Numerical Modelling for Geophysical Electromagnetic Methods

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Outline

I. 1-D

II. EM rules & regulations

III. Finite difference

IV. Finite element

V. Integral equation

III. Finite difference

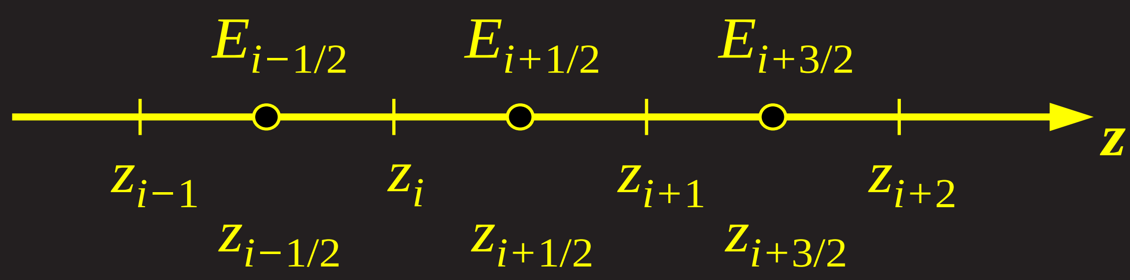
1. Revisiting the 1-D example from Lecture I

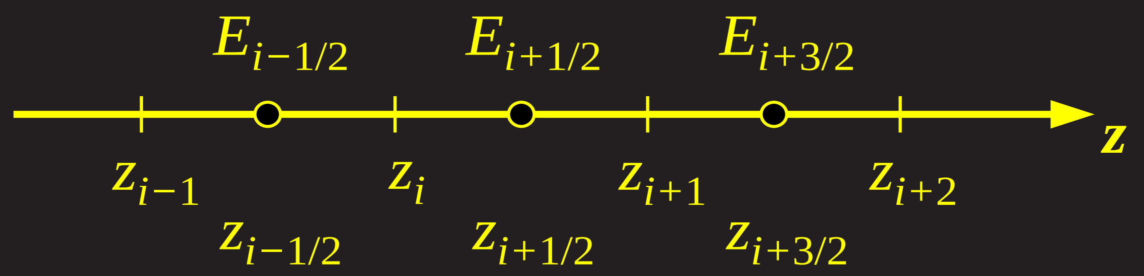
Remembering the 1-D example ...

The electric-field differential equation was:

$$\frac{\partial^2 E_x}{\partial z^2} + i\omega\mu_0\sigma(z)\, E_x = 0$$

Consider the $(i + \frac{1}{2})$th node and its neighbours:





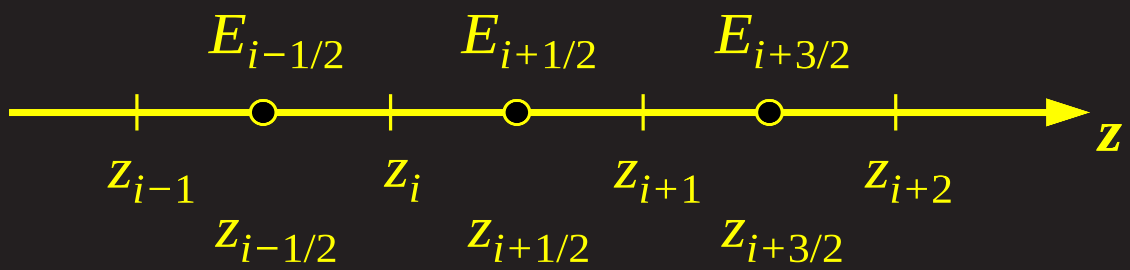
Approximating derivatives by the "obvious" finite differences (assuming layers of equal thickness Δ) gave:

$$\left.\frac{\partial E}{\partial z}\right|_{z_i} \approx \frac{E_{i+\frac{1}{2}} - E_{i-\frac{1}{2}}}{\Delta z} \quad \text{and} \quad \left.\frac{\partial E}{\partial z}\right|_{z_{i+1}} \approx \frac{E_{i+\frac{3}{2}} - E_{i+\frac{1}{2}}}{\Delta z}$$

This is fine. It gave good results, after all.

But let's try to be a bit more rigorous. Also, "obvious" only works for this simplest of examples.

