Numerical Modelling for Geophysical Electromagnetic Methods

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Outline

I. 1-D

II. EM rules & regulations

III. Finite difference

IV. Finite element

V. Integral equation

V. Integral equation

1. Introduction

The integral-equation method was the mostly widely used approach for early 3-D geophysical EM numerical modelling. It's a classic!

This was because it gave rise to small systems of equations, relative to finite-difference and finite-element methods. The integral-equation method was therefore tractable on computer technology a couple of decades ago.

It has it's issues. (Don't all EM modelling techniques?)

It is still relevant, especially to mineral exploration (delineation) using down-hole EM through and around ore-bodies of interesting shapes.

1. Introduction

As for the finite-difference and finite-element methods, I want to look in reasonable detail at a 1-D example. This will introduce the concepts that are used in 3-D.

2. A 1-D example

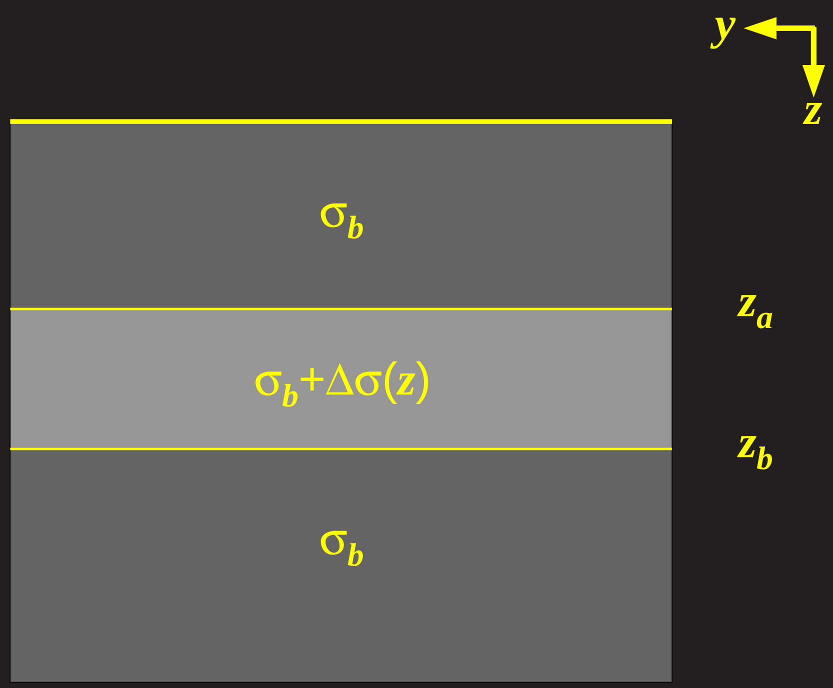
Remembering the 1-D example . . .

The electric-field differential equation was:

$$\frac{\partial^2 E_x}{\partial z^2} + i\omega\mu_0\sigma(z)\,E_x = 0$$

Now, suppose the Earth can be represented by the following model . . .

2. A 1-D example (contd.)



That is, the subsurface comprises an "anomalous" layer within a nice and simple "background" conductivity.

This is the kind of model to which the integral-equation approach is suited.

3. Background model and field

Returning to the PDE, now thinking about this particular style of model:

$$\frac{\partial^2 E_x}{\partial z^2} + i\omega\mu_0(\sigma_b + \Delta\sigma)\, E_x = 0$$

The background model needs to be a model in which we can easily compute the electric fields. Typically it is a homogeneous halfspace, sometimes a layered halfspace.

In other words, we need the "background" or "primary" field E_b such that:

$$\frac{\partial^2 E_b}{\partial z^2} + i\omega\mu_0\sigma_b\, E_b = 0$$

4. The scattered field

The difference between the actual or "total" electric field E and the background field E_b is the "scattered" or "secondary" field:

$$E = E_b + E_s$$

Substituting the above into the electric-field PDE gives:

$$\frac{\partial^2 (E_b + E_s)}{\partial z^2} + i\omega\mu_0(\sigma_b + \Delta\sigma)\,(E_b + E_s) = 0$$

Separating out almost all the terms:

$$\frac{\partial^2 E_b}{\partial z^2} + \frac{\partial^2 E_s}{\partial z^2} + i\omega\mu_0\sigma_b E_b + i\omega\mu_0\sigma_b E_s + i\omega\mu_0\Delta\sigma(E_b + E_s) = 0$$

4. The scattered field (contd.)

Using the information about which PDE the background field satisfies gives:

$$\frac{\partial^2 E_s}{\partial z^2} + i\omega\mu_0\sigma_b E_s = -i\omega\mu_0\Delta\sigma(E_b + E_s)$$

This is an inhomogeneous PDE for the scattered electric field.

