# Introduction to 3D MT inversion code *x*3D*i*

Anna Avdeeva (aavdeeva@ifm-geomar.de), Dmitry Avdeev and Marion Jegen

31 March 2011

# Outline

# Outline

# How 3D inverse problem is commonly solved



Traditionally solution is sought as a stationary point of a penalty function

$$\varphi(\mathbf{m}, \lambda) = \varphi_d(\mathbf{m}) + \lambda \varphi_s(\mathbf{m}) \underset{\mathbf{m}}{\longrightarrow} \min \tag{1}$$

$\varphi_d(\mathbf{m})$ data misfit

$\varphi_s(\mathbf{m})$ Tikhonov-type stabilizer

$\lambda$ regularization parameter

$\mathbf{m}$ vector of model parameters (conductivities)

$$\varphi_d(\mathbf{m}) = \frac{1}{2} \left\| d^{obs} - F(\mathbf{m}) \right\|^2 \tag{2}$$

$F(\mathbf{m})$ is a forward problem mapping

$$\varphi_s(\mathbf{m}) = \frac{1}{2} \left\| \mathbf{W}(\mathbf{m} - \mathbf{m}^{ref}) \right\|^2 \tag{3}$$

$\mathbf{m}^{ref}$ vector of model parameters, for some reference model, which usually include some a priori information.

# Where are the differences between 3D inversion codes?



1. model parameters ($\sigma$, $\rho$, $\log\rho$, $\log\sigma$ etc.)
2. forward problem solver (FD, FE or IE)
3. optimization method (GN, QN, LMQN, NLCG etc.)
4. form of the data misfit $\varphi_d$
5. form of the stabilizer $\varphi_s$

# Where are the differences between 3D inversion codes?

1. model parameters ($\sigma$, $\rho$, $\log\rho$, $\log\sigma$ etc.)
2. forward problem solver (FD, FE or IE)
3. optimization method (GN, QN, LMQN, NLCG etc.)
4. form of the data misfit $\varphi_d$
5. form of the stabilizer $\varphi_s$

# Where are the differences between 3D inversion codes?



1. model parameters ($\sigma$, $\rho$, $\log\rho$, $\log\sigma$ etc.)
2. forward problem solver (FD, FE or IE) - *X3D*
3. optimization method (GN, QN, LMQN, NLCG etc.)
4. form of the data misfit $\varphi_d$
5. form of the stabilizer $\varphi_s$

# Where are the differences between 3D inversion codes?



1. model parameters ($\sigma$, $\rho$, $\log\rho$, $\log\sigma$ etc.)
2. forward problem solver (FD, FE or IE) - *X3D*
3. optimization method (GN, QN, LMQN, NLCG etc.)
4. form of the data misfit $\varphi_d$
5. form of the stabilizer $\varphi_s$

# Optimization method



All iterative methods work at the same manner. At each iteration $l$ they find:

1. a search direction vector $\mathbf{p}^{(l)}$
2. a step length $\alpha^{(l)}$ using an inexact line search along $\mathbf{p}^{(l)}$
3. the next, improved model given by $\mathbf{m}^{(l+1)} = \mathbf{m}^{(l)} + \alpha^{(l)}\mathbf{p}^{(l)}$

The difference is in how a specific method finds the search direction and in what price is paid for this.

NLCG $\mathbf{p}^{(l)} = -\mathbf{g}^{(l)} + \gamma^{(l)}\mathbf{p}^{(l-1)}$, where $\gamma^{(l)} = \frac{\mathbf{g}^{(l)} \cdot \mathbf{g}^{(l)}}{\mathbf{g}^{(l-1)} \cdot \mathbf{g}^{(l-1)}}$

Newton, GN $\mathbf{H}^{(l)}\mathbf{p}^{(l)} = -\mathbf{g}^{(l)}$

QN $\{\mathbf{m}^{(i)}, \mathbf{g}^{(i)} : i = 1, ..., l\} \to \mathbf{p}^{(l)}$

LMQN $\{\mathbf{m}^{(i)}, \mathbf{g}^{(i)} : i = l - n_{cp}, ..., l\} \to \mathbf{p}^{(l)}$

$$H_{ij}^{(l)} = \frac{\partial^2 \varphi}{\partial m_i \partial m_j}(\mathbf{m}^{(l)}), \quad g_i^{(l)} = \frac{\partial \varphi}{\partial m_i}(\mathbf{m}^{(l)})$$

# Optimization method

All iterative methods work at the same manner. At each iteration $l$ they find:

1. a search direction vector $\mathbf{p}^{(l)}$
2. a step length $\alpha^{(l)}$ using an inexact line search along $\mathbf{p}^{(l)}$
3. the next, improved model given by $\mathbf{m}^{(l+1)} = \mathbf{m}^{(l)} + \alpha^{(l)}\mathbf{p}^{(l)}$

The difference is in how a specific method finds the search direction and in what price is paid for this.

