Numerical Modelling for Geophysical Electromagnetic Methods

A short course given at the Dublin Institute of Advanced Studies
6th-14th July 2009

Colin G. Farquharson

Department of Earth Sciences,
Memorial University of Newfoundland,
St. John's, NL, Canada.

MEMORIAL
UNIVERSITY

Outline

I. 1-D

II. EM rules & regulations

III. Finite difference

IV. Finite element

V. Integral equation

II. EM rules & regulations

1. Introduction

The behaviour of electromagnetic fields is described by many laws and rules: Maxwell's equations, Ohm's law, conservation of charge, constitutive relations, ...

True EM fields satisfy all the rules and regulations.

Approximate EM fields cannot satisfy all the rules and regulations – something must give.

It's critical that the rules that end up being violated by the approximate EM fields are not the ones that specify the most important behaviour.

Let's spend some time looking at all the rules that we would really like our approximate EM fields to obey ...

2. Faraday's law

In differential form, valid at some point:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

In integral form, explicitly thinking of a surface (S) and its bounding curve (C):

$$\int_C \mathbf{E} \cdot \hat{\mathbf{t}}\, dl = -\frac{d}{dt} \int_S \mathbf{B} \cdot \hat{\mathbf{n}}\, ds$$

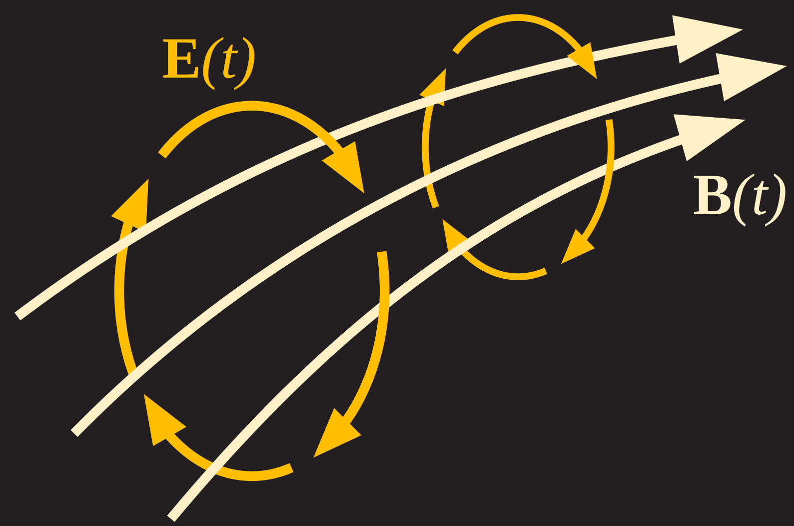
In qualitative terms:

A time-varying magnetic field generates
"curly" electric fields.

2. Faraday's law (contd.)

"Circulating about any time-varying magnetic field there exists an electric field ...

"... such that the total emf generated around any closed path C is proportional to the negative rate of change the magnetic flux passing through C."



(Grant & West)

2. Faraday's law (contd.)

Assuming a time-dependence of $e^{-i\omega t}$ (for both the electric and magnetic fields, of course!) gives:

$$\nabla \times \mathbf{E} = i\omega\, \mathbf{B}$$

(So only operational for time-varying fields.)

The electric field exists anywhere: current only flows if you're in a conductor. (Ohm's law coming in a moment.)

Any partial differential equation (PDE) will involve the differential form.

For deriving a numerical solution, maybe the differential form is good, or maybe the integral form is good . . . ?

3. Ampère's law

In differential form, valid at some point:

$$\nabla \times \mathbf{H} \;=\; \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

In integral form, explicitly thinking of a surface (S) and its bounding curve (C):

$$\int_C \mathbf{H} \cdot \hat{\mathbf{t}} \, dl = \int_S \left\{ \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right\} \cdot \hat{\mathbf{n}} \, ds$$

In qualitative terms:

An electric current generates "curly" magnetic fields.