2. Derivation of finite differences via Taylor's series



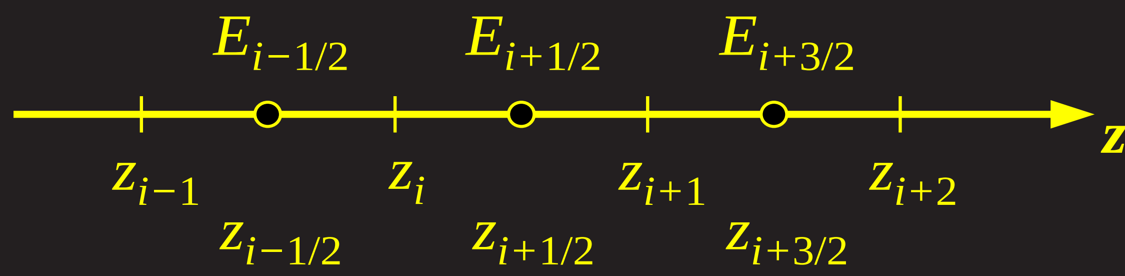
Expanding the electric field about $z = z_i$ gives:

$$E(z) = E_i + \left.\frac{\partial E}{\partial z}\right|_{z_i} (z - z_i) + O\big((z - z_i)^2\big)$$

This expression can be used to approximate the electric field at $z_{i-\frac{1}{2}}$ and $z_{i+\frac{1}{2}}$:

$$E_{i+\frac{1}{2}} = E_i + \left.\frac{\partial E}{\partial z}\right|_{z_i} (z_{i+\frac{1}{2}} - z_i) + O\big(\Delta^2\big)$$

$$E_{i-\frac{1}{2}} = E_i + \left.\frac{\partial E}{\partial z}\right|_{z_i} (z_{i-\frac{1}{2}} - z_i) + O\big(\Delta^2\big)$$



$$E_{i+\frac{1}{2}} = E_i + \left.\frac{\partial E}{\partial z}\right|_{z_i} (z_{i+\frac{1}{2}} - z_i) + O(\Delta^2)$$

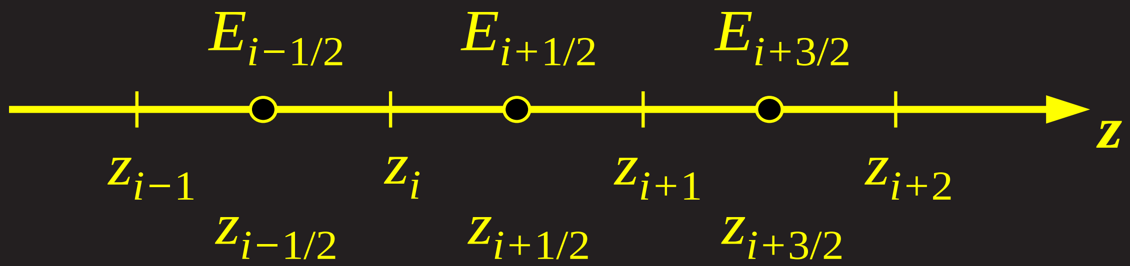
$$E_{i-\frac{1}{2}} = E_i + \left.\frac{\partial E}{\partial z}\right|_{z_i} (z_{i-\frac{1}{2}} - z_i) + O(\Delta^2)$$

Subtracting the second of these expressions from the first gives:

$$E_{i+\frac{1}{2}} - E_{i-\frac{1}{2}} = \left.\frac{\partial E}{\partial z}\right|_{z_i} (z_{i+\frac{1}{2}} - z_{i-\frac{1}{2}}) + O(\Delta^2)$$

and hence

$$\frac{E_{i+\frac{1}{2}} - E_{i-\frac{1}{2}}}{z_{i+\frac{1}{2}} - z_{i-\frac{1}{2}}} = \left.\frac{\partial E}{\partial z}\right|_{z_i} + O(\Delta)$$

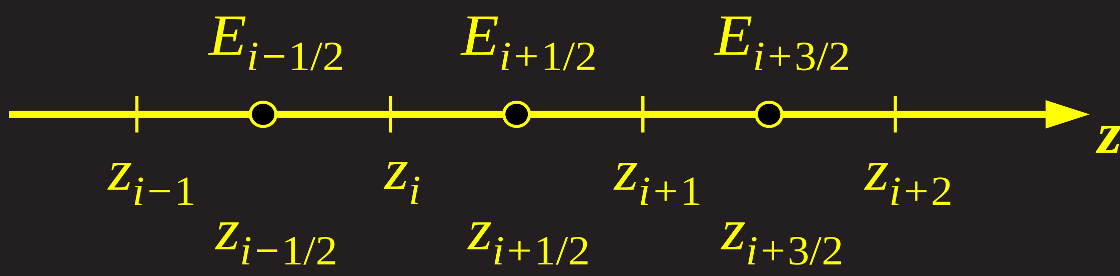


That is,

$$\left.\frac{\partial E}{\partial z}\right|_{z_i} = \frac{E_{i+\frac{1}{2}} - E_{i-\frac{1}{2}}}{\Delta z} + O(\Delta)$$

