Numerical Modelling for Geophysical Electromagnetic Methods

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Outline

I. 1-D

II. EM rules & regulations

III. Finite difference

IV. Finite element

V. Integral equation

IV. Finite element

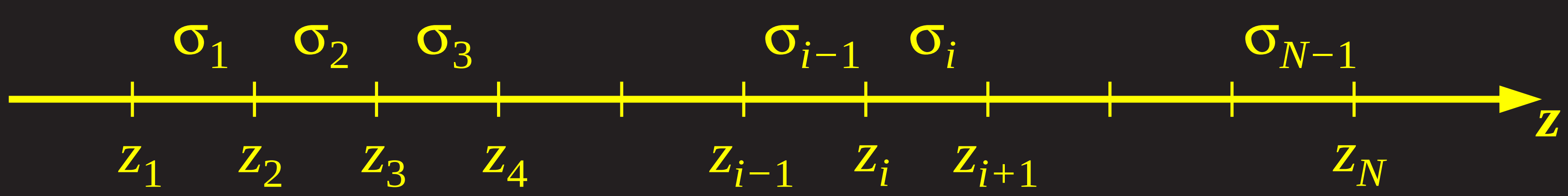
1. A 1-D example

Remembering the 1-D example . . .

The electric-field differential equation was:

$$\frac{\partial^2 E_x}{\partial z^2} + i\omega\mu_0\sigma(z)\, E_x = 0$$

Our discretized Earth model was:



2. Discretize the electric field

In the finite-element method there is an explicit specification of what the approximate electric field is, e.g.:

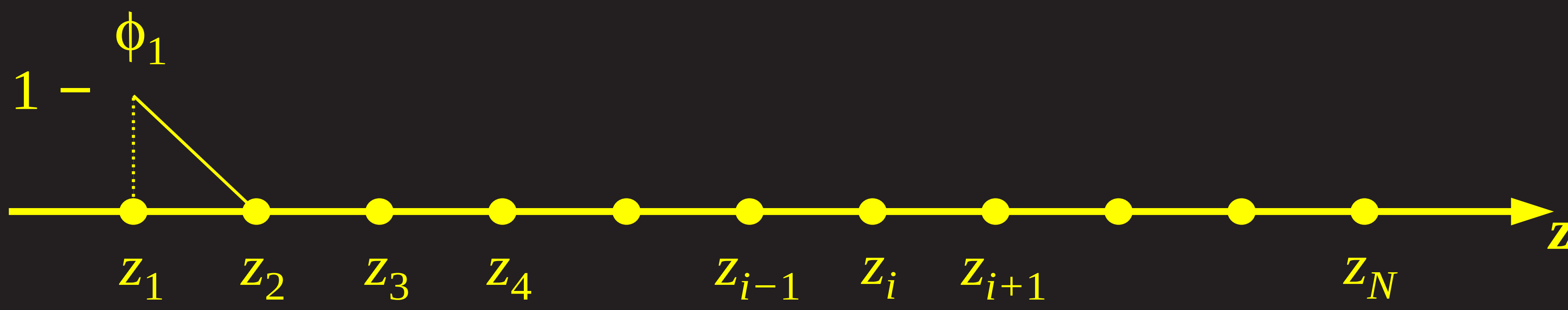
$$\tilde{E}_x(z) \;=\; \sum_{i=1}^{N} E_i\, \phi_i(z)$$

where

- $\longrightarrow$ $\phi_i(z)$ are basis functions
- $\longrightarrow$ E_i are the coefficients in this expansion
- $\longrightarrow$ with appropriate choice for ϕ_i, E_i can be the values of the approximate electric field at the nodes.