3. Ampère's law

"Around any current field there circulates a magnetic field ...

"... such that the integral of the tangential H-field around any path is proportional to the total current which flows through it."

(Grant & West)

3. Ampère's law (contd.)

For the quasi-static régime (more on this later; geophysical EM) Ampère's law becomes:

$$\nabla \times \mathbf{H} \ = \ \mathbf{J}$$

(This doesn't need time variations to be operational.)

Faraday's and Ampère's laws are almost symmetric ... except for $-d/dt$, $\mathbf{B}$ vs. $\mathbf{H}$, and $\mathbf{E}$ vs. $\mathbf{J}$!

Again, differential form and integral form can both be appropriate for numerical modelling.

4. Constitutive relations

The effects of materials get into Maxwell's equations via:

$$\mathbf{B} = \mu \, \mathbf{H} \qquad \text{and} \qquad \mathbf{D} = \epsilon \, \mathbf{E}$$

Some prospecting EM systems have seen effects of magnetic susceptibility.

The polarization effects measured by the Induced Polarization (IP) method of exploration geophysics enter into Maxwell's equations through ϵ .

But, for most EM geophysics, the above two constitutive relations need only be:

$$\mathbf{B} = \mu_0 \, \mathbf{H} \qquad \text{and} \qquad \mathbf{D} = \epsilon_0 \, \mathbf{E}$$

and the only "constitutive relation" of consequence is . . .

5. Ohm's law

$$\mathbf{J} = \sigma\, \mathbf{E}$$

⋆ This is how *The Earth* gets into Maxwell's equations. The electrical conductivity is assumed to be a function of position. Perhaps a 1-D or 2-D function, or fully 3-D. Perhaps maybe anisotropic.

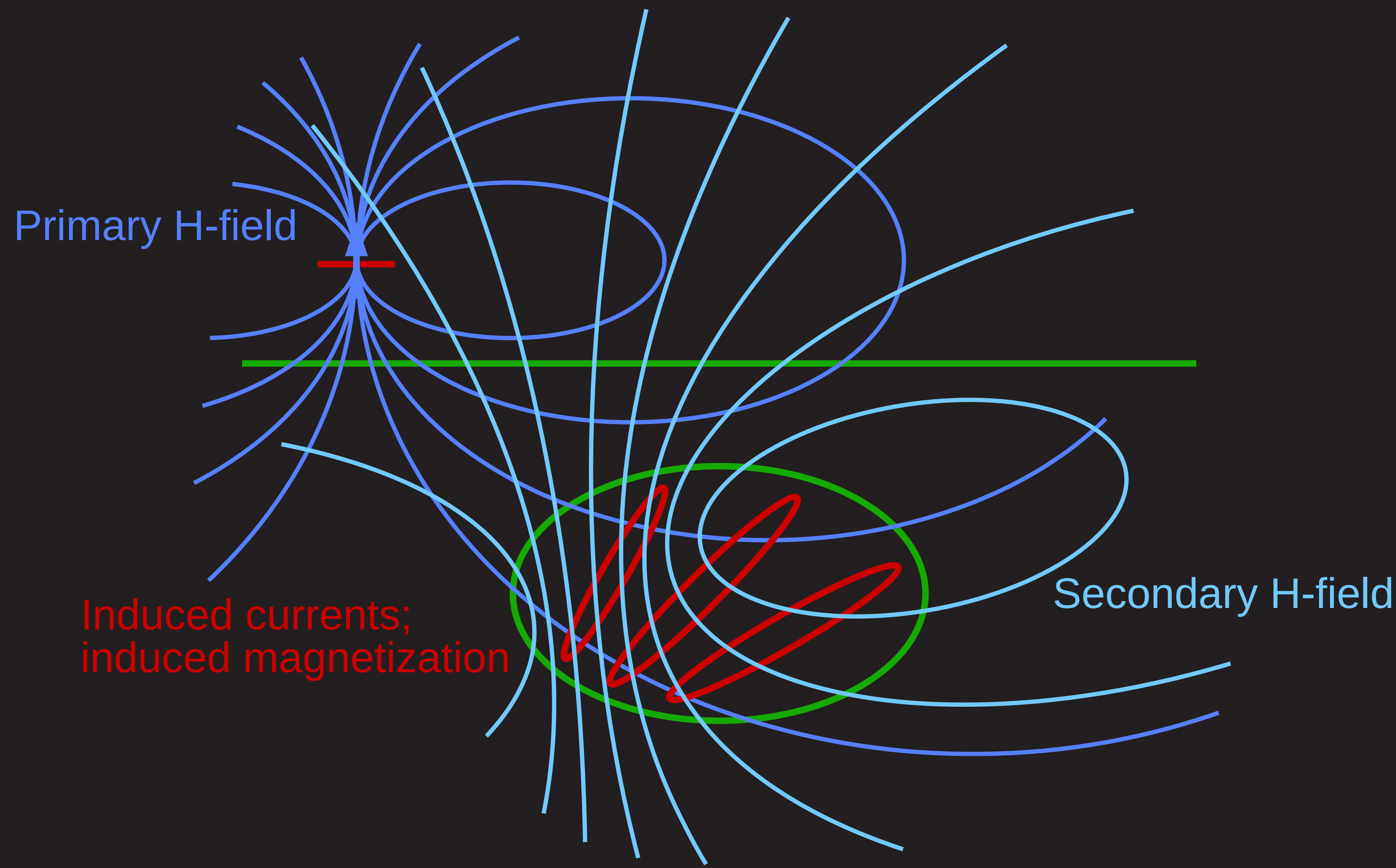
⋆ Ohm's law provides the linkage between Faraday's law and Ampère' law:

$$\nabla \times \mathbf{E} = i\omega\mathbf{B} \qquad \text{and} \qquad \nabla \times \mathbf{H} = \mathbf{J}$$

become

$$\nabla \times \mathbf{E} = i\omega\mu_0\mathbf{H} \qquad \text{and} \qquad \nabla \times \mathbf{H} = \sigma\mathbf{E}$$

5. Ohm's law (contd.)



5. Ohm's law (contd.)

⋆ And Ohm's law, and the fact that The Earth has non-negligible conductivity, means the physics of EM geophysics is not waves but (vector) diffusion.

That is, we're in the quasi-static régime.

6. The quasi-static régime

I like John Weaver's analysis of the quasi-static situation.

Start from the non-quasi-static "curly" equations of Maxwell:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{and} \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Taking the curl of Faraday's law above:

$$\nabla \times \nabla \times \mathbf{E} = -\frac{\partial}{\partial t} \nabla \times \mathbf{B}$$

and using Ampère's law and Ohm's law to get $\mathbf{B}$ and $\mathbf{J}$ in terms of $\mathbf{E}$, respectively, gives:

$$\nabla \times \nabla \times \mathbf{E} \;=\; -\mu_0 \sigma \frac{\partial \mathbf{E}}{\partial t} \;-\; \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

6. The quasi-static régime (contd.)

Introduce scaled dimensionless variables. For example, do the transformation to the new variable t' such that:

$$t' = \frac{1}{T} t$$

where T is a typical period for the fields being investigated. This means that, for the periods of interest, t' will have "sensible" values.

For example, if one is dealing with periods of, say, 100 to 200 seconds, and so $t \sim 100 \longrightarrow 200$ s, taking $T = 100$ s would mean that $t' \sim 1 \longrightarrow 2$.

Conversely, if one is dealing with frequencies of 1 kHz, and so $t \sim 0.001$ s, taking $T = 0.001$ s would mean that $t' \sim 1$.

