3D MT simulation using vector finite elements on unstructured grids

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Outline

1. MT boundary value problem
2. Finite element method
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    - MT3DINV forward model

Maxwell's equations

- harmonic time dependency $e^{i\omega t}$

$$\begin{aligned}
\nabla \times \mathbf{H} &= \mathbf{j} + i\omega \mathbf{D} \\
\nabla \times \mathbf{E} &= -i\omega \mathbf{B} \\
\nabla \cdot \mathbf{D} &= \rho \\
\nabla \cdot \mathbf{B} &= 0
\end{aligned}$$

Electromagnetic potentials

- magnetic vector potential $\mathbf{A}$, electric scalar potential V

$$\mathbf{H} = \mu^{-1}(\nabla \times \mathbf{A}) \qquad \mathbf{E} = -\nabla V - i\omega \mathbf{A}$$

MT boundary value problem I

- equation of induction

$$\nabla \times \mu^{-1}(\nabla \times \mathbf{A}) + (i\omega\sigma - \omega^2\varepsilon)\mathbf{A} + (\sigma + i\omega\varepsilon)\nabla V = 0 \quad \text{in } \Omega$$

- equation of continuity

$$-\nabla \cdot \big((i\omega\sigma - \omega^2\varepsilon)\mathbf{A} + (\sigma + i\omega\varepsilon)\nabla V\big) = 0 \quad \text{in } \Omega$$

- outer boundary conditions

$$\begin{aligned} \mathbf{n} \times \mathbf{H} = 0, \quad \mathbf{n} \cdot \mathbf{j} = 0 \quad &\text{on} \quad \Gamma_{||} \\ \mathbf{n} \times \mathbf{A} = 0, \quad V = 0 \quad &\text{on} \quad \Gamma_{\perp} \\ H_{\perp} = H(z) \quad &\text{on} \quad \Gamma_{bottom}, \Gamma_{top} \end{aligned}$$

- inner boundary conditions

$$\mathbf{n}_1 \times \mathbf{H}_1 + \mathbf{n}_2 \times \mathbf{H}_2 = 0 \quad \text{on} \quad \Gamma_{int}$$

MT boundary value problem II

- choosing $\tilde{\mathbf{A}} = \mathbf{A} - \nabla\Psi$, $\tilde{V} = V - \dot{\Psi}$ and the gauge condition $\Psi = -iV/\omega$:

$$\tilde{\mathbf{A}} = \mathbf{A} - \frac{i}{\omega}\nabla V,\ \ \tilde{V} = 0$$

- equation of induction

$$\nabla \times \mu^{-1}\nabla \times \tilde{\mathbf{A}} + (i\omega\sigma - \omega^2\varepsilon)\tilde{\mathbf{A}} = 0 \quad \text{in } \Omega$$

- boundary conditions

$$\mathbf{n} \times \mathbf{H} = 0 \quad \text{on} \quad \Gamma_{||}$$
$$\mathbf{n} \times \tilde{\mathbf{A}} = 0 \quad \text{on} \quad \Gamma_{\perp}$$
$$H_{\perp} = H(z) \quad \text{on} \quad \Gamma_{bottom}, \Gamma_{top}$$

$$\mathbf{n}_1 \times \mathbf{H}_1 + \mathbf{n}_2 \times \mathbf{H}_2 = 0 \quad \text{on} \quad \Gamma_{int}$$

More formulations of the MT boundary value problem

in terms of

- the electric field
- the magnetic field
- the secondary magnetic vector potential

Is there a most efficient formulation of the MT boundary value problem regarding computational effort and accuracy?

Weak formulation

Find $\mathbf{A} \in U$ such that

$$\int_\Omega (\mu^{-1}(\nabla \times \mathbf{A}) \cdot \nabla \times \bar{\mathbf{v}} + (i\omega\sigma - \omega^2\epsilon)\mathbf{A} \cdot \bar{\mathbf{v}})\, dV + \underbrace{\int_{\partial\Omega} \mathbf{n} \times (\mu^{-1}\nabla \times \mathbf{A}) \cdot \bar{\mathbf{v}}\, dS}_{0} = 0 \quad \forall \mathbf{v} \in V,$$

with

$$\begin{aligned} U &:= \{\mathbf{A} \in H(\mathrm{curl}, \Omega) : \mathbf{A} = \mathbf{A}(x, y, z) \text{ on } \Gamma_D\}, \\ V &:= \{\mathbf{v} \in H(\mathrm{curl}, \Omega) : \mathbf{v} \equiv 0 \text{ on } \Gamma_D\} \quad \text{and} \\ H(\mathrm{curl}, \Omega) &:= \{\mathbf{u} \in (L^2(\Omega))^3, \nabla \times \mathbf{u} \in (L^2(\Omega))^3\} \quad \text{(Nédélec)}. \end{aligned}$$

