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All-at-once Inversion Approach for MT on a Finite Difference Grid

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Outline

Governing equations

Curl-curl equation

- electrical field for 3D earth

$$\nabla \times \nabla \times \mathbf{E} + i\omega\mu_0\sigma\mathbf{E} = 0$$

Helmholtz equation

- electrical field for 1D layered earth

$$\partial^2{}_{zz}E_x(z) - i\omega\mu_0\sigma(z)E_x(z) = 0$$

1D case

Formulation

- forward modelling: $\mathbf{A}(\boldsymbol{m})\boldsymbol{u} = \boldsymbol{b}$

  $\mathbf{A}$ – forward operator or system matrix
  $\boldsymbol{m}$ – model parameter, $\boldsymbol{m} = \log\boldsymbol{\sigma}$
  $\boldsymbol{b}$ – source terms and boundary values
  $\boldsymbol{u}$ – real and imaginary parts of the electrical field and its first spatial derivative
- solution $\boldsymbol{u}$ for the whole discretized region and different frequencies
- measurements only at the earth's surface for each frequency $\rightarrow \boldsymbol{d}$
- compare modelled and measured data: $\boldsymbol{r} = \mathbf{Q}\boldsymbol{u} - \boldsymbol{d}$
  projection matrix $\mathbf{Q}$ is necessary

Newton method

- nonlinear systems of equations
- linearisation of the optimisation problem
- solving a linear system of equations in each iteration
- first and second derivatives of objective function are used

Lagrange multipliers

Lagrangian

$$\mathcal{L}(\boldsymbol{u}, \boldsymbol{m}, \boldsymbol{\lambda}) = \frac{1}{2}||\mathbf{Q}\boldsymbol{u} - \boldsymbol{d}||^2 + \frac{\beta}{2}||\mathbf{W}(\boldsymbol{m} - \boldsymbol{m}_{\text{ref}})||^2 + \boldsymbol{\lambda}^T[\mathbf{A}(\boldsymbol{m})\boldsymbol{u} - \boldsymbol{b}] \rightarrow \min$$

gradient of the Lagrangian: $\nabla\mathcal{L}$

$$\nabla\mathcal{L} = \begin{pmatrix} \mathcal{L}_{\boldsymbol{u}} \\ \mathcal{L}_{\boldsymbol{m}} \\ \mathcal{L}_{\boldsymbol{\lambda}} \end{pmatrix}$$

$$\begin{aligned} \mathcal{L}_{\boldsymbol{u}} &= \mathbf{Q}^T(\mathbf{Q}\boldsymbol{u} - \boldsymbol{d}) + \mathbf{A}^T\boldsymbol{\lambda} \\ \mathcal{L}_{\boldsymbol{m}} &= \beta\mathbf{W}^T\mathbf{W}(\boldsymbol{m} - \boldsymbol{m}_{\text{ref}}) + \mathbf{G}^T\boldsymbol{\lambda} \\ \mathcal{L}_{\boldsymbol{\lambda}} &= \mathbf{A}\boldsymbol{u} - \boldsymbol{b} \end{aligned}$$

with $\mathbf{G} = \partial_m\mathbf{A} \cdot \boldsymbol{u}$

Second derivatives of $\mathcal{L}$, Hessian $\mathcal{H}_{kkt}$

KKT system

$$\mathcal{H}_{kkt} \cdot \begin{pmatrix} \delta \boldsymbol{u} \\ \delta \boldsymbol{m} \\ \delta \boldsymbol{\lambda} \end{pmatrix} = - \begin{pmatrix} \mathcal{L}_{\boldsymbol{u}} \\ \mathcal{L}_{\boldsymbol{m}} \\ \mathcal{L}_{\boldsymbol{\lambda}} \end{pmatrix}$$