NLCG $\mathbf{p}^{(l)} = -\mathbf{g}^{(l)} + \gamma^{(l)}\mathbf{p}^{(l-1)}$, where $\gamma^{(l)} = \frac{\mathbf{g}^{(l)} \cdot \mathbf{g}^{(l)}}{\mathbf{g}^{(l-1)} \cdot \mathbf{g}^{(l-1)}}$

Newton, GN $\mathbf{H}^{(l)}\mathbf{p}^{(l)} = -\mathbf{g}^{(l)}$

QN $\{\mathbf{m}^{(i)}, \mathbf{g}^{(i)} : i = 1, ..., l\} \rightarrow \mathbf{p}^{(l)}$

LMQN $\{\mathbf{m}^{(i)}, \mathbf{g}^{(i)} : i = l - n_{cp}, ..., l\} \rightarrow \mathbf{p}^{(l)}$

$$H_{ij}^{(l)} = \frac{\partial^2 \varphi}{\partial m_i \partial m_j}(\mathbf{m}^{(l)}), \quad g_i^{(l)} = \frac{\partial \varphi}{\partial m_i}(\mathbf{m}^{(l)})$$

# Adjoint approach



- Calculation of gradients requires 2 forward modellings at each frequency
- Important: Straightforward calculation of the gradients would require $N+1$ forward modellings, i.e. in $\approx N/2$ times more

**Example: number of model parameters = 3000**

Single forward modelling requires $\approx$ 4 min on PC
$\Rightarrow$ Straightforward calculation of the single gradient requires 8 days
Usually $\approx$ 200 iterations(gradients) is needed
$\Rightarrow$ Total inversion time is $\approx$ 4 years.

# 3D data misfit



$$\varphi_d(\boldsymbol{\sigma}) = \frac{1}{2}\sum_{i=1}^{N_S}\sum_{j=1}^{N_T}\beta_{ij}\ tr\left[\bar{\mathbf{A}}_{ij}^T(\boldsymbol{\sigma})\mathbf{A}_{ij}(\boldsymbol{\sigma})\right] \qquad (4)$$

$\boldsymbol{\sigma} = (\sigma_1, ..., \sigma_N)^T$ the vector of the electrical conductivities of the cells, $N$ number of cells

$N_S$ number of MT sites $\mathbf{r}_i = (x_i, y_i, z = 0)$

$N_T$ number of frequencies $\omega_j$

$\mathbf{A}_{ij} = \mathbf{Z}_{ij} - \mathbf{D}_{ij}$ $2 \times 2$ matrices

$\mathbf{Z}_{ij}$ complex-valued predicted $\mathbf{Z}(\mathbf{r}_i, \omega_j)$ impedance

$\mathbf{D}_{ij}$ complex-valued observed $\mathbf{D}(\mathbf{r}_i, \omega_j)$ impedance

$\beta_{ij}$ some positive weights

$$tr\left[\bar{\mathbf{A}}^T\mathbf{A}\right] = \bar{A}_{11}A_{11} + \bar{A}_{12}A_{12} + \bar{A}_{21}A_{21} + \bar{A}_{22}A_{22}$$

# Gradients: Adjoint approach



$$\frac{\partial \varphi_d}{\partial \sigma_k} = Re\left\{\sum_{j=1}^{N_T}\sum_{p=1}^{2}\int_{V_k}\left(\mathbf{u}_x^{(p)}\mathbf{E}_x^{(p)} + \mathbf{u}_y^{(p)}\mathbf{E}_y^{(p)} + \mathbf{u}_z^{(p)}\mathbf{E}_z^{(p)}\right) dV\right\} \quad (5)$$

Maxwell's equation

$$\nabla \times \nabla \times \mathbf{E}_j^{(p)} - \sqrt{-1}\omega_j\mu\sigma(\mathbf{r})\mathbf{E}_j^{(p)} = \sqrt{-1}\omega_j\mu\mathbf{J}_j^{(p)} \quad (6)$$