$\longrightarrow$ The differential "operator" is the same as for the background electric field.

$\longrightarrow$ The "source" term on the right-hand side is kind of like an anomalous current density that exists because of the total electric field in the anomalous layer.

5. The Green's function

Okay, we're now in a position of having to solve an inhomogeneous differential equation.

Introduce the Green's function $G(z; z')$ such that:

$$\frac{\partial^2 G}{\partial z^2} + i\omega\mu_0\sigma_b G = \delta(z - z')$$

The differential operator above is the same as that for the background electric field.

The Dirac delta function is such that:

$$\delta(z - z') = 0 \quad \text{for} \quad z \neq z' \qquad \text{and}$$

$$\int_{z=z_a}^{z_b} f(z)\,\delta(z - z')\,dz = f(z') \quad \text{if} \quad z_a < z' < z_b$$

6. The integral equation

Consider G times the scattered-field equation and E_s times the Green's function equation:

$$G\,\frac{\partial^2 E_s}{\partial z^2} \;+\; i\omega\mu_0\sigma_b\, G\, E_s \;=\; -i\omega\mu_0\Delta\sigma\, G\, E$$
$$E_s\,\frac{\partial^2 G}{\partial z^2} \;+\; i\omega\mu_0\sigma_b\, E_s\, G \;=\; E_s\,\delta(z-z')$$

Subtracting one from the other:

$$G\,\frac{\partial^2 E_s}{\partial z^2} \;-\; E_s\,\frac{\partial^2 G}{\partial z^2} \;=\; -i\omega\mu_0\Delta\sigma\, G\, E \;-\; E_s\,\delta(z-z')$$

6. The integral equation

Now integrate this equation over the whole domain:

$$\int_{z=-\infty}^{\infty} \left\{ G \, \frac{\partial^2 E_s}{\partial z^2} - E_s \, \frac{\partial^2 G}{\partial z^2} \right\} dz \; = \; - \, i\omega\mu_0 \int_{z=-\infty}^{\infty} \Delta\sigma \, G \, E \, dz \; - \; \int_{z=-\infty}^{\infty} E_s \, \delta(z - z') \, dz$$

The left-hand side vanishes. Try integration-by-parts on the first half of the integrand:

$$\int_{z=-\infty}^{\infty} G \, \frac{\partial^2 E_s}{\partial z^2} \, dz \; = \; \left[G \, \frac{\partial E_s}{\partial z} \right]_{z=-\infty}^{\infty} \; - \; \int_{z=-\infty}^{\infty} \frac{\partial G}{\partial z} \, \frac{\partial E_s}{\partial z} \, dz$$

The [ ] term vanishes since everything vanishes as $z \to \pm\infty$.

6. The integral equation (contd.)

So:

$$\int_{z=-\infty}^{\infty} G\,\frac{\partial^2 E_s}{\partial z^2}\,dz \;=\; -\int_{z=-\infty}^{\infty} \frac{\partial G}{\partial z}\,\frac{\partial E_s}{\partial z}\,dz$$

Continuing in this direction leads to:

$$\int_{z=-\infty}^{\infty} G\,\frac{\partial^2 E_s}{\partial z^2}\,dz \;=\; \int_{z=-\infty}^{\infty} \frac{\partial^2 G}{\partial z^2}\,E_s\,dz$$

Hence:

$$-i\omega\mu_0 \int_{z=-\infty}^{\infty} \Delta\sigma\,G\,E\,dz \;-\; \int_{z=-\infty}^{\infty} E_s\,\delta(z-z')\,dz \;=0$$

6. The integral equation (contd.)

Finally, doing the delta function magic gives:

$$-i\omega\mu_0 \int_{z=-\infty}^{\infty} \Delta\sigma \, G \, E \, dz \; - \; E_s(z') \; = \; 0$$

That is:

$$E_s(z') \; = \; -i\omega\mu_0 \int_{z=-\infty}^{\infty} \Delta\sigma(z) \, G(z; z') \, E(z) \, dz$$

This is the integral equation for the scattered electric field. (The scattered electric field also appears in the integrand of the integral.)