2. Discretize the electric field (contd.)

Let's choose a piecewise linear approximation for the electric field. Specifically:

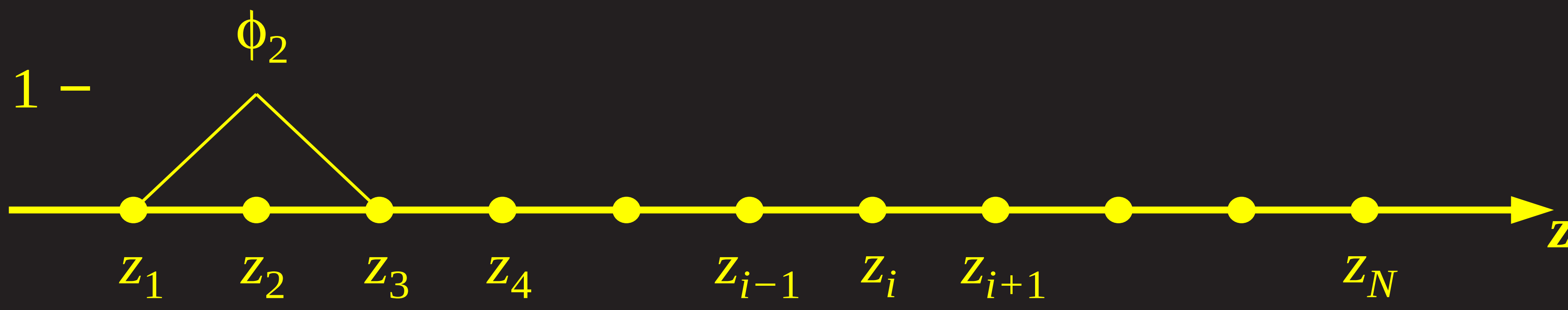


$$\phi_1(z) = \begin{cases} (z_2 - z)/(z_2 - z_1), & z_1 \le z < z_2; \\ 0, & \text{otherwise.} \end{cases}$$

With this choice:

$$\phi_1(z_1) = 1 \qquad \text{and} \qquad \phi_1(z_2) = 0$$

And:



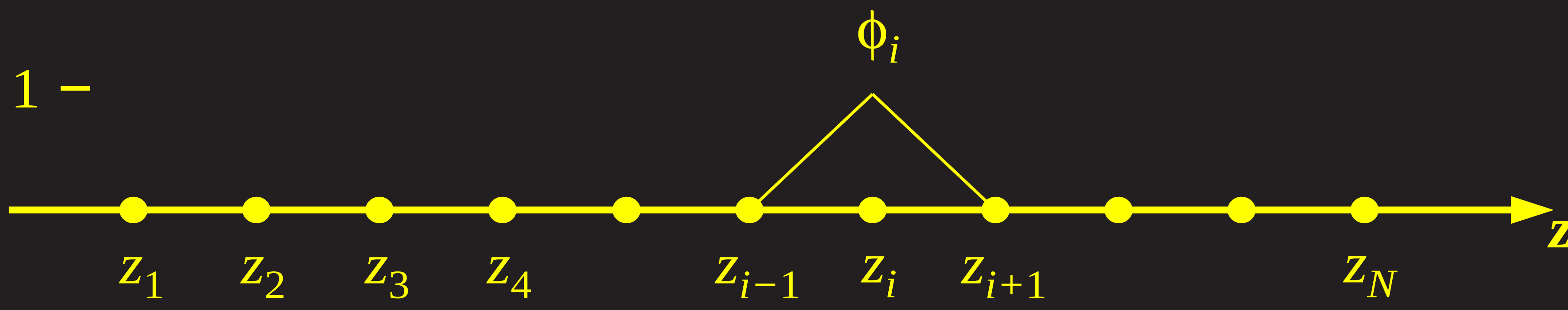
$$\phi_2(z) = \begin{cases} (z - z_1)/(z_2 - z_1), & z_1 \le z < z_2; \\ (z_3 - z)/(z_3 - z_2), & z_2 \le z < z_3; \\ 0, & \text{otherwise.} \end{cases}$$

With this choice:

$$\phi_2(z_1) = 0 \quad \text{and} \quad \phi_2(z_2) = 1 \quad \text{and} \quad \phi_2(z_3) = 0$$

and $\phi_2(z)$ is continuous at z_1, z_2, and z_3.

Similarly:



$$\phi_i(z) = \begin{cases} (z - z_{i-1})/(z_i - z_{i-1}), & z_{i-1} \leq z < z_i; \\ (z_{i+1} - z)/(z_{i+1} - z_i), & z_i \leq z < z_{i+1}; \\ 0, & \text{otherwise.} \end{cases}$$

With this choice:

$$\phi_i(z_{i-1}) = 0 \quad \text{and} \quad \phi_i(z_i) = 1 \quad \text{and} \quad \phi_i(z_{i-1}) = 0$$

and $\phi_i(z)$ is continuous at z_{i-1}, z_i, and z_{i+1}.