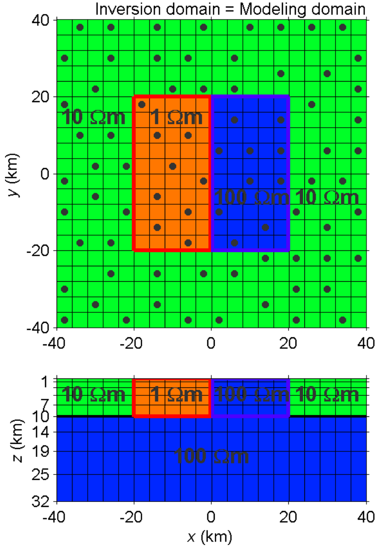
Adjoint Maxwell's equation

$$\nabla \times \nabla \times \mathbf{u}_j^{(p)} - \sqrt{-1}\omega_j\mu\sigma(\mathbf{r})\mathbf{u}_j^{(p)} = \sqrt{-1}\omega_j\mu\left(\mathbf{j}_j^{(p)} + \nabla \times \mathbf{h}_j^{(p)}\right) \quad (7)$$

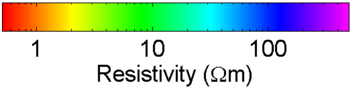
$\mathbf{j}_j^{(p)}$ and $\mathbf{h}_j^{(p)}$ - horizontal electric and magnetic dipoles at the MT sites
$p = 1, 2$ - polarization

The forward modelling is performed with *X3D* by (Avdeev *et al.*, 2002).

# Grid and MT sites

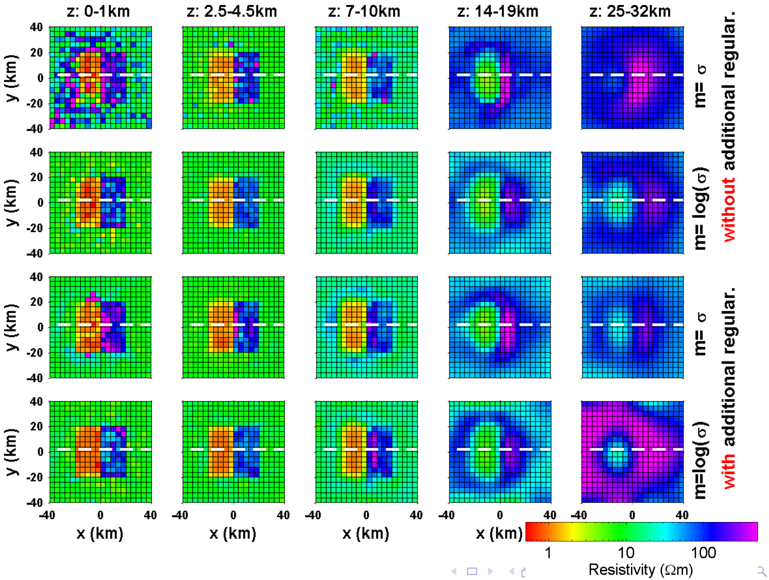



$N = 20 \times 20 \times 9$ cells
$dx = dy = 4$ km
$N_S = 80$ MT sites
$N_T = 3$ periods:
100, 300, 1000 s
noise - 1%
initial guess model:
50 Ωm halfspace



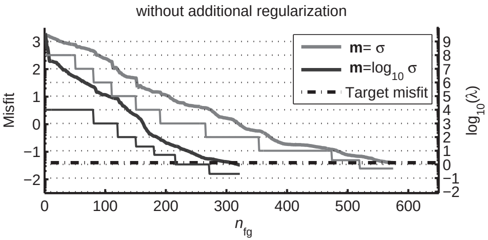
# Logarithmic parametrization vs conductivities: inversion results

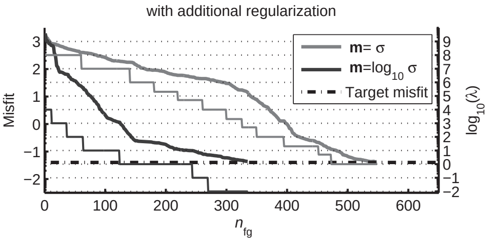




# Logarithmic parametrization vs conductivities: convergence curves





# Regularization

Tikhonov-type stabilizers:

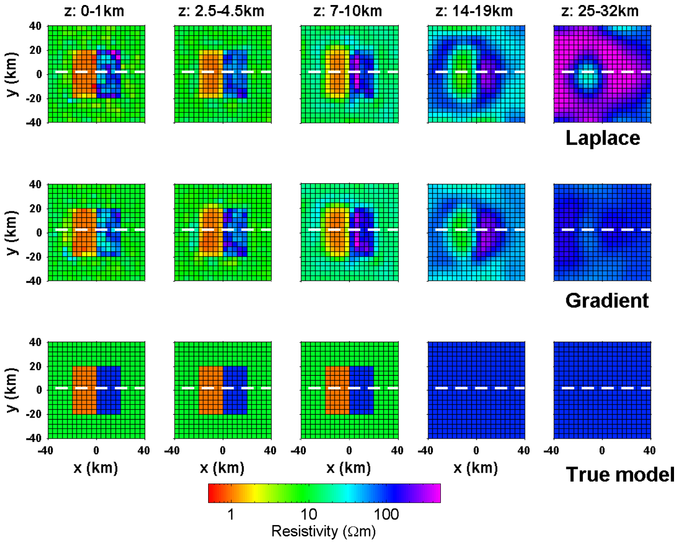
- Laplace

$$\varphi_s = dxdy \sum_{\alpha\beta\gamma} \left[ \frac{\partial^2 m}{\partial x^2} + \frac{\partial^2 m}{\partial y^2} + \frac{\partial^2 m}{\partial z^2} \right]^2_{\alpha\beta\gamma} dz_\gamma \quad (8)$$

- Gradient

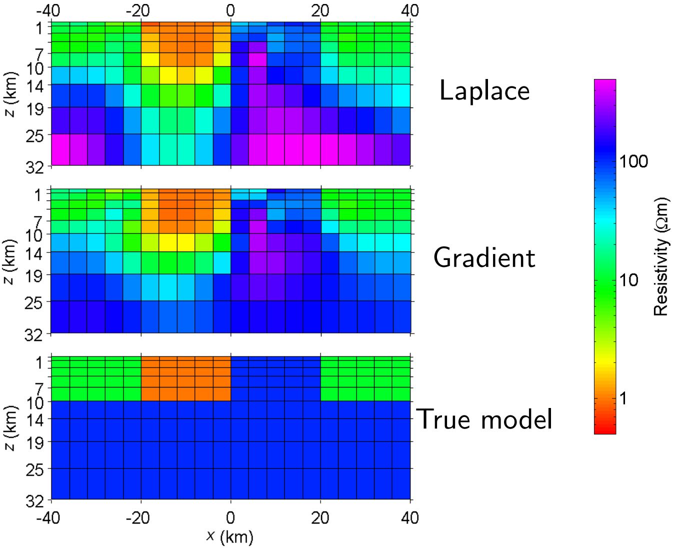
$$\varphi_s = dxdy \sum_{\alpha\beta\gamma} \left[ \left( \frac{\partial m}{\partial x} \right)^2 + \left( \frac{\partial m}{\partial y} \right)^2 + \left( \frac{\partial m}{\partial z} \right)^2 \right]_{\alpha\beta\gamma} dz_\gamma \quad (9)$$