Similarly,

$$\left.\frac{\partial E}{\partial z}\right|_{z_{i+1}} = \frac{E_{i+\frac{3}{2}} - E_{i+\frac{1}{2}}}{\Delta z} + O(\Delta)$$



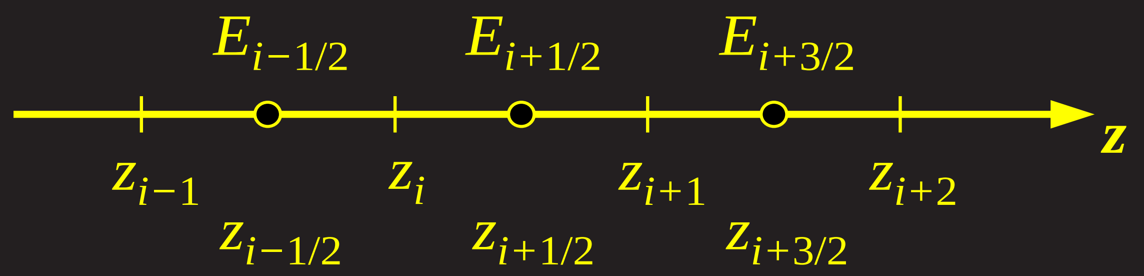
Also, expanding the derivative of the electric field about $z = z_{i+\frac{1}{2}}$ gives:

$$\frac{\partial E(z)}{\partial z} = \left.\frac{\partial E}{\partial z}\right|_{z_{i+\frac{1}{2}}} + \left[\frac{\partial}{\partial z}\frac{\partial E}{\partial z}\right]_{z_{i+\frac{1}{2}}} (z - z_{i+\frac{1}{2}}) + O\big((z - z_i)^2\big)$$

Hence:

$$\left.\frac{\partial E}{\partial z}\right|_{z_{i+1}} = \left.\frac{\partial E}{\partial z}\right|_{z_{i+\frac{1}{2}}} + \left(\frac{\partial^2 E}{\partial z^2}\right)_{z_{i+\frac{1}{2}}} (z_{i+1} - z_{i+\frac{1}{2}}) + O(\Delta^2)$$

$$\left.\frac{\partial E}{\partial z}\right|_{z_i} = \left.\frac{\partial E}{\partial z}\right|_{z_{i+\frac{1}{2}}} + \left(\frac{\partial^2 E}{\partial z^2}\right)_{z_{i+\frac{1}{2}}} (z_i - z_{i+\frac{1}{2}}) + O(\Delta^2)$$

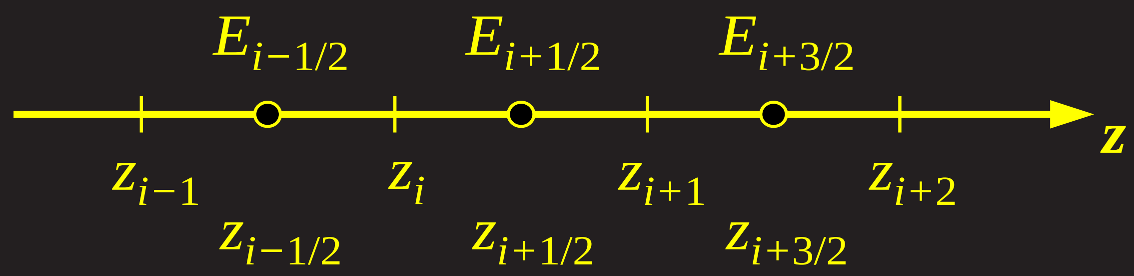


Subtracting the two preceding equations gives:

$$\left.\frac{\partial E}{\partial z}\right|_{z_{i+1}} - \left.\frac{\partial E}{\partial z}\right|_{z_i} = \left(\frac{\partial^2 E}{\partial z^2}\right)_{z_{i+\frac{1}{2}}} \left(z_{i+1} - z_i\right) + O\left(\Delta^2\right)$$

Rearranging:

$$\frac{1}{z_{i+1} - z_i}\left\{\left.\frac{\partial E}{\partial z}\right|_{z_{i+1}} - \left.\frac{\partial E}{\partial z}\right|_{z_i}\right\} = \left(\frac{\partial^2 E}{\partial z^2}\right)_{z_{i+\frac{1}{2}}} + O(\Delta)$$