2. Discretize the electric field (contd.)

The preceding description of the approximate electric field is

$\longrightarrow$ linear between each pair of nodes

$\longrightarrow$ continuous at each node.

(Continuous at a node, eh? That automatically satisfies the continuity of the tangential electric field at a material boundary. Good.)

(Our finite-difference solutions also had a continuous electric field across the layer interfaces, explicitly when we chose the nodes to be at the layer interfaces, and implicitly when we chose the nodes at the layer centres (because of the implicit reliance on Taylor's series).)

4. Discretize the PDE, sort of

Returning to the PDE:

$$\frac{\partial^2 E_x}{\partial z^2} + i\omega\mu_0\sigma(z)\, E_x = 0$$

Substituting our explicit representation of the approximate electric field for the true electric field gives, simply:

$$\sum_{i=1}^{N} E_i \left\{ \frac{\partial^2 \phi_i}{\partial z^2} + i\omega\mu_0\sigma(z)\, \phi_i \right\} + R = 0$$

R is the residual or remainder that we have because we introduced an approximate electric field.

4. Discretize the PDE, sort of (contd.)

Reiterating:

$$\sum_{i=1}^{N} E_i \left\{ \frac{\partial^2 \phi_i}{\partial z^2} + i\omega\mu_0\sigma(z)\, \phi_i \right\} + R \;=\; 0$$

This is one equation in the N unknown coefficients E_i.

There are a number of (related) ways to create a system of equations from which the coefficients E_i can be determined.

I like the *method of weighted residuals*. (I find it simpler to understand. (Although not as "rigorous"?))

5. The Method of Weighted Residuals

Reiterating:

$$\sum_{i=1}^{N} E_i \left\{ \frac{\partial^2 \phi_i}{\partial z^2} + i\omega\mu_0\sigma(z)\,\phi_i \right\} + R \;=\; 0$$

The method of weighted residuals:

$\rightarrow$ choose some weight functions $\psi_j(z)$, $j = 1, \ldots, M$

$\rightarrow$ multiply the PDE by $\psi_j(z)$:

$$\sum_{i=1}^{N} E_i \left\{ \psi_j \, \frac{\partial^2 \phi_i}{\partial z^2} + i\omega\mu_0\sigma(z)\,\psi_j\,\phi_i \right\} + \psi_j\,R \;=\; 0$$

$\rightarrow$ integrate over the whole domain ...

5. The Method of Weighted Residuals (contd.)

$$\sum_{i=1}^{N} E_i \left\{ \int_{z_1}^{z_N} \psi_j \, \frac{\partial^2 \phi_i}{\partial z^2} \, dz + i\omega\mu_0 \int_{z_1}^{z_N} \sigma \, \psi_j \, \phi_i \, dz \right\} + \int_{z_1}^{z_N} \psi_j \, R \, dz = 0$$

$\longrightarrow$ and require, argue, hope that the residual R is orthogonal to the weight functions, that is:

$$\int_{z_1}^{z_N} \psi_j \, R \, dz = 0 \qquad j = 1, \ldots, M$$

Hence . . .

5. The Method of Weighted Residuals (contd.)

... the system of equations to be solved for the finite-element solution to the electric field PDE is:

$$\sum_{i=1}^{N} E_i \left\{ \int_{z_1}^{z_N} \psi_j \frac{\partial^2 \phi_i}{\partial z^2} \, dz + i\omega\mu_0 \int_{z_1}^{z_N} \sigma \, \psi_j \, \phi_i \, dz \right\} = 0$$

This system contains M equations in N unknowns.

In the particular approach called the Galerkin method, the basis functions are used as the weight functions.

This makes a square system with N equations in N unknowns.

5. The Method of Weighted Residuals (contd.)

Reiterating ...

$$\sum_{i=1}^{N} \left\{ \int_{z_1}^{z_N} \phi_j \frac{\partial^2 \phi_i}{\partial z^2} \, dz + i\omega\mu_0 \int_{z_1}^{z_N} \sigma \, \phi_j \, \phi_i \, dz \right\} E_i \;=\; 0$$

$$j = 1, \ldots, N$$

Okay, so the integrals above become the elements in the matrix equation. Piece of cake ...