6. The quasi-static régime (contd.)

Also, do the transformation to the new spatial variable x' such that:

$$x' = \frac{1}{L}x$$

where L is a typical spatial dimension for the situation being investigated. This means that, for the geological features of interest, x' will have "sensible" values.

For example, if one is dealing with a region of the Earth of, say, 10 km in diameter and so $x \sim 10,000$ m, taking $L = 10,000$ m would mean that $x' \sim 1$.

The preceding two changes-of-variables means the derivatives change as follows:

$$\frac{\partial}{\partial t} = \frac{1}{T}\frac{\partial}{\partial t'} \qquad \text{and} \qquad \frac{\partial}{\partial x} = \frac{1}{L}\frac{\partial}{\partial x'}$$

6. The quasi-static régime (contd.)

$$\nabla \times \nabla \times \mathbf{E} \;=\; -\mu_0 \sigma \frac{\partial \mathbf{E}}{\partial t} \;-\; \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Doing the change of variables on the general electric-field PDE (repeated above) gives:

$$\nabla' \times \nabla' \times \mathbf{E} \;=\; -\eta_\sigma \frac{\partial \mathbf{E}}{\partial t'} \;-\; \eta_\epsilon \frac{\partial^2 \mathbf{E}}{\partial t'^2}$$

where

$$\eta_\sigma = \frac{\mu_0 \sigma L^2}{T} \qquad \text{and} \qquad \eta_\epsilon = \frac{\mu_0 \epsilon_0 L^2}{T^2} = \frac{L^2}{c^2 T^2}$$

Let's try some numbers ...

6. The quasi-static régime (contd.)

$$\eta_\sigma = \frac{\mu_0 \sigma L^2}{T} \qquad \text{and} \qquad \eta_\epsilon = \frac{L^2}{c^2 T^2}$$

$L = 10\,\text{km}$, $T = 10\,\text{s}$, $\sigma = 0.01\,\text{S/m}$ ...

$$\eta_\sigma \sim \left(\frac{10^{-6} 10^{-2} 10^{8}}{10}\right) \sim 0.1 \qquad \eta_\epsilon \sim \left(\frac{10^{8}}{10^{17} 10^{2}}\right) \sim 10^{-11}$$

$L = 100\,\text{m}$, $T = 10^{-4}\,\text{s}$, $\sigma = 10^{-2}\,\text{S/m}$ ...

$$\eta_\sigma \sim \left(\frac{10^{-6} 10^{-2} 10^{4}}{10^{-4}}\right) \sim 1 \qquad \eta_\epsilon \sim \left(\frac{10^{4}}{10^{17} 10^{-8}}\right) \sim 10^{-5}$$

6. The quasi-static régime (contd.)

Also consider the ratio:

$$\frac{\eta_\sigma}{\eta_\epsilon} = \mu_0 c^2 \sigma\, T = 10^{-6} 10^{17} \sigma\, T = 10^{11} \sigma\, T$$

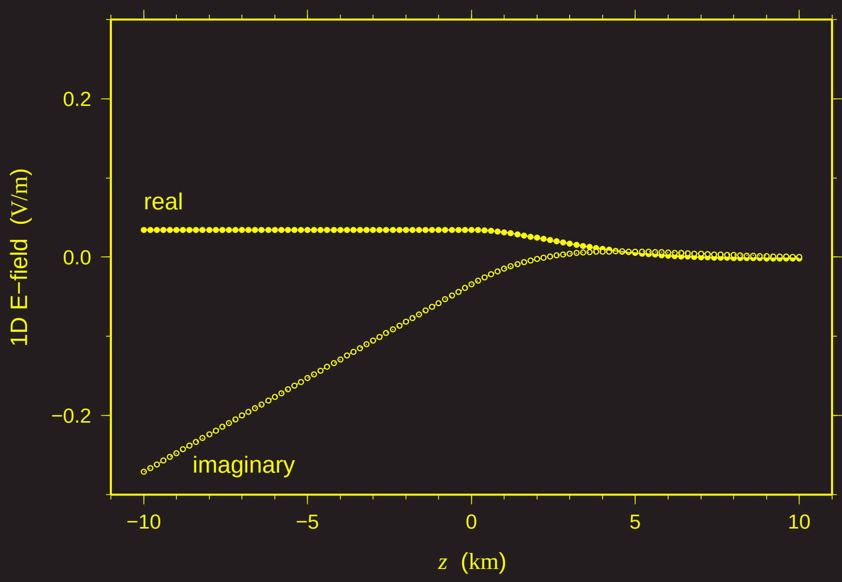
$\star$ So conductivity and period have to be amazingly small for the wavy term to be anywhere near the same size as the diffusion term.