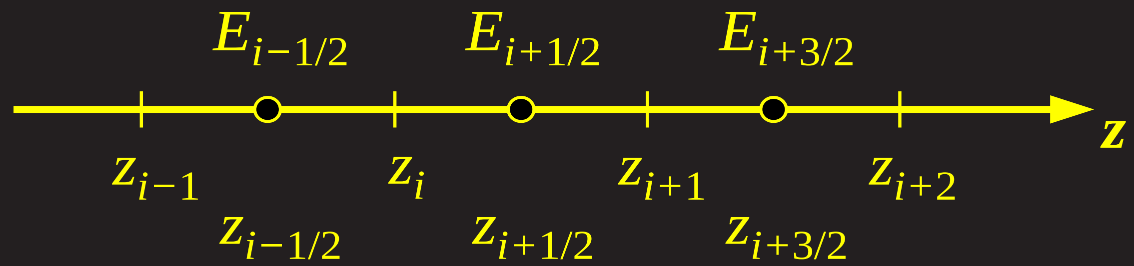
$$\left(\frac{\partial^2 E}{\partial z^2}\right)_{z_{i+\frac{1}{2}}} = \frac{1}{\Delta z}\left\{\frac{\partial E}{\partial z}\bigg|_{z_{i+1}} - \frac{\partial E}{\partial z}\bigg|_{z_i}\right\} + O(\Delta)$$

Substituting the approximations for the first-order derivatives:

$$\left(\frac{\partial^2 E}{\partial z^2}\right)_{z_{i+\frac{1}{2}}} =$$

$$\frac{1}{\Delta z}\left\{\frac{E_{i+\frac{3}{2}} - E_{i+\frac{1}{2}}}{\Delta z} - \frac{E_{i+\frac{1}{2}} - E_{i-\frac{1}{2}}}{\Delta z} + O(\Delta)\right\}$$

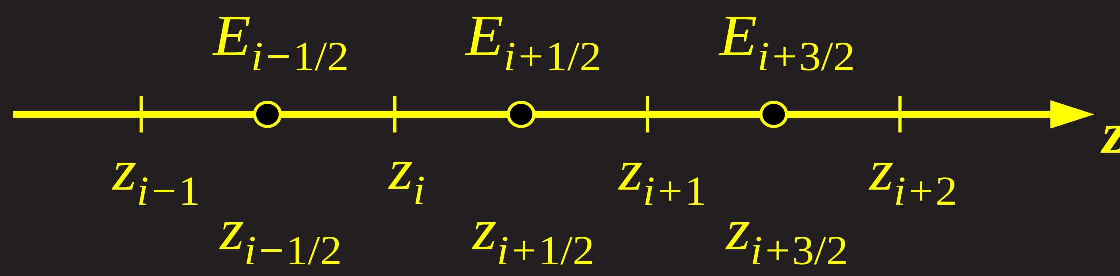
$$+\,O(\Delta)$$



$$\left.\frac{\partial^2 E}{\partial z^2}\right|_{z_{i+\frac{1}{2}}} = \frac{1}{\Delta z^2}\left\{E_{i+\frac{3}{2}} - E_{i+\frac{1}{2}} - E_{i+\frac{1}{2}} + E_{i-\frac{1}{2}}\right\} + \frac{1}{\Delta z}O(\Delta) + O(\Delta)$$

That is:

$$\left.\frac{\partial^2 E}{\partial z^2}\right|_{z_{i+\frac{1}{2}}} = \frac{1}{\Delta z^2}\left\{E_{i+\frac{3}{2}} - 2E_{i+\frac{1}{2}} + E_{i-\frac{1}{2}}\right\} + O(1)$$



So, an approximation of the PDE that is centred on the $(i+\frac{1}{2})$th node is:

$$\frac{1}{\Delta z^2}\left\{E_{i+\frac{3}{2}} - 2E_{i+\frac{1}{2}} + E_{i-\frac{1}{2}}\right\} + \; i\omega\mu_0\sigma_i\, E_{i+\frac{1}{2}} + O\big(1\big) \;=\; 0$$