$$\begin{pmatrix} \mathcal{L}_{\boldsymbol{u},\boldsymbol{u}} & \mathcal{L}_{\boldsymbol{u},\boldsymbol{m}} & \mathcal{L}_{\boldsymbol{u},\boldsymbol{\lambda}} \\ \mathcal{L}_{\boldsymbol{m},\boldsymbol{u}} & \mathcal{L}_{\boldsymbol{m},\boldsymbol{m}} & \mathcal{L}_{\boldsymbol{m},\boldsymbol{\lambda}} \\ \mathcal{L}_{\boldsymbol{\lambda},\boldsymbol{u}} & \mathcal{L}_{\boldsymbol{\lambda},\boldsymbol{m}} & \mathcal{L}_{\boldsymbol{\lambda},\boldsymbol{\lambda}} \end{pmatrix} \cdot \begin{pmatrix} \delta \boldsymbol{u} \\ \delta \boldsymbol{m} \\ \delta \boldsymbol{\lambda} \end{pmatrix} = - \begin{pmatrix} \mathcal{L}_{\boldsymbol{u}} \\ \mathcal{L}_{\boldsymbol{m}} \\ \mathcal{L}_{\boldsymbol{\lambda}} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{Q}^T\mathbf{Q} & \mathbf{K} & \mathbf{A}^T \\ \mathbf{K}^T & \beta \mathbf{W}^T\mathbf{W} + \mathbf{R} & \mathbf{G}^T \\ \mathbf{A} & \mathbf{G} & \mathbf{0} \end{pmatrix} \cdot \begin{pmatrix} \delta \boldsymbol{u} \\ \delta \boldsymbol{m} \\ \delta \boldsymbol{\lambda} \end{pmatrix} = - \begin{pmatrix} \mathcal{L}_{\boldsymbol{u}} \\ \mathcal{L}_{\boldsymbol{m}} \\ \mathcal{L}_{\boldsymbol{\lambda}} \end{pmatrix}$$

$$\mathbf{G} = \partial_m \mathbf{A} \cdot \boldsymbol{u}, \qquad \mathbf{K} = \partial_m \mathbf{A}^T \cdot \boldsymbol{\lambda}, \qquad \mathbf{R} = \partial_m \mathbf{G}^T \cdot \boldsymbol{\lambda}$$

All-at-once

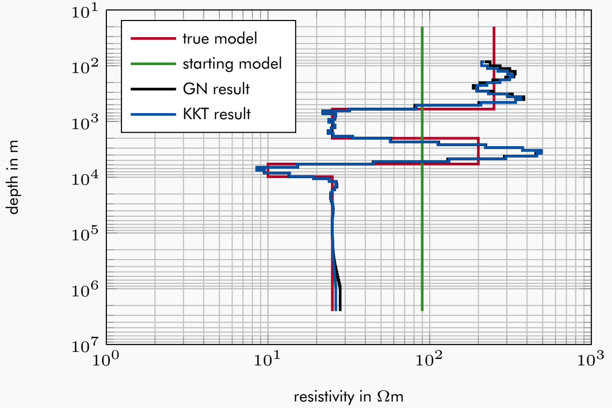
update after each iteration

$$\begin{pmatrix} \boldsymbol{u}_{n+1} \\ \boldsymbol{m}_{n+1} \\ \boldsymbol{\lambda}_{n+1} \end{pmatrix} = \begin{pmatrix} \boldsymbol{u}_{n} \\ \boldsymbol{m}_{n} \\ \boldsymbol{\lambda}_{n} \end{pmatrix} + \begin{pmatrix} \boldsymbol{\delta u} \\ \boldsymbol{\delta m} \\ \boldsymbol{\delta \lambda} \end{pmatrix}$$

- no forward problem has to be solved to get data for the new model

Results

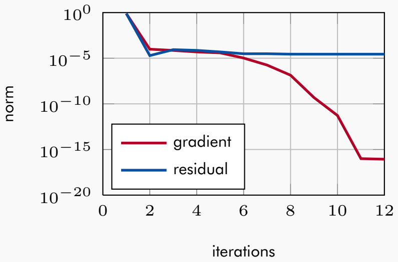
Synthetic data

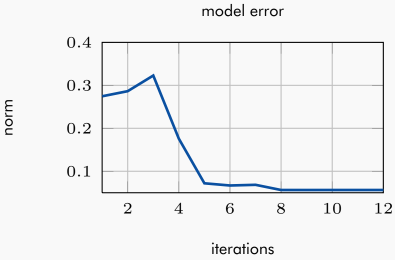


21 frequencies between 10^{-4} and 10^3 Hz
regularisation $\beta = 4.5 \cdot 10^{-14}$