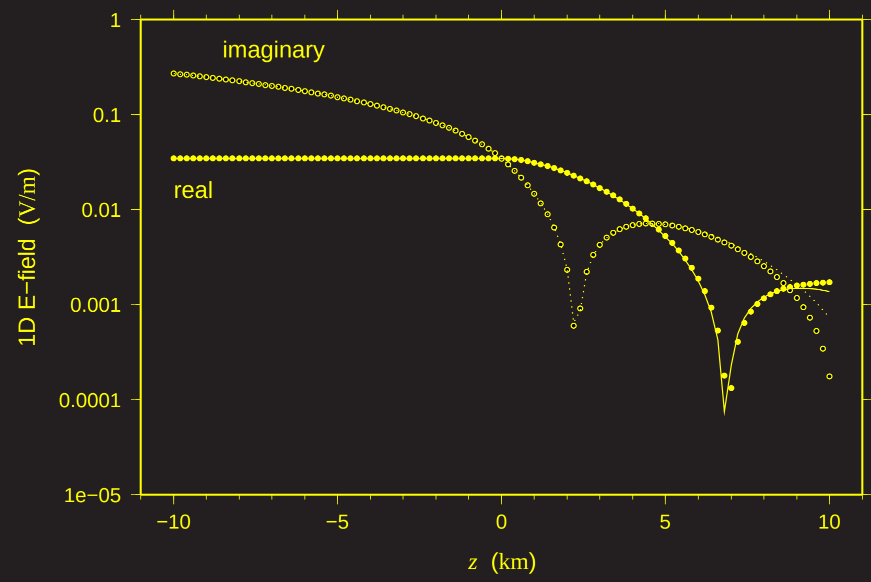
$\star$ This is the "quasi-static" régime.

Effectively, the time taken for light to travel across the volume of the Earth we're studying is much much smaller than the time variations of the fields we're interested in.

Equivalently, the wavelengths at our periods of interest are much much longer than any of the geological features we're studying.

6. The quasi-static régime (contd.)





For numerical modelling, have to pay attention to *skin depth* relative to cell size and domain size.

Don't need to worry about a wavelength relative to cell size.
Don't need to worry about reflections from domain boundaries.

6. The quasi-static régime (contd.)

One final comment ...

η_σ can be small relative to 1 for low frequencies.

This means that the conductivity term in the general electric-field PDE:

$$\nabla \times \nabla \times \mathbf{E} \;=\; -\mu_0 \sigma \frac{\partial \mathbf{E}}{\partial t} \;-\; \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

is small relative to the $\nabla \times \nabla \times \mathbf{E}$ term.

This means that even the conductivity information is dropping out of this E-field PDE!

7. Conservation of charge

Conservation of charge (in the quasi-static régime) is:

$$\nabla \cdot \mathbf{J} \;=\; 0$$

(in a source-free region).

This is a statement that charge is neither created nor destroyed.