Finite element analysis

A discrete approximation $\mathbf{A}^h \in U_h$ of $\mathbf{A} \in U$

$$\mathbf{A}^h = \sum_{i=1}^{N} a_i \boldsymbol{\phi}_i$$

and discrete test functions $\mathbf{v}_i = \boldsymbol{\phi}_i$ yield the discretized boundary value problem

$$\tilde{\mathbf{K}}\mathbf{A}^h = \mathbf{L}$$

with

$$\begin{aligned} K_{i,j} &= \int_\Omega (\mu^{-1}(\nabla \times \boldsymbol{\phi}_i) \cdot \nabla \times \bar{\boldsymbol{\phi}}_j + (i\omega\sigma - \omega^2\epsilon)\boldsymbol{\phi}_i \cdot \bar{\boldsymbol{\phi}}_j)dV, \\ L_i &= 0. \end{aligned}$$

Incorporation of the inhomogeneous Dirichlet boundary conditions

Taking the known boundary values $\mathbf{A}_{\Gamma_D}$ and

$$\mathbf{A}^h = \mathbf{A}^h_{\Omega\setminus\Gamma_D} + \mathbf{A}^h_{\Gamma_D}$$

we obtain a modified system of equations for all interior values $\mathbf{A}^h_{\Omega\setminus\Gamma_D}$

$$\tilde{\mathbf{K}}\mathbf{A}^h_{\Omega\setminus\Gamma_D} = -\tilde{\mathbf{K}}\mathbf{A}^h_{\Gamma_D}$$

with

$$\begin{aligned} K_{i,j} &= \int_\Omega (\mu^{-1}(\nabla\times\boldsymbol{\phi}_i)\cdot\nabla\times\bar{\boldsymbol{\phi}}_j + (i\omega\sigma-\omega^2\epsilon)\boldsymbol{\phi}_i\cdot\bar{\boldsymbol{\phi}}_j)dV, \\ L_i &= K_{i,j}A^{h,\Gamma_{Dj}}. \end{aligned}$$

To solve the system of equations we apply a direct Gauss-type solver.

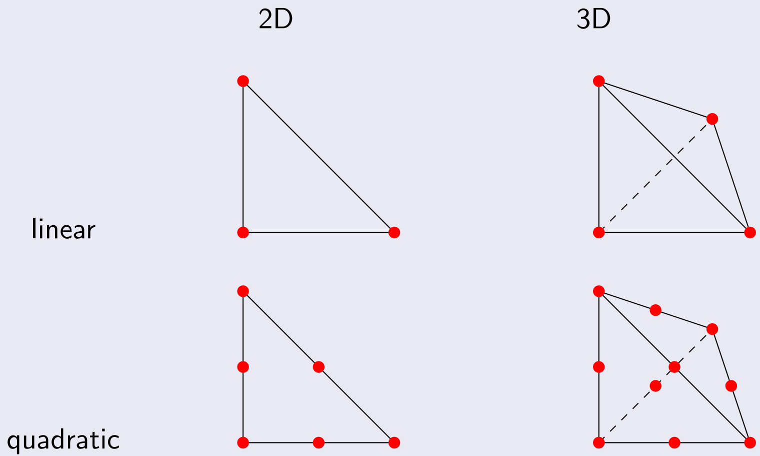
Basis functions

... ϕ_i are polynomials of arbitrary degree, mostly linear or quadratic. To the so-called degrees of freedom (DOF) l_n

$$l_n(\phi_j) = \delta_{n,j},$$

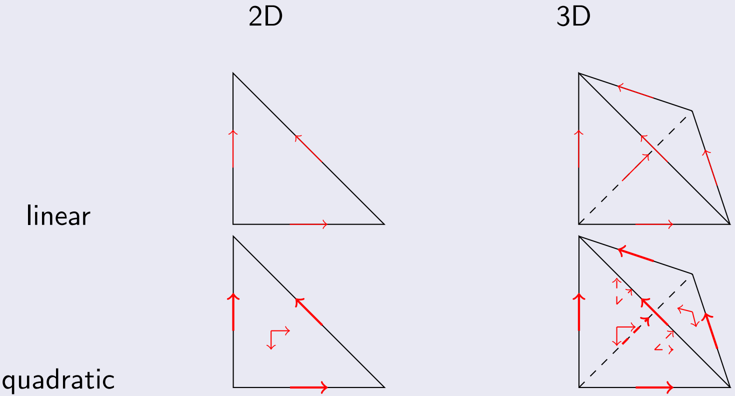
applies.