6. The integral equation (contd.)

It is common to add the background field at z' to both sides of the equation. This gives an integral equation for the total electric field E:

$$E(z') \;=\; E_b(z') \;-\; i\omega\mu_0 \int_{z=-\infty}^{\infty} \Delta\sigma(z)\, G(z;z')\, E(z)\, dz$$

This equation is valid for z' inside or outside the anomalous layer.

This is now the starting point for the numerical stuff. (Just as the electric-field PDE was the starting point for the finite-difference and finite-element approaches.)

7. Discretizing the electric field

Introduce an approximate electric field:

$$\tilde{E}_x = \sum_{i=1}^{N} E_i \, \phi_i(z)$$

Substituting this into the integral equation gives:

$$\sum_{i=1}^{N} E_i \, \phi_i(z') =$$

$$E_b(z') - i\omega\mu_0 \int_{z=-\infty}^{\infty} \Delta\sigma(z) \, G(z; z') \left\{ \sum_{i=1}^{N} E_i \, \phi_i(z) \right\} dz$$

8. Discretizing the integral equation

Rearranging slightly:

$$\sum_{i=1}^{N} E_i \, \phi_i(z') \; = \;$$

$$E_b(z') \; - \; \sum_{i=1}^{N} \left\{ i\omega\mu_0 \int_{z=-\infty}^{\infty} \Delta\sigma(z) \, G(z; z') \, \phi_i(z) \, dz \right\} E_i$$

In succinct, cryptic, deceptive notation:

$$\sum_{i=1}^{N} E_i \, \phi_i(z') \; = \; E_b(z') \; - \; \sum_{i=1}^{N} \Gamma_i(z') \, E_i$$

where $\Gamma_i(z')$ is the stuff in braces in the preceding equation.

8. Discretizing the integral equation (contd.)

This is the approximation to the electric-field integral equation. Reiterating:

$$\sum_{i=1}^{N} E_i \, \phi_i(z') = E_b(z') - \sum_{i=1}^{N} \Gamma_i(z') \, E_i + R(z')$$

where the residual R should really be included since this is an approximate equation.

This is one equation in N unknowns.

A numerical solution to this equation can be achieved via the method of weighted residuals, just as for the finite-element approach. (And the finite-difference approach, actually.)

Multiplying the approximate integral equation by weight functions ψ_j, and integrating over the whole domain gives:

$$\sum_{i=1}^{N} E_i \left\langle \psi_j, \phi_i \right\rangle \;=\; \left\langle \psi_j, E_b \right\rangle \;-\; \sum_{i=1}^{N} \left\langle \psi_j, \Gamma_i \right\rangle E_i \;+\; \left\langle \psi_j, R \right\rangle$$

where $\langle \psi, \phi \rangle = \int_{-\infty}^{\infty} \psi(z)\, \phi(z)\, dz$.

As before, let's think of the residual as being orthogonal to the weight functions, so its term can be dropped.

Rearranging:

$$\sum_{i=1}^{N} \Big(\left\langle \psi_j, \phi_i \right\rangle + \left\langle \psi_j, \Gamma_i \right\rangle \Big) E_i \;=\; \left\langle \psi_j, E_b \right\rangle$$

9. The system of equations

The preceding system of M equations, where M is the number of weight functions, can be written as the matrix equation:

$$\left(\underline{\mathbf{K}} + \underline{\mathbf{G}}\right)\tilde{\mathbf{E}} = \mathbf{r}$$

$\tilde{\mathbf{E}}$ is the vector containing the coefficients in the expansion for the approximate electric field (dimension N).

$$r_j = \int_{-\infty}^{\infty} \psi_j(z)\, E_b(z)\, dz$$

$$K_{ji} = \int_{-\infty}^{\infty} \psi_j(z)\, \phi_i(z)\, dz$$

$$G_{ji} = \int_{-\infty}^{\infty} \psi_j(z)\, \Gamma_i(z)\, dz$$

$$\Gamma_i(z) = i\omega\mu_0 \int_{z'=-\infty}^{\infty} \Delta\sigma(z')\, G(z';z)\, \phi_i(z')\, dz'$$

9. The system of equations (contd.)

That last term, $G_{ji} = \langle \phi_j, \Gamma_i \rangle$, is rather complicated.

The typical choices for basis and weight functions, especially when integral-equation methods were popular, were:

- → "pulse" basis functions, i.e., the electric field is uniform in each cell (or layer);
- → delta function weight functions – "collocation" – meaning the approximate integral equation is considered to be exact at a specified points.

Also, the anomalous conductivity was assumed to be uniform within each cell (or layer).