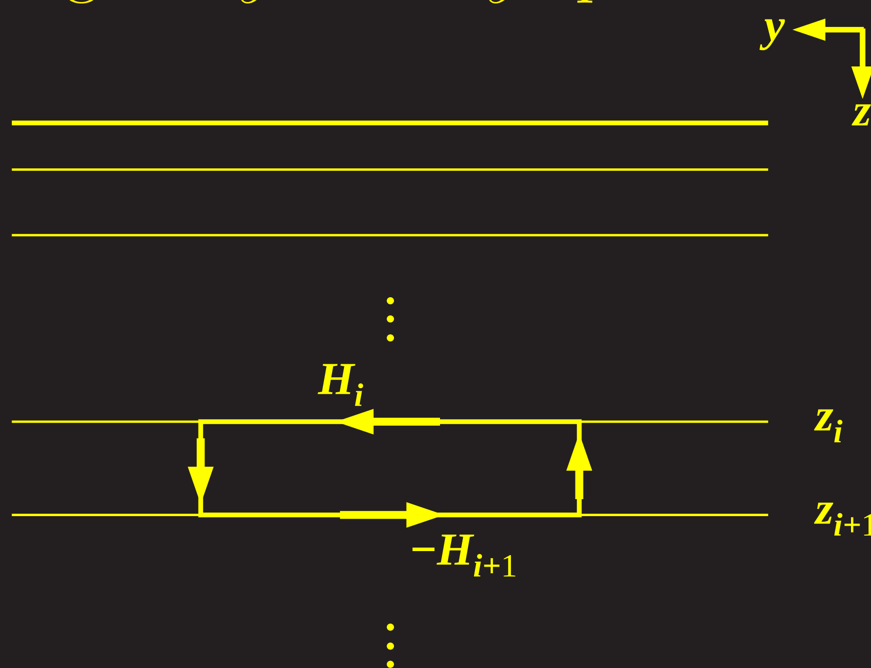
(where σ_i is the conductivity of the ith layer, i.e., between z_i and z_{i+1}.)

Woo-hoo! We've now got an approximation to the PDE complete with accuracy analysis.

But not so fast! This is okay for the derivatives. But what about the σ term?

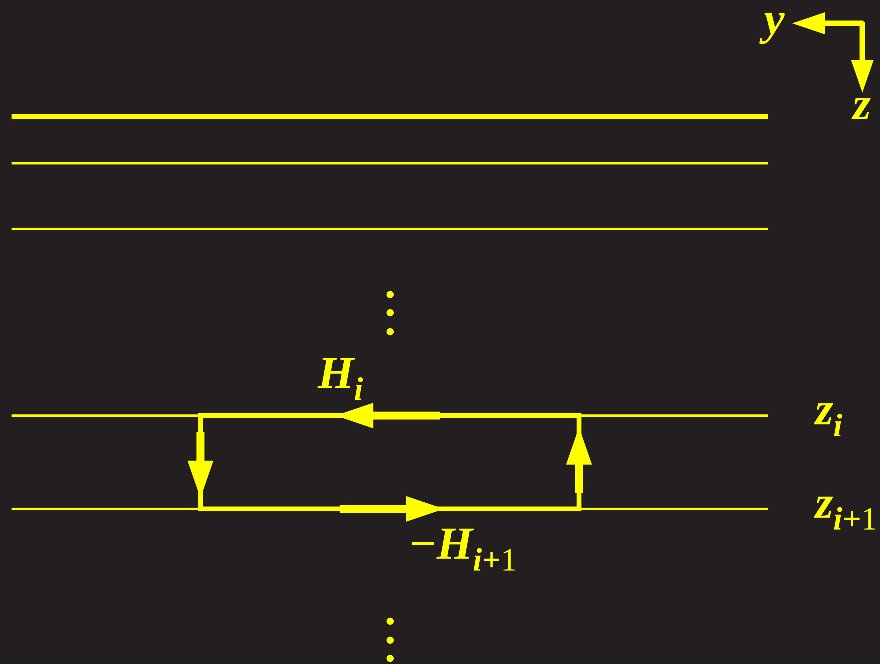
3. Derivation via Maxwell's equations and Taylor's series

Consider a loop of length Δy in the yz-plane:



And let's consider the integral form of Ampère's law around this loop:

$$\int_C \mathbf{H} \cdot \hat{\mathbf{t}}\, dl \;=\; \int_S \mathbf{J} \cdot \hat{\mathbf{n}}\, ds$$