It can also be obtained by taking the divergence of Ampère's law:

$$\begin{aligned} \nabla \cdot \nabla \times \mathbf{H} &= \nabla \cdot \mathbf{J} \\ 0 &= \nabla \cdot \mathbf{J} \end{aligned}$$

7. Conservation of charge (contd.)

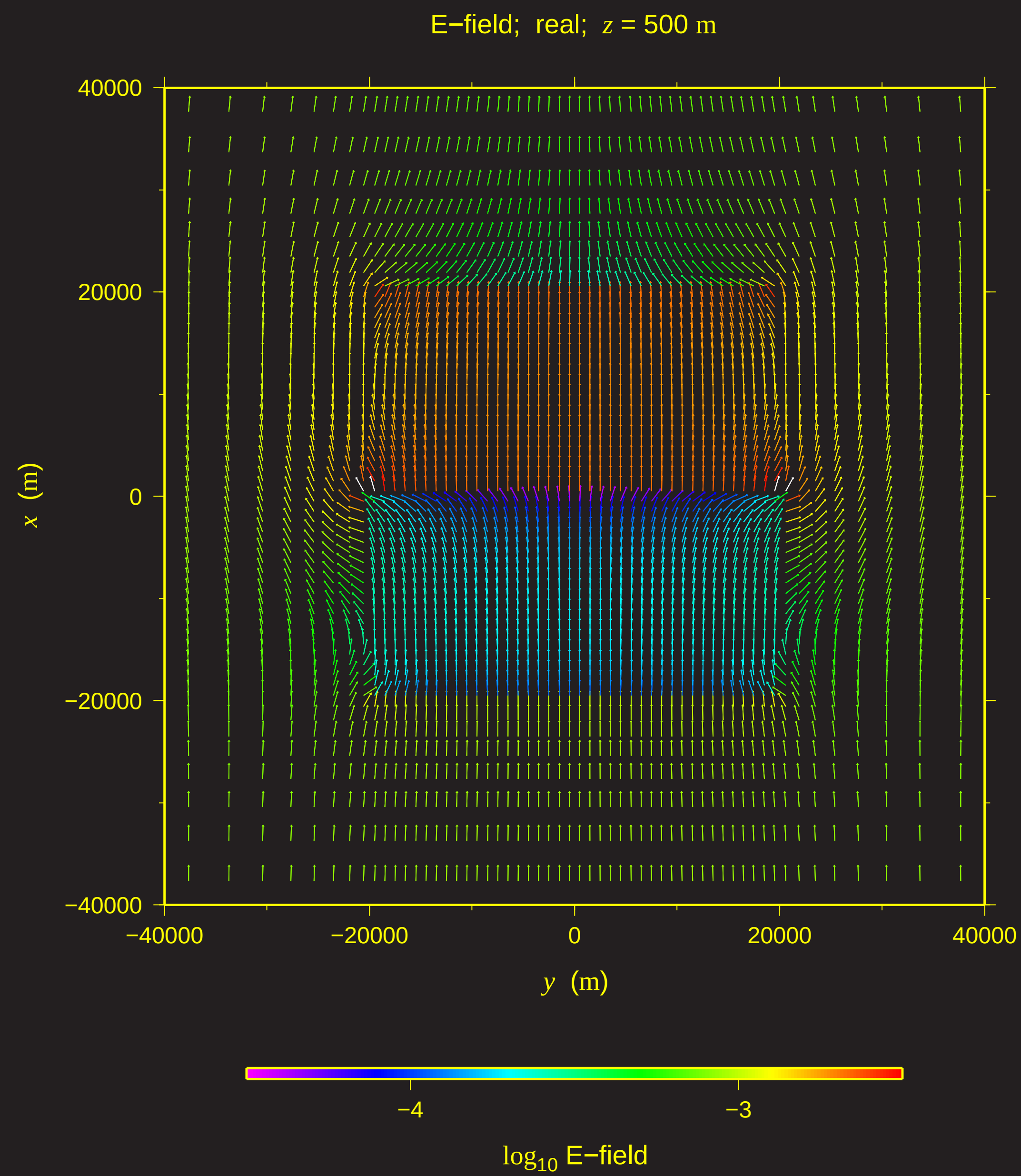
$$\nabla \cdot \mathbf{J} = 0 \quad \longrightarrow \quad \nabla \cdot (\sigma \mathbf{E}) = 0$$

This is the governing equation for DC resistivity (at zero frequency; it is also valid for our slowly varying electric fields).