But wait, our basis functions are linear functions. So the second derivative in the integrand of the first integral above is zero.

That isn't good because we'd loose half of our equation.

Also, if it were the true electric field in there instead of ϕ_i, the second derivative wouldn't be zero.

5. The Method of Weighted Residuals (contd.)

Well, have to do something about that. Let's integrate the awkward integral by parts (considering just layer i):

$$\int_{z_i}^{z_{i+1}} \phi_j \frac{\partial^2 \phi_i}{\partial z^2}\, dz = -\int_{z_i}^{z_{i+1}} \frac{\partial \phi_j}{\partial z} \frac{\partial \phi_i}{\partial z}\, dz + \left[\phi_j \frac{\partial \phi_i}{\partial z} \right]_{z_i}^{z_{i+1}}$$

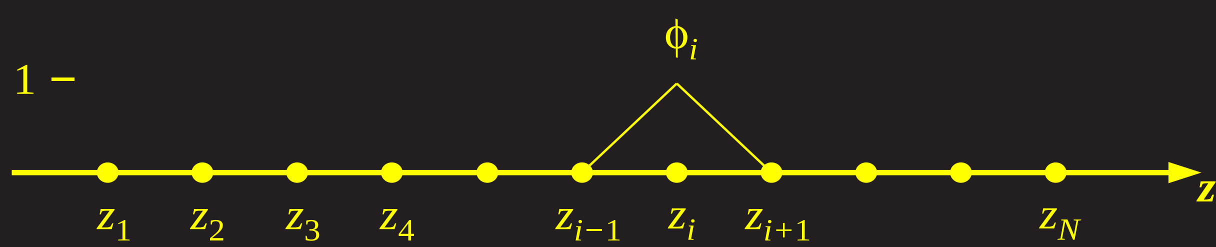
The first integral on the right-hand side involves only first-order derivatives of the basis functions. That's okay.

Regarding the other term on the right-hand side above, i.e., the "surface" term …

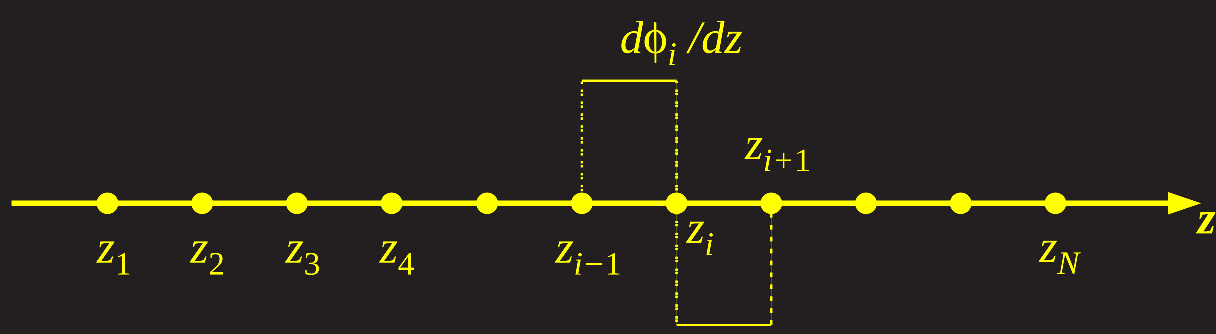
5. The Method of Weighted Residuals (contd.)

$$\int_{z_i}^{z_{i+1}} \phi_j \, \frac{\partial^2 \phi_i}{\partial z^2} \, dz = - \int_{z_i}^{z_{i+1}} \frac{\partial \phi_j}{\partial z} \, \frac{\partial \phi_i}{\partial z} \, dz + \left[\phi_j \, \frac{\partial \phi_i}{\partial z} \right]_{z_i}^{z_{i+1}}$$

$\longrightarrow$ ϕ_j is continuous at z_i and z_{i+1}



$\longrightarrow$ $\partial \phi_i / \partial z$ is probably not continuous at z_i and z_{i+1}



5. The Method of Weighted Residuals (contd.)

$$\int_{z_i}^{z_{i+1}} \phi_j \frac{\partial^2 \phi_i}{\partial z^2} \, dz = - \int_{z_i}^{z_{i+1}} \frac{\partial \phi_j}{\partial z} \frac{\partial \phi_i}{\partial z} \, dz + \left[\phi_j \frac{\partial \phi_i}{\partial z} \right]_{z_i}^{z_{i+1}}$$

$\longrightarrow$ However, if it were $\partial E / \partial z$ in the surface term, this would be H_y, and this would be continuous at z_i and z_{i+1}.

