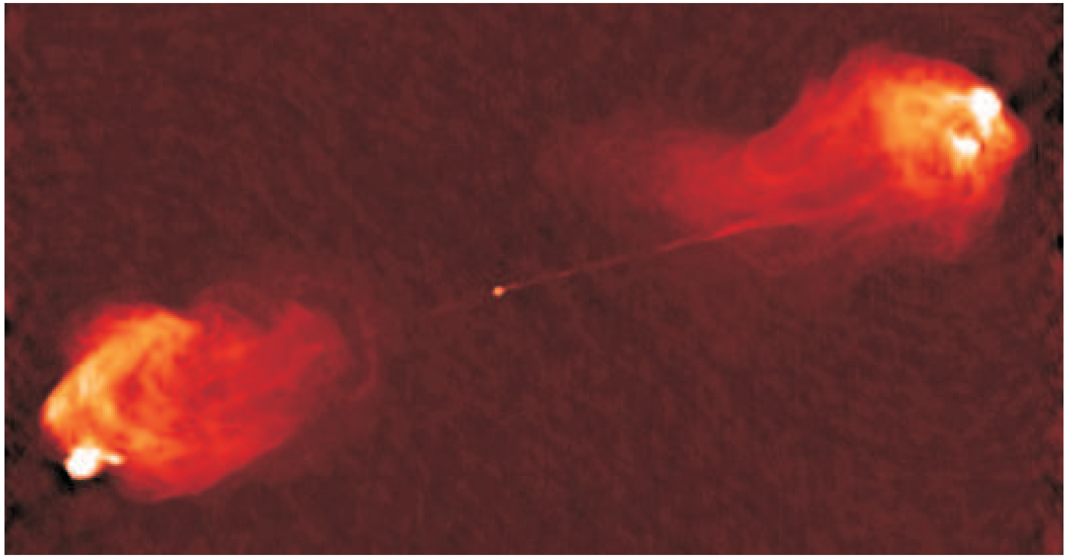
It is cumbersome to have to go through the procedure of working out $G(\alpha)$ to estimate the minimum energy requirements and magnetic field strengths. A simplified calculation can be performed as follows. If we assume that the spectral index $\alpha = 0.75$, we can neglect the upper limit ν_{max} in comparison with ν_{min} in evaluating $G(\alpha)$. Then, if we know the luminosity $L(\nu)$ at a certain frequency ν, we obtain a lower limit to the energy requirements if we set $\nu = \nu_{\mathrm{min}}$. Making these simplifications, we find that the minimum energy requirement is:

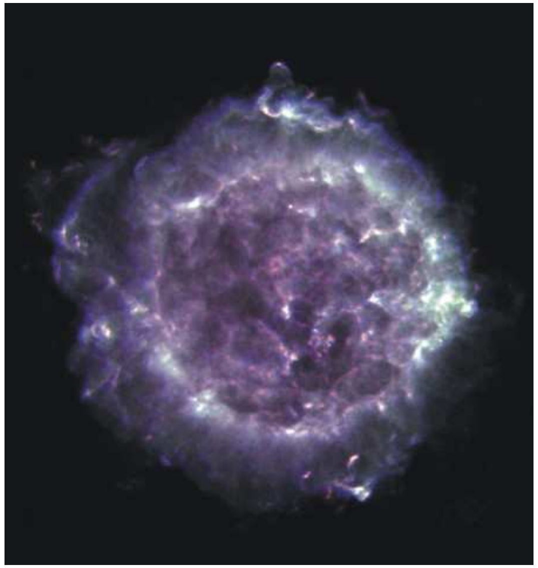
$$W_{\mathrm{min}} \approx 3.0 \times 10^{6}\, \eta^{4/7} V^{3/7} \nu^{2/7} L_{\nu}^{4/7} \quad \mathrm{J}, \tag{70}$$

where the volume of the source V is measured in m^3, the luminosity $L(\nu)$ in $\mathrm{W\,Hz^{-1}}$ and the frequency ν in Hz. In the same units, the minimum magnetic field strength is:

$$B_{\mathrm{min}} = 1.8 \left(\frac{\eta L_{\nu}}{V}\right)^{2/7} \nu^{1/7} \quad \mathrm{T}. \tag{71}$$

Examples





A good example is provided by the radio source Cygnus A. On the large scale, the source consists of two components roughly 100 kpc in diameter. The source had luminosity roughly 8×10^{28} W Hz^{-1} at 178 MHz. The minimum total energy is $2 \times 10^{52} \eta^{4/7}$ J, corresponding to a rest mass energy of $3 \times 10^{5} \eta^{4/7}$ $M_{\odot}$. Thus, a huge amount of mass has to be converted into relativistic particle energy and ejected from the nucleus of the galaxy into enormous radio lobes.

Performing the same calculation for the supernova remnant Cassiopeia A, the magnetic flux density corresponding to the minimum energy requirements is $B = 10\eta^{2/7}$ nT and the minimum total energy is $W_{\rm min} = 2 \times 10^{41} \eta^{4/7}$ J. This can be compared with the kinetic energy of the filaments which amounts to about 2×10^{44} J.