DC resistivity $\longrightarrow$ “galvanic” effects. Current channelled into better conductors; current avoids poorer conductors.

Picture . . .

7. Conservation of charge (contd.)



7. Conservation of charge (contd.)

There are consequences of conservation of charge directly relevant to numerical modelling ...

$\star$ At an interface between two regions of different conductivities, $\mathbf{J} \cdot \hat{\mathbf{n}}$ is continuous. This tells us what the jump in the normal electric field should be.

Also, product rule differentiation means:

$$\nabla \cdot (\sigma \mathbf{E}) = \sigma \nabla \cdot \mathbf{E} + \mathbf{E} \cdot \nabla \sigma = 0$$

$\star$ In a uniform region, i.e., where σ is independent of position, $\nabla \cdot \mathbf{E} = 0$.

8. Other interface conditions

Across an interface between two regions of different conductivities:

$\longrightarrow$ normal $\mathbf{J}$ is continuous

$\longrightarrow$ tangential $\mathbf{E}$ is continuous

$\longrightarrow$ tangential $\mathbf{H}$ is continuous.

These can be derived from Maxwell's equations.

9. Electric field in homogeneous regions

The curl-curl term in the electric-field PDE:

$$\nabla \times \nabla \times \mathbf{E} \; - \; i\omega\mu_0\sigma\mathbf{E} \; = \; 0$$

can be rewritten to give:

$$\nabla\big(\nabla \cdot \mathbf{E}\big) \; - \; \nabla^2\mathbf{E} \; - \; i\omega\mu_0\sigma\mathbf{E} \; = \; 0$$

In a region of uniform conductivity, $\nabla \cdot \mathbf{E} = 0$. So:

$$\nabla^2\mathbf{E} \; + \; i\omega\mu_0\sigma\mathbf{E} \; = \; 0$$

Taking 2-D Fourier transform:

$$\frac{\partial^2\mathbf{E}}{\partial z^2} \; - \; \left(k_x^2 + k_y^2 - i\omega\mu_0\sigma\right)\mathbf{E} \; = \; 0$$

9. Electric field in homogeneous regions (contd.)

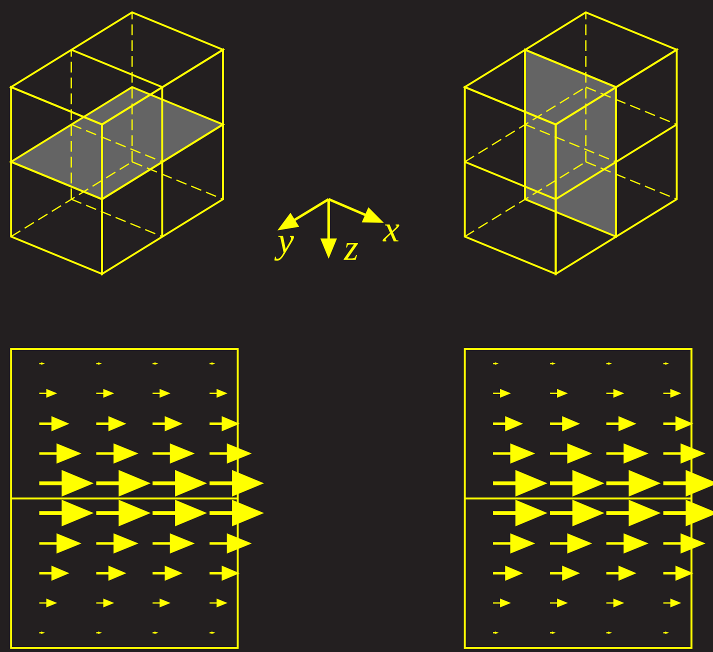
$$\frac{\partial^2 \mathbf{E}}{\partial z} - (k_x^2 + k_y^2 - i\omega\mu_0\sigma)\,\mathbf{E} = 0$$