$\longrightarrow$ So, one can argue that the contributions from this surface term to the awkward integral for layer i should get cancelled by the corresponding contributions for the awkward integrals for the neighbouring layers.

$\longrightarrow$ Let's therefore throw away this surface term and go with:

$$\int_{z_i}^{z_{i+1}} \phi_j \frac{\partial^2 \phi_i}{\partial z^2} \, dz = - \int_{z_i}^{z_{i+1}} \frac{\partial \phi_j}{\partial z} \frac{\partial \phi_i}{\partial z} \, dz$$

6. The finite-element system of equations

Hence, the system of equations for this finite-element solution for the 1-D electric field PDE is:

$$\sum_{i=1}^{N} \left\{ \int_{z_1}^{z_N} \frac{\partial \phi_j}{\partial z} \frac{\partial \phi_i}{\partial z}\, dz \;+\; i\omega\mu_0 \int_{z_1}^{z_N} \sigma\, \phi_j\, \phi_i\, dz \right\} E_i \;=\; 0$$

$$j = 1, \ldots, N$$

Succinctly:

$$\left(\underline{\mathbf{L}} + i\omega\mu_0\, \underline{\mathbf{S}} \right) \tilde{\mathbf{E}} \;=\; \mathbf{r}$$

where

$\mathbf{r}$ is the right-hand side vector (dimension N);

$\tilde{\mathbf{E}}$ is . . .

6. The finite-element system of equations (contd.)

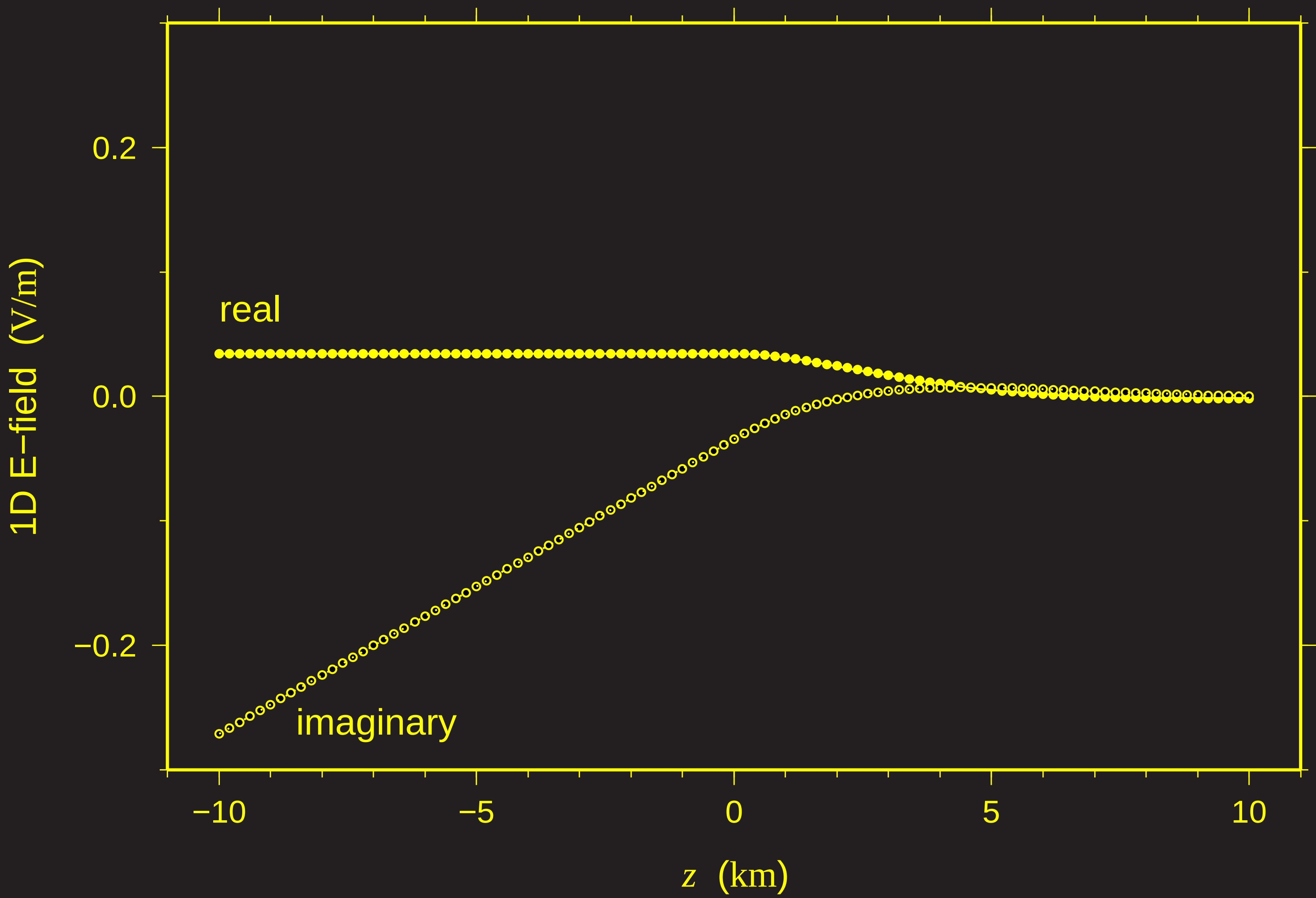
$\tilde{\mathbf{E}}$ is the vector containing the coefficients in the finite-element expansion for the approximate electric field, which, in this case, are the values of the approximate field at the nodes (dimension N);