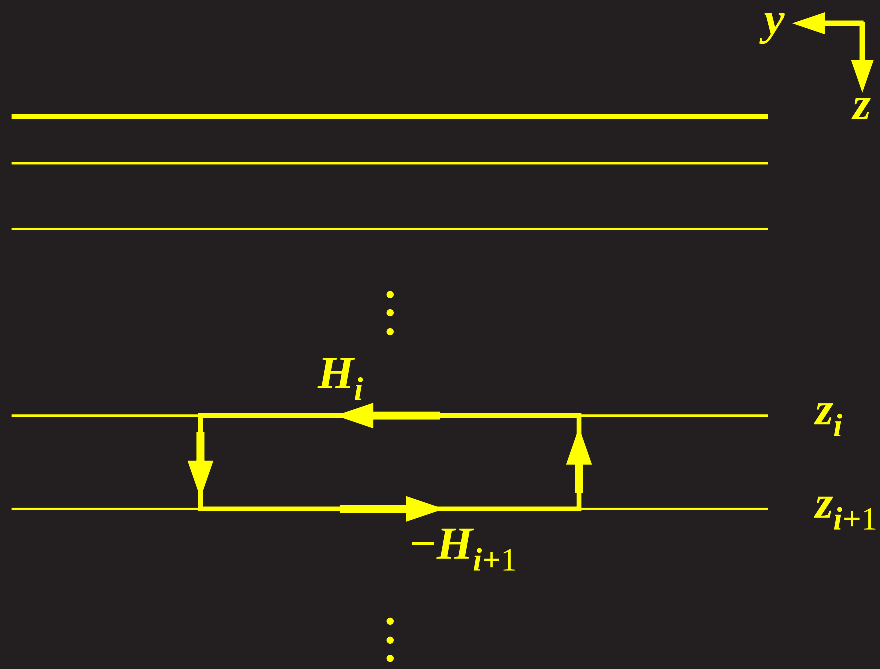


→ 1-D situation, so no vertical H-field.

→ 1-D situation, so H_i is constant along the top segment, and $H_{i+\frac{1}{2}}$ is constant along the bottom segment.

So:

$$\int_C \mathbf{H} \cdot \hat{\mathbf{t}}\, dl = H_i \Delta y - H_{i+1} \Delta y$$



For the area integral:

$$\int_S \mathbf{J} \cdot \hat{\mathbf{n}}\, ds \;=\; \int_{\Delta y} \int_{z_i}^{z_{i+1}} \sigma\, E_x(z)\, dy\, dz \;=\; \Delta y\, \sigma_i \int_{z_i}^{z_{i+1}} E_x(z)\, dz$$

Use a Taylor's series approximation for $E(z)$ about the centre of the layer:

$$E(z) \;=\; E_{i+\frac{1}{2}} \;+\; \left.\frac{\partial E}{\partial z}\right|_{z_{i+\frac{1}{2}}} \left(z - z_{i+\frac{1}{2}}\right) \;+\; O\Big(\big(z - z_{i+\frac{1}{2}}\big)^2\Big)$$

Using this approximation in the integral gives:

$$\begin{aligned}\int_{z_i}^{z_{i+1}} E(z)\,dz &= E_{i+\frac{1}{2}} \int_{z_i}^{z_{i+1}} dz \\ &\quad + \left.\frac{\partial E}{\partial z}\right|_{z_{i+\frac{1}{2}}} \int_{z_i}^{z_{i+1}} (z - z_{i+\frac{1}{2}})\,dz \\ &\quad + O\big((z - z_{i+\frac{1}{2}})^2\big) \int_{z_i}^{z_{i+1}} dz \\ &= E_{i+\frac{1}{2}} \Delta z + O\big(\Delta z^3\big)\end{aligned}$$

Hence:

$$\int_S \mathbf{J} \cdot \hat{\mathbf{n}}\, ds = \sigma_i\, E_{i+\frac{1}{2}} \Delta y \Delta z + O\big(\Delta z^3\big)$$

Putting together the two sides of Ampère's law for the loop gives:

$$H_i \Delta y \;-\; H_{i+1} \Delta y \;=\; \sigma_i \, E_{i+\frac{1}{2}} \Delta y \Delta z \;+\; O\left(\Delta z^3\right)$$

Dividing through by Δz (and by Δy) gives:

$$\frac{H_i - H_{i+1}}{\Delta z} \;=\; \sigma_i \, E_{i+\frac{1}{2}} \;+\; O\left(\Delta z^2\right)$$

From Faraday's law for this 1-D situation:

$$H_y \;=\; -\frac{i}{\omega \mu_0} \frac{\partial E_x}{\partial z}$$

So . . .

$$\frac{1}{\Delta z}\left\{\left.\frac{\partial E}{\partial z}\right|_{z_i} - \left.\frac{\partial E}{\partial z}\right|_{z_{i+1}}\right\} = i\omega\mu_0\sigma_i\, E_{i+\frac{1}{2}} + O(\Delta z^2)$$

From our previous analysis for the derivative terms:

$$\left.\frac{\partial E}{\partial z}\right|_{z_i} = \frac{E_{i+\frac{1}{2}} - E_{i-\frac{1}{2}}}{\Delta z} + O(\Delta)$$

$$\left.\frac{\partial E}{\partial z}\right|_{z_{i+1}} = \frac{E_{i+\frac{3}{2}} - E_{i+\frac{1}{2}}}{\Delta z} + O(\Delta)$$

So the approximation of the PDE is ...