Results

Synthetic data – inversion process

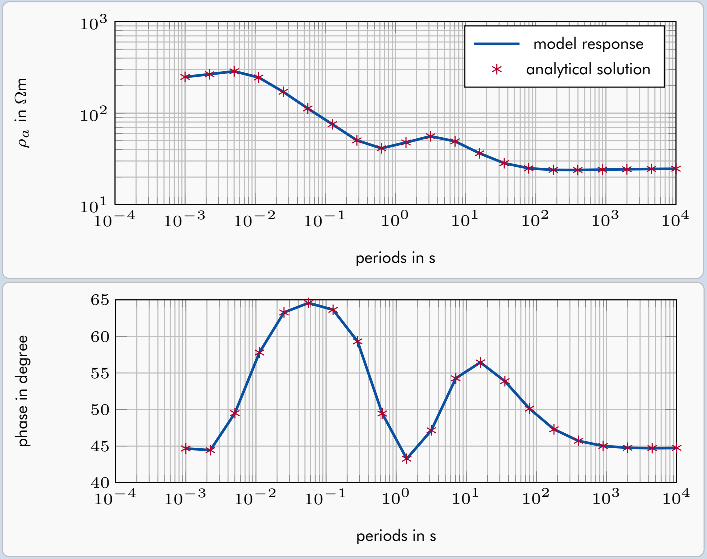




gradient $||\nabla\mathcal{L}||$ and residual norm

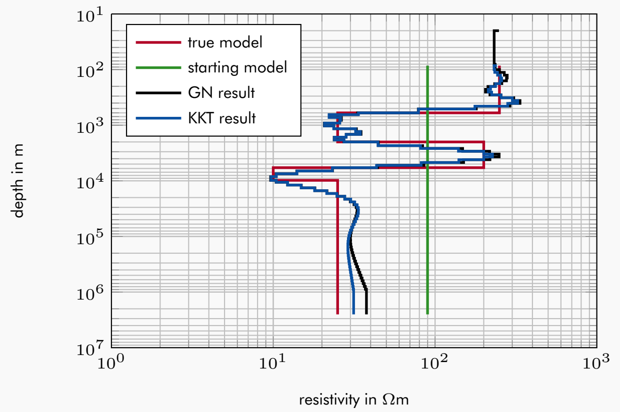
model parameter residual norm

Sounding curves



Results

Synthetic data with noise

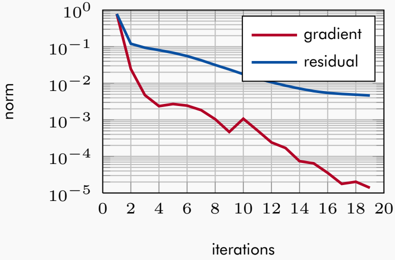


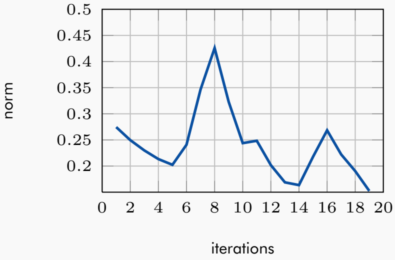
21 frequencies between 10^{-4} and 10^3 Hz