The solution to this is of the form:

$$\mathbf{E} \sim \sum_{k_x,k_y} C(k_x, k_y) \exp\left\{\pm\sqrt{k_x^2 + k_y^2 - i\omega\mu_0\sigma}\; z\right\}$$

10. Example of rules and numerical modelling

Edge-element finite elements:



$\longrightarrow$ Tangential $\tilde{E}$ is continuous from one cell to the next.

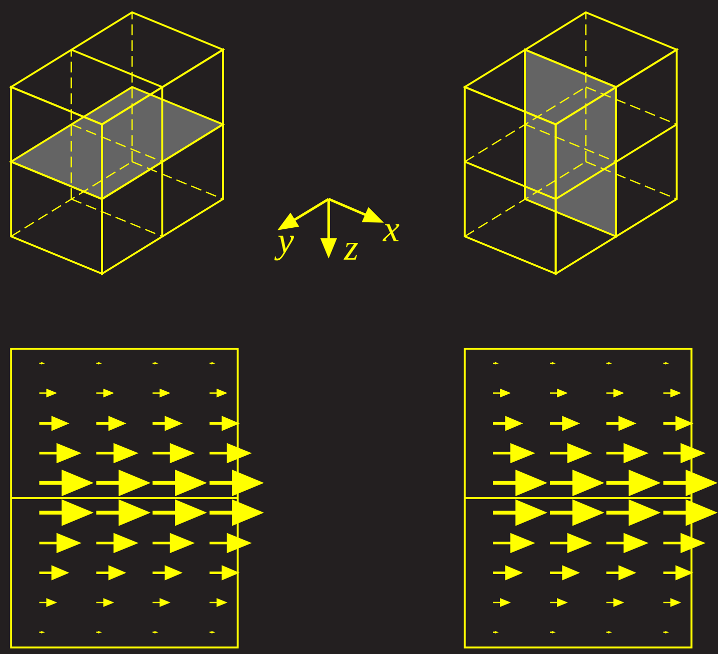
$\longrightarrow$ $\tilde{\mathbf{E}}$ is divergence-free within each cell.

$\longrightarrow$ $\nabla \times \tilde{\mathbf{E}}$, and hence $\tilde{\mathbf{H}}$, is non-zero in a cell.

$\longrightarrow$ Normal $\tilde{E}$ can be discontinuous between cells.

10. Example of rules and numerical modelling

Edge-element finite elements:



→ But $\tilde{\mathbf{H}}$ is constant in a cell.

→ But $\tilde{\mathbf{E}}$ only varies linearly in perpendicular directions and is constant in the direction in which it points, so doesn't satisfy PDE in homogeneous regions.

11. Potentials!

In the general case,

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \qquad \text{and} \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

where ϕ and $\mathbf{A}$ are scalar and vector potentials respectively (with, e.g., $\nabla \cdot \mathbf{A} = 0$).

For steady currents and fields, ϕ can quantify DC resistivity physics, i.e., "galvanic" effects.

The magnetic field depends only on $\mathbf{A}$, and so $\mathbf{A}$ can be associated with magnetic-field linkages, i.e., "inductive" effects.

So, in general time-varying case, can divide things up using ϕ and $\mathbf{A}$ into "galvanic" and "inductive" effects.

12. "Take-home" points

- ⋆ There are many laws that true electromagnetic fields obey.
- ⋆ The approximate EM fields computed using a numerical method will not satisfy all these rules.
- ⋆ The "trick" in numerical modelling is to make sure that the important rules are satisfied, and that the rules that are violated are the less important ones.