So the approximation of the PDE centred at $z_{i+\frac{1}{2}}$ is:

$$\frac{1}{\Delta z^2}\left\{-E_{i-\frac{1}{2}} + 2E_{i+\frac{1}{2}} - E_{i+\frac{3}{2}}\right\} + O\big(1\big) = i\omega\mu_0\sigma_i\, E_{i+\frac{1}{2}} + O\big(\Delta^2\big)$$

That is,

$$\frac{1}{\Delta z^2}\left\{E_{i+\frac{3}{2}} - 2E_{i+\frac{1}{2}} + E_{i-\frac{1}{2}}\right\} + i\omega\mu_0\sigma_i\, E_{i+\frac{1}{2}} + O\big(1\big) = 0$$

Yipee! We've got exactly the same result as before. But at least we now know the accuracies (or rather inaccuracies) coming from both parts of the PDE. (And we justified the "obvious" average conductivity around $z_{i+\frac{1}{2}}$.)

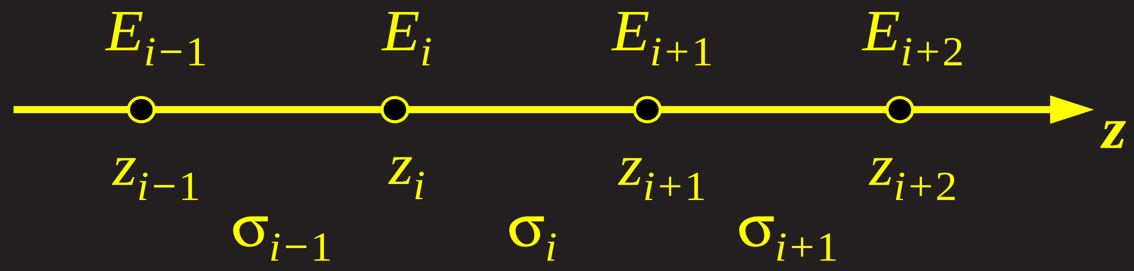
4. More on this “finite-volume” approach

The preceding analysis using Ampère’s law around a specific loop is an example of the “finite-volume” approach to the finite-difference method.

In our 1-D example so far, it’s been obvious how to approximate the derivatives.

It is not so obvious in higher dimensions. And definitely not so if one were to consider non-rectilinear meshes.

For a simple example, consider the 1-D situation, but with the approximate electric field specified at the layer interfaces ...

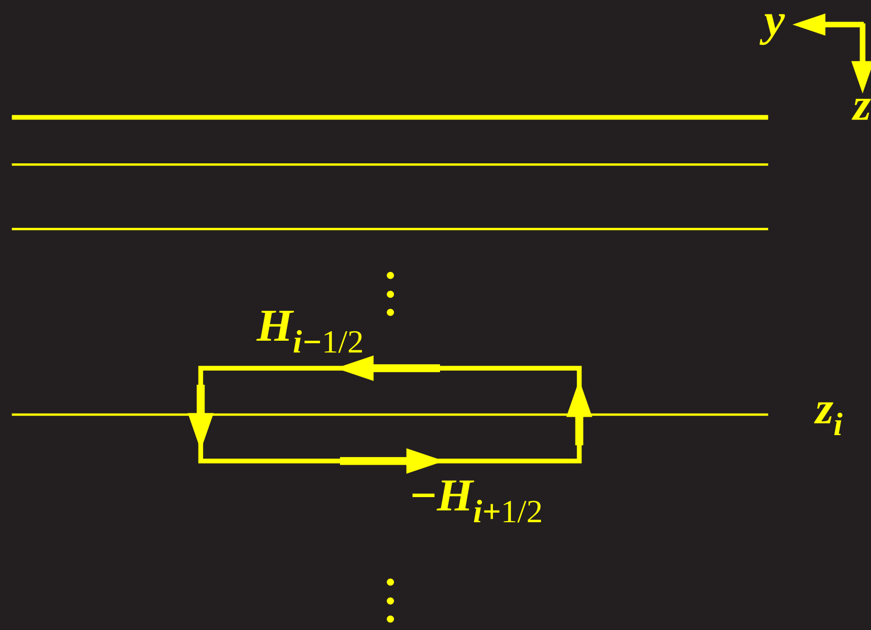


(This is a sensible discretization (so was the previous one) because E_x is continuous across the layer interfaces, i.e., at the nodes.)