For these choices . . .

9. The system of equations (contd.)

$$r_j = \int_{-\infty}^{\infty} \psi_j(z)\, E_b(z)\, dz \;=\; \int_{-\infty}^{\infty} \delta(z - z_j)\, E_b(z)\, dz \;=\; E_b(z_j)$$

$$K_{ji} \;=\; \int_{-\infty}^{\infty} \psi_j(z)\, \phi_i(z)\, dz \;=\; \int_{\Delta z_i} \delta(z - z_j)\, dz \;=\; \delta_{ji}$$

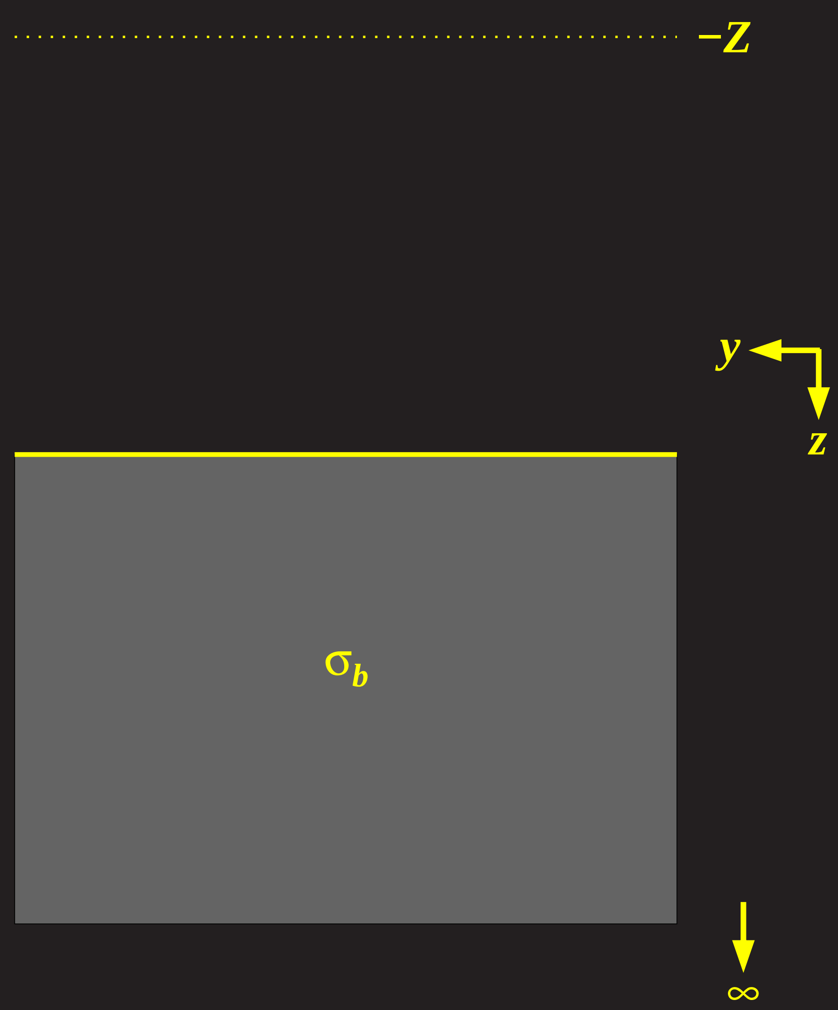
$$G_{ji} \;=\; \int_{-\infty}^{\infty} \psi_j(z)\, \Gamma_i(z)\, dz \;=\; \int_{-\infty}^{\infty} \delta(z - z_j)\, \Gamma_i(z)\, dz \;=\; \Gamma_i(z_j)$$

$$\Gamma_i(z_j) \;=\; i\omega\mu_0\, \Delta\sigma_i \int_{\Delta z_i} G(z'; z_j)\, dz'$$

$\tilde{\mathbf{E}}$ is the vector containing the values of the approximate electric field in the N cells (or layers).

10. The 1-D example

Suppose the background model is a homogeneous halfspace.



The differential equation for the background electric field is:

$$\frac{\partial^2 E_b}{\partial z^2} + i\omega\mu_0 \sigma_b E_b = 0$$

with $\sigma_b = 10^{-8}\,\mathrm{S/m}$, for example, in the air.