And the $N \times N$ matrices $\underline{\mathbf{L}}$ and $\underline{\mathbf{S}}$ are:

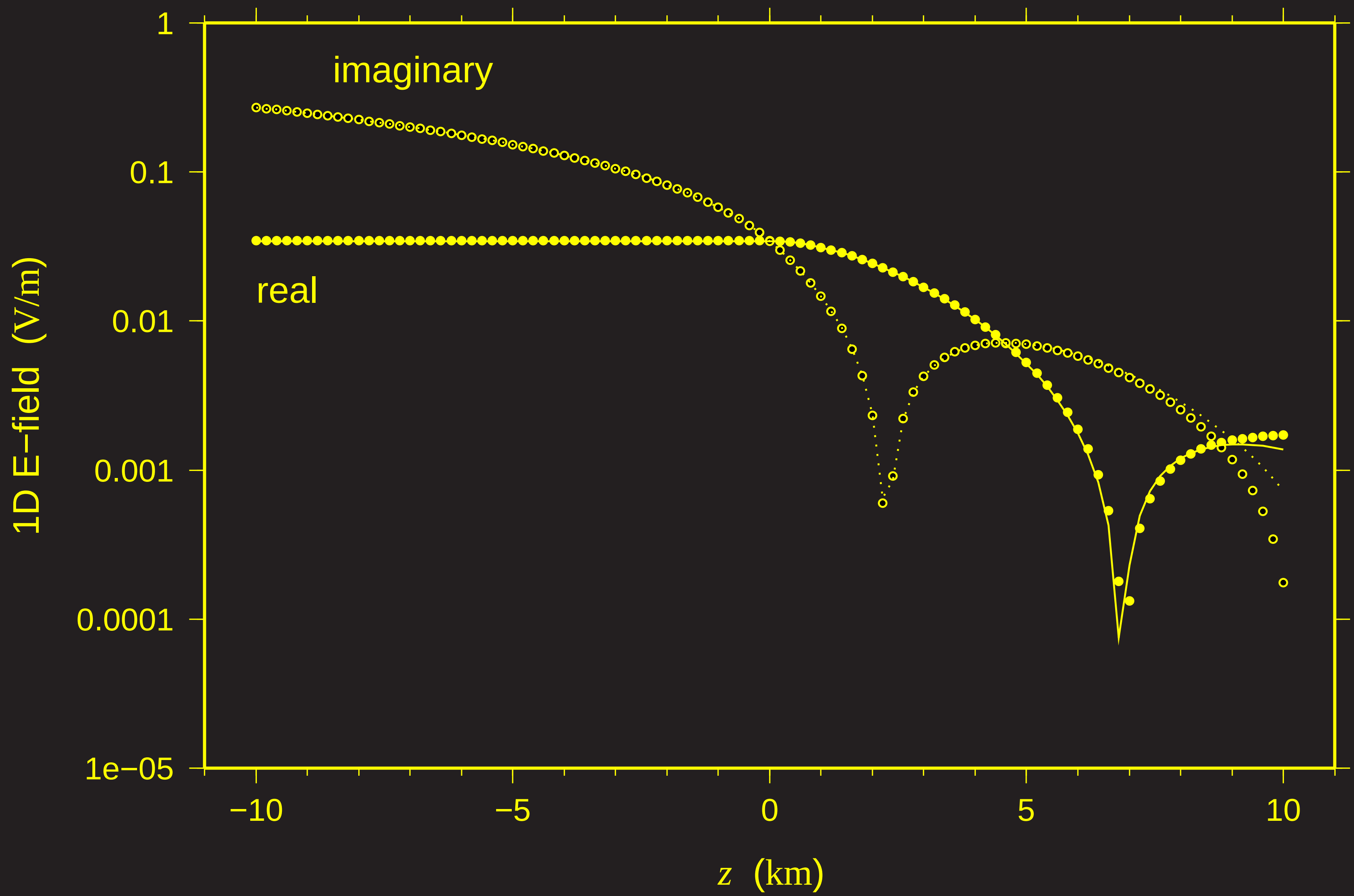
$$L_{ji} = \int_{z_1}^{z_N} \frac{\partial \phi_j}{\partial z} \frac{\partial \phi_i}{\partial z} \, dz \qquad \text{and} \qquad S_{ji} = \int_{z_1}^{z_N} \sigma \, \phi_j \, \phi_i \, dz$$