regularisation $\beta = 0.45^{\text{iteration}} \cdot 10^{-4}$

Results

Synthetic data with noise – inversion process

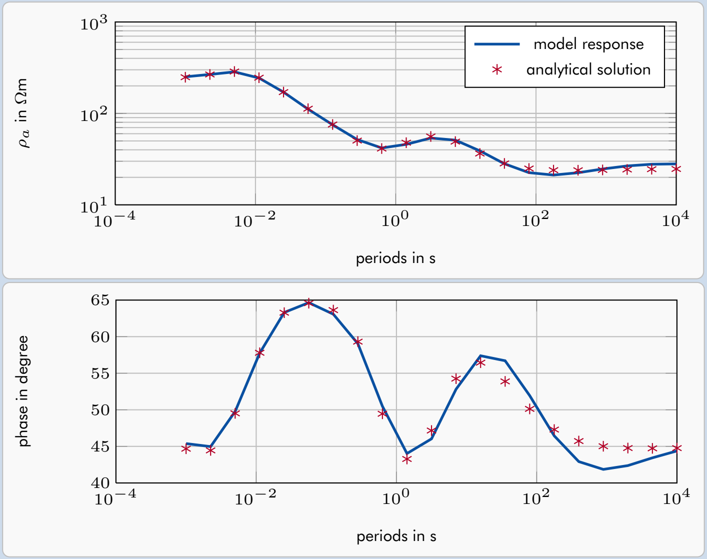




gradient $||\nabla\mathcal{L}||$ and residual norm

model parameter residual norm

Sounding curves



Outlook

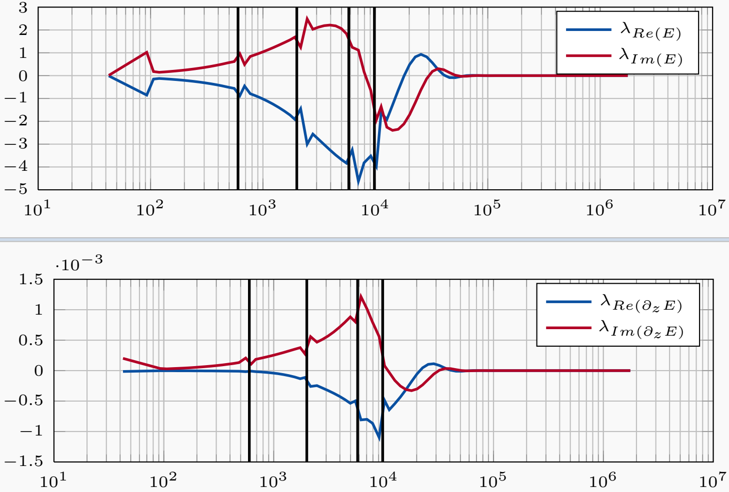
- 2D magnetotelluric problem
- 3D magnetotelluric problem

Thank you for your attention!

References

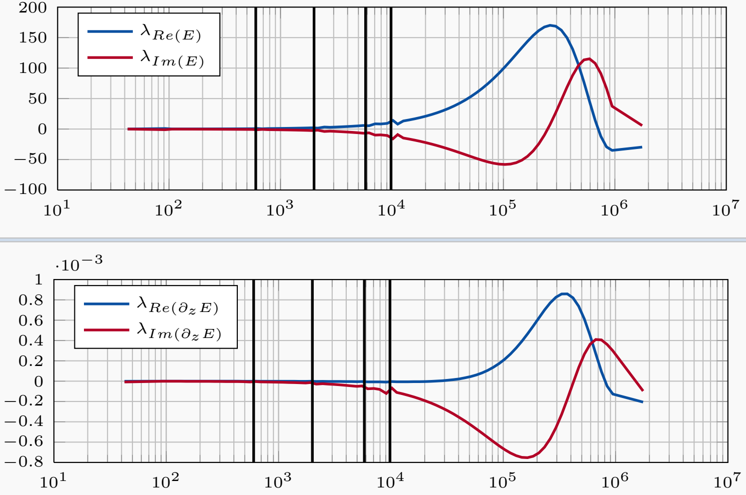
Haber, E., Asher, U. M., Oldenburg, D., On optimization techniques for solving nonlinear inverse problems, Inverse Problems 16, 2000.

Behaviour of Lagrange multipliers



frequency=0.14 Hz

Behaviour of Lagrange multipliers



frequency=0.0001 Hz

Regularisation parameter noisy data