# Gradient vs Laplace: inversion results

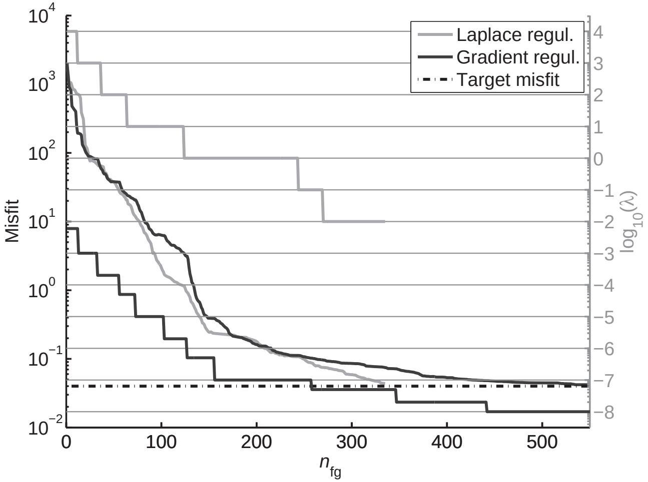


# Gradient vs Laplace: inversion results





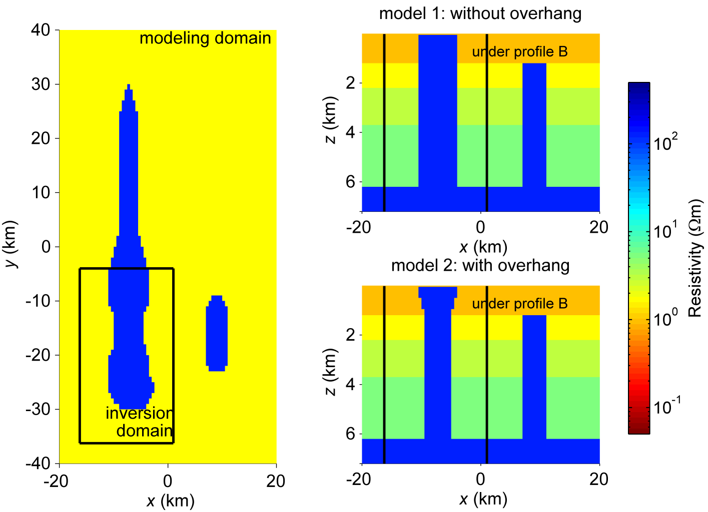
# Gradient vs Laplace: convergence curves



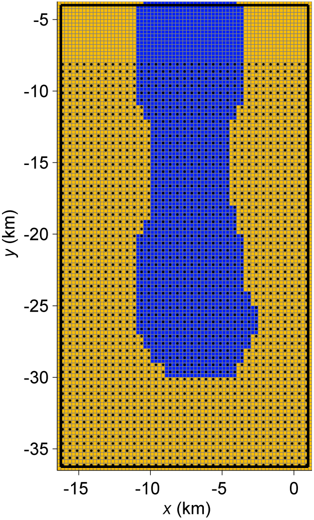
# Outline

# 3D Model of a salt wall





# 1995 MT sites




$N$= 129 $\times$ 69 $\times$ 13 cells

$dx = dy = 0.25$ km

$N_S$=1995 MT sites

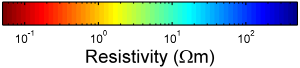
$N_T$=5 frequencies:

$10^{-3}$ - $10^{1}$ Hz

noise - 5%

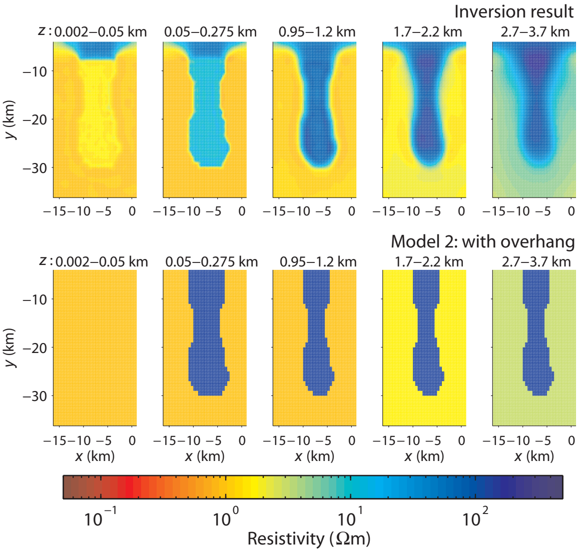
initial guess model:

11 $\Omega$m halfspace



# Inversion result for model 2 with overhang. 1995 MT sites. Horizontal slices

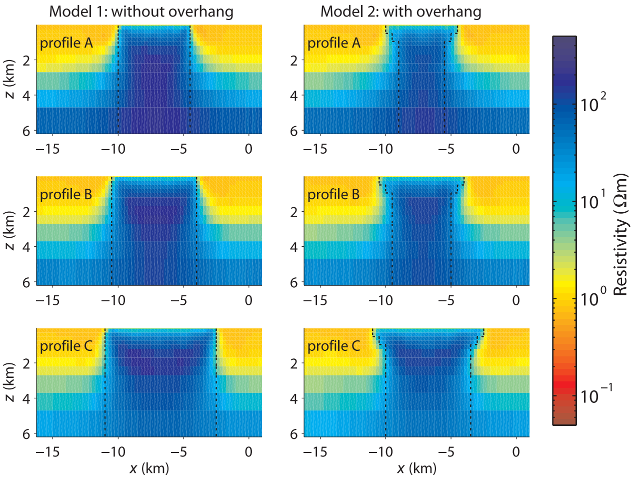




# Inversion result: model 1 vs model 2. 1995 MT sites. Vertical slices

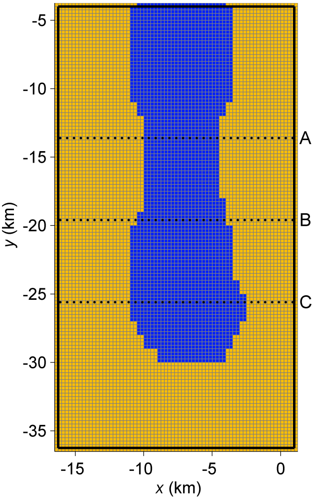




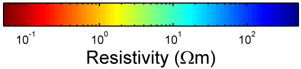


# 105 MT sites along three profiles



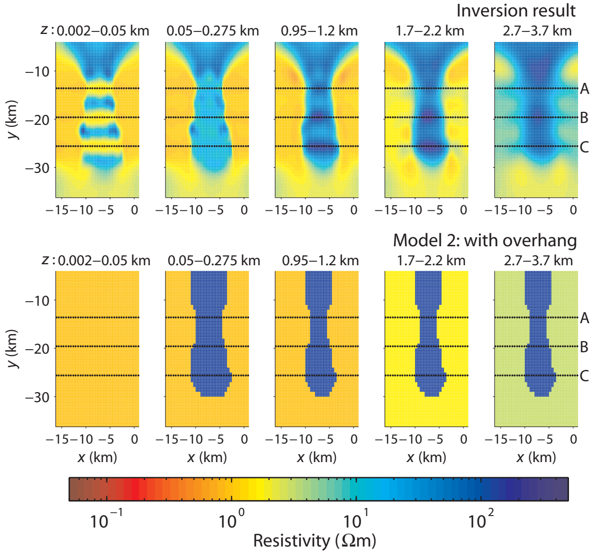



$N = 129 \times 69 \times 13$ cells
$dx = dy = 0.25$ km
$N_S$=105 MT sites
$N_T$=5 frequencies:
$10^{-3}$ - $10^{1}$ Hz
noise - 5%
initial guess model:
11 Ωm halfspace



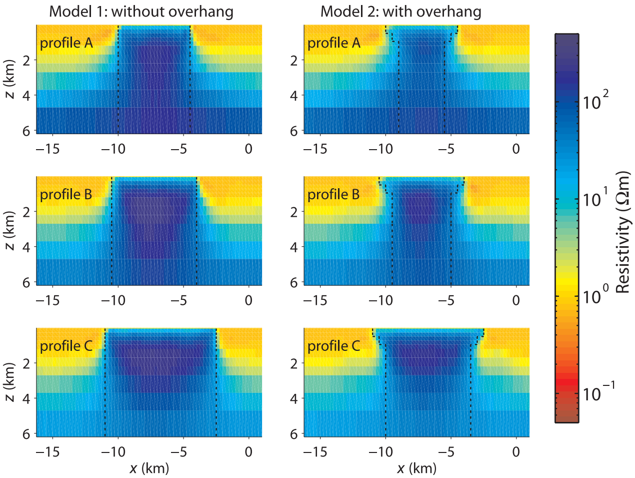
# Inversion result for model 2 with overhang. 105 MT sites. Horizontal slices





# Inversion result: model 1 vs model 2. 105 MT sites. Vertical slices





# Outline

# Conclusions

- Logarithmic parametrization is beneficial in terms of inversion result and computational time
- Regularization based on Gradient suppresses spatial resistivity gradients
- The *x*3D*i* code produces encouraging results for salt dome overhang detectability

# Acknowledgements

We would like to thank

- Wintershall Holding AG, who funded the inversion study and 20 ocean bottom MT instruments

# Thanks for your attention



Avdeev, D. B. and Avdeeva, A. D. (2009). Three-dimensional magnetotelluric inversion using a limited-memory QN optimization. *Geophysics*, **74(3)**, F45–F57.

Avdeev, D. B., Kuvshinov, A. V., Pankratov, O. V., and Newman, G. A. (2002). Three-dimensional induction logging problems, Part I: An integral equation solution and model comparisons. *Geophysics*, **67**, 413–426.

Avdeeva, A. D. and Avdeev, D. B. (2006). A limited-memory quasi-newton inversion for 1D magnetotellurics. *Geophysics*, **71**, G191–G196.