$\underline{\mathbf{L}}$ and $\underline{\mathbf{S}}$ are straight-forward to evaluate given the specific form of ϕ_j.

Boundary conditions can be implemented in the same way as was done for the finite-difference example in Lecture I.



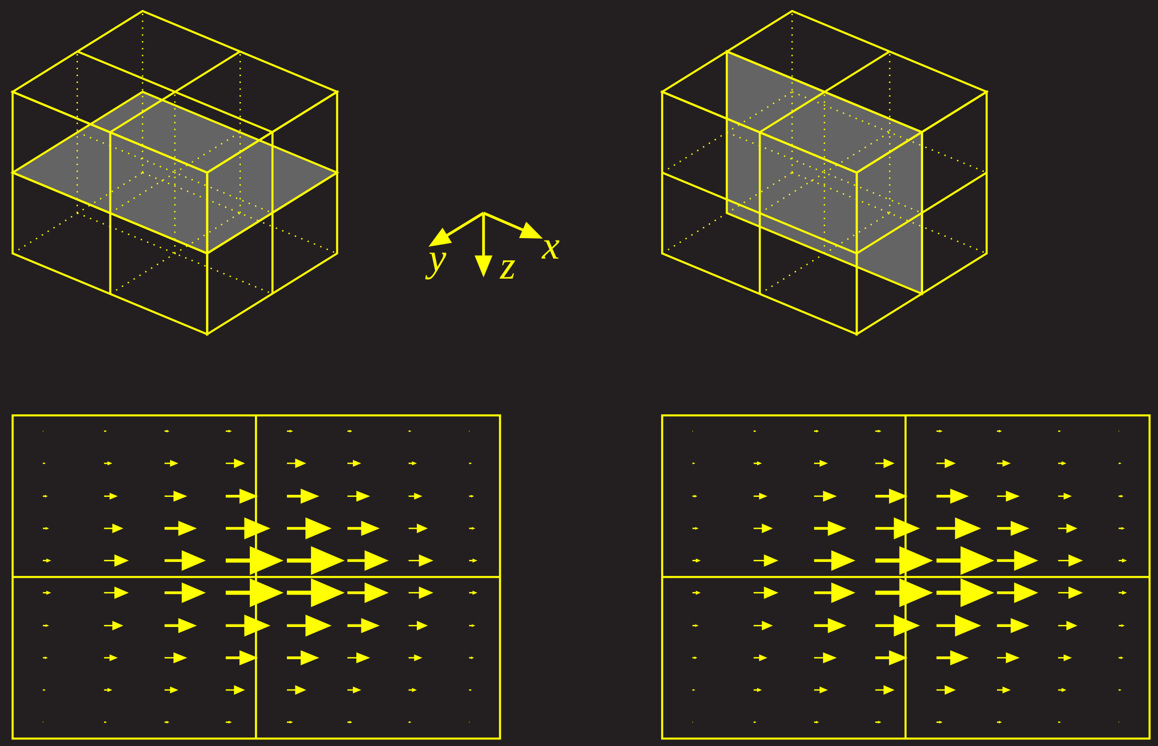
(Homogeneous halfspace of 0.01 S/m; "air" of 10^{-8}S/m; 3 Hz.)