The background electric field is therefore of the form:

$$E_b = A\,e^{kz} + B\,e^{-kz}, \qquad k^2 = -i\omega\mu_0\sigma_b$$

The boundary and interface conditions on E_b are:

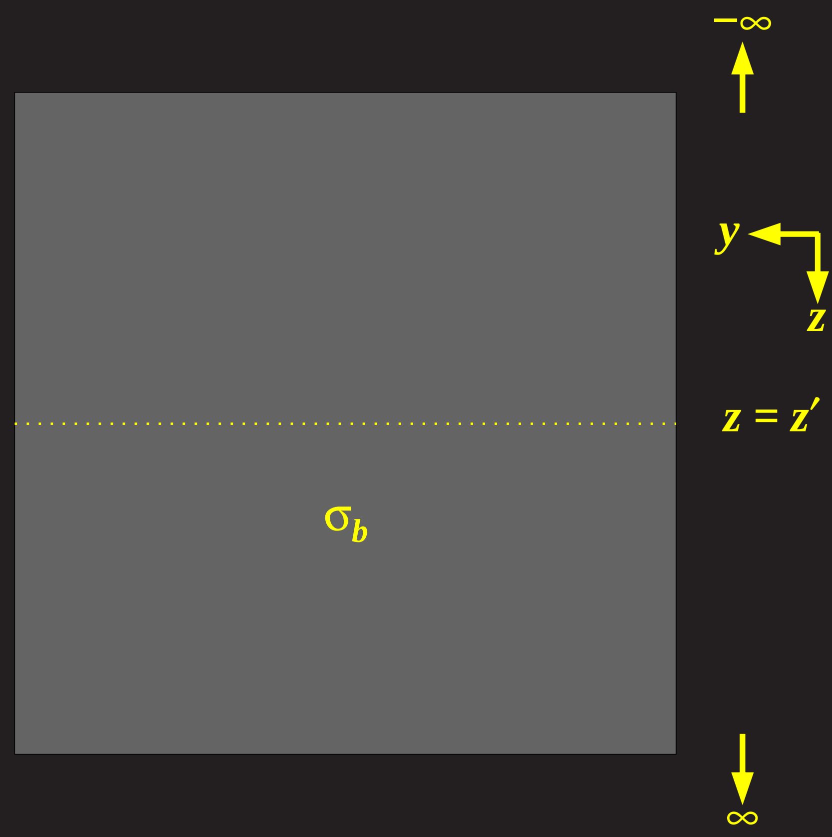
$$\frac{\partial E_b}{\partial z} = i\omega\mu_0 \quad \text{at} \quad z = -Z$$

$$E_b,\ \frac{\partial E_b}{\partial z} \quad \text{continuous at} \quad z = 0$$

$$E_b \to 0 \quad \text{as} \quad z \to \infty$$

It is pretty straight-forward to determine A & B in the air and ground that satisfy these boundary conditions.

For the Green's function, consider a wholespace:



This is usually an adequate approximation if the anomalous layer is not too close to the Earth's surface.

The differential equation for the Green's function is:

$$\frac{\partial^2 G(z;z')}{\partial z^2} + i\omega\mu_0\sigma_b G(z;z') = \delta(z-z')$$

The boundary conditions on the Green's function are that it vanishes as $z \to \pm\infty$.

The form of the Green's function is therefore:

$$G(z;z') = \begin{cases} A\,e^{kz}, & z < z'; \\ B\,e^{-kz}, & z' < z. \end{cases}$$

The conditions at $z = z'$ are that G is continuous, and $\partial G/\partial z$ is discontinuous by an amount equal to 1.

Again, it is straight-forward to determine A & B from the above information.

11. Exercise

Write a program that uses the integral-equation method to calculate the MT electric field in an Earth model comprising an anomalous layer in an otherwise homogeneous halfspace.

12. A comment on the 3-D situation

The traditional formulation in 3-D involving pulse basis functions (and collocation) failed for conductivity contrasts greater than about 300:1.

This is because of the difficulty in trying to keep both the inductive and galvanic physics in the approximate equations.

SanFilipo, Newman & Hohmann (in two separate papers) incorporated hard-wired current loops extending over the whole anomalous region.

I used edge-element basis functions, and (importantly) continuity of normal current density between cells within the anomalous region.

13. “Take-home” points

- ⋆ The integral-equation method is best suited to Earth models comprising a localized anomalous region in a background for which we can readily compute electric fields. (Limiting, but still relevant.)

- ⋆ The systems of equations are small (but full) as only the anomalous region and the field within it require discretization.

- ⋆ Quite involved analysis (Green’s functions – eek!) to figure out what the elements of the matrix equation are, certainly compared to the finite-difference method.