The derivatives could be approximated in the same way as before.

But now what about the σ term, since there can be different conductivities on either side of a node?

One can repeat the analysis with the Ampèrian loop ...

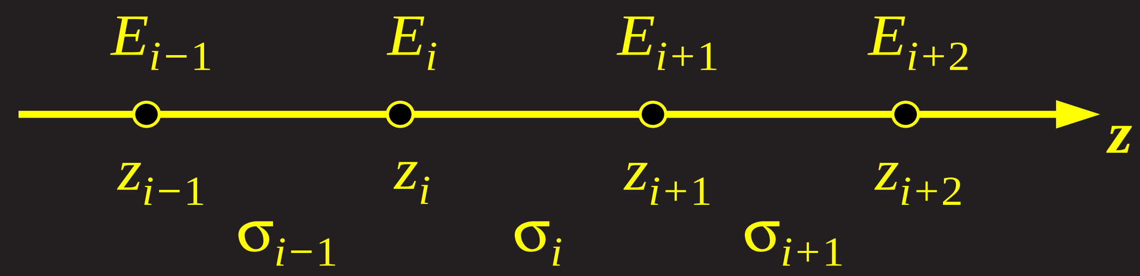


For the area integral:

$$\int_S \mathbf{J}\cdot\hat{\mathbf{n}}\,ds \;=\; \Delta y \int_{z_{i-\frac{1}{2}}}^{z_{i+\frac{1}{2}}} \sigma(z)\,E_x(z)\,dz$$

$$\;=\; \Delta y \int_{z_{i-\frac{1}{2}}}^{z_i} \sigma_{i-1}\,E_x(z)\,dz \;+\; \Delta y \int_{z_i}^{z_{i+\frac{1}{2}}} \sigma_i\,E_x(z)\,dz$$

$$\;=\; \left(\frac{\sigma_i}{2}+\frac{\sigma_{i-1}}{2}\right) E_i\,\Delta y\,\Delta z \;+\; O(\Delta^3)$$



This would lead to the following approximation for the electric-field PDE:

$$\frac{1}{\Delta z^2}\left\{E_{i+1} - 2E_i + E_{i-1}\right\} + i\omega\mu_0\left(\frac{\sigma_{i-1}}{2} + \frac{\sigma_i}{2}\right) E_i + O(1) = 0$$

Proceed as before: assemble matrix equation from approximations to PDE at all nodes, impose boundary condition information, and solve using your favourite solver.

5. Continuing the accuracy analysis

Maintaining the accuracy analysis to the stage of the matrix equation leads to:

$$\underline{\mathbf{A}}\,\mathbf{x} = \mathbf{b} + O(1)$$

(This kind of accuracy analysis is not definitive, but it's better than nothing.)

The matrix $\underline{\mathbf{A}}$ is full of second-order spatial finite differences. These are of the form $1/\Delta z^2$, and so are $O(\Delta^{-2})$.

So, the inverse of $\underline{\mathbf{A}}$, i.e., $\underline{\mathbf{A}}^{-1}$, is $O(\Delta^2)$.

Hence:

$$\mathbf{x} = \underline{\mathbf{A}}^{-1}\mathbf{b} + O(\Delta^2)O(1) = \underline{\mathbf{A}}^{-1}\mathbf{b} + O(\Delta^2)$$

5. Continuing the accuracy analysis (contd.)

$$\mathbf{x} = \underline{\mathbf{A}}^{-1}\,\mathbf{b} + O\left(\Delta^2\right)$$

So, one might expect the solution to vary as Δ^2.

That is, if the separation between nodes is halved, i.e., the number of nodes is doubled, the difference between the computed electric field and the true electric field would decrease by a factor of four.

6. Exercise 1

For the 1-D solution you coded up in the exercise for Lecture I, try different numbers and separations of nodes and see if the accuracy of the solution varies as predicted by our analysis.

7. Exercise 2

With the "finite-volume" technique that uses Ampèrian loops specifically in mind, read a paper that describes a finite-difference solution to the 3-D MT forward-modelling problem.

For example,

Mackie, Madden & Wannamaker, 1993, Geophysics, v.58(2), p.215-226;

Smith, 1996, Geophysics, v.61(5), p.1308-1318.

8. “Take-home” points

- ⋆ Taylor’s series provides a means to be somewhat rigorous in the derivation of a finite-difference solution.

- ⋆ The integral forms of Maxwell’s equations provide the means to figure out the correct averaging of physical properties that vary from cell to cell.

- ⋆ (To contrast with finite-element methods …)
  Finite-difference methods *implicitly* assume linear spatial dependence of the approximate electric field between nodes.