(Looks suspiciously similar to the finite-difference results!)

7. A quick look at 3-D

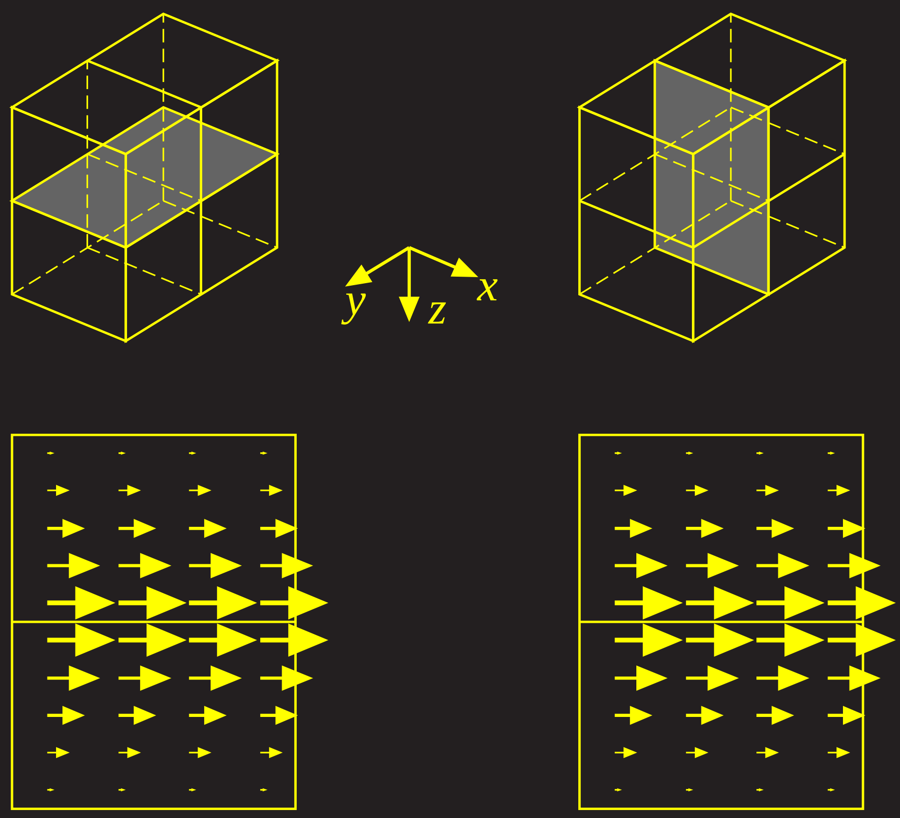
At first, people used nodal elements for the electric field in 3-D, mimicking the situation in 1-D:



But this gives continuous normal electric field, by construction, even if there's a discontinuity in conductivity.

7. A quick look at 3-D (contd.)

Then "edge-element" vector basis functions were introduced:



These basis functions can give a discontinuous normal electric field.

8. Exercise

Code up a finite-element solution for the 1-D MT electric-field PDE. (Well, at least do the integrals for L_{ji} and S_{ji} to see that they result in the same elements as for the finite-difference case.)

9. "Take-home" points

- ⋆ The finite-element method uses an *explicit* representation of the approximate electric field. (Not just linear basis functions.)

- ⋆ The system of equations is constructed by inner products with weighting functions ...

- ⋆ ... and the sleight-of-hand of integration-by-parts.

- ⋆ The finite-element method is more complicated than the finite-difference method for simple problems, but is more readily extended to non-rectilinear meshes.