Lagrange elements



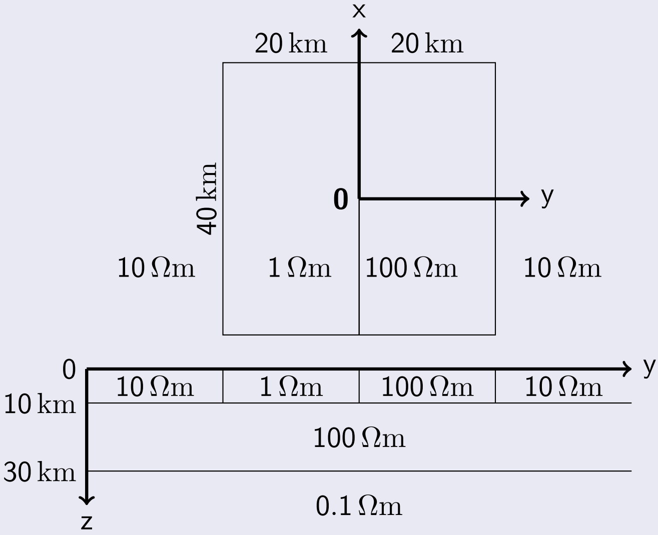
DOF: $l_i(V) = V(\mathbf{x}_i)$

Nédélec elements

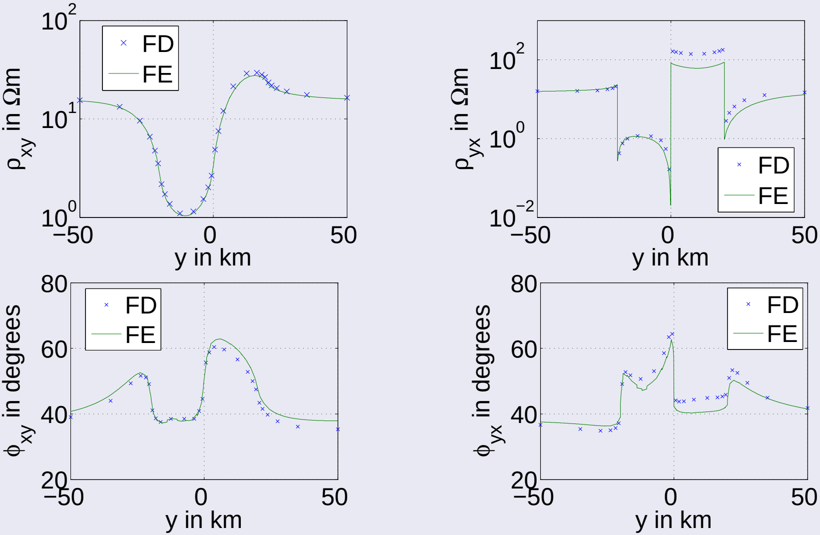


DOF: Integrals over **A** along edges, over faces and the volume

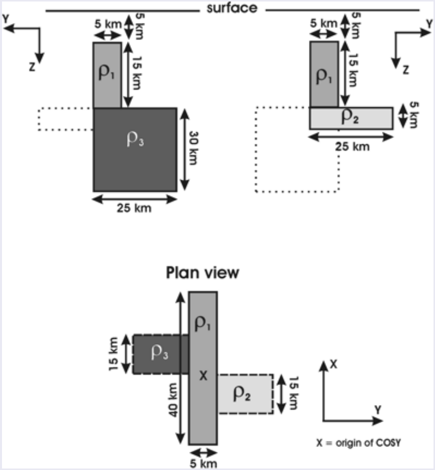
COMMEMI model 3D-2



Comparison of apparent resistivity and phase with Randy Mackie's FD code, T=100 s



MT3DINV forward model



$$\begin{aligned} \rho_1 &= 10\,\Omega\,\mathrm{m} \\ \rho_2 &= 1\,\Omega\,\mathrm{m} \\ \rho_3 &= 10000\,\Omega\,\mathrm{m} \\ \rho_{\mathrm{halfspace}} &= 100\,\Omega\,\mathrm{m} \end{aligned}$$

Challenges

- large effort to administer the unstructured tetrahedral grids
- optimum trade-off between fitting the boundary conditions and minimizing the discretization error
  $\longrightarrow$ smart distribution of DOF is needed

Advantages of the FE method in conjunction with unstructured grids

- precise parametrization of arbitrary model geometries including surface and seafloor topography
- adaptive mesh refinement based on *a posteriori* error estimators

Discretized MT3DINV forward model for $10\,\mathrm{s} \leq T < 100\,\mathrm{s}$