Finite-element Forward Solver for Unstructured Tetrahedral Grids

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1. Brief explanation of *EM PDE* equations: induction and continuity
2. Discretization of the computational domain
3. Finite-element method
4. Calculating and coding the inner product terms
5. Code is under development: Future plans

- Considering a time dependence of $e^{i\omega t}$ in the *Quasi-Static* regime

$$\nabla \times \nabla \times \mathbf{E} + i\omega\mu\tilde{\sigma}\mathbf{E} = i\omega\mu\mathbf{J}^s \tag{1}$$

$\tilde{\sigma} = \tilde{\sigma}(x, y, z)$ and $\mathbf{E} = \mathbf{E}(x, y, z, t)$

A-ϕ *decomposition*: To Visualize the *Inductive* and *Galvanic* nature of the EM fields

$$\mathbf{E} = -i\omega\mathbf{A} - \nabla\phi \tag{2}$$

$$\boxed{\nabla \times \nabla \times \mathbf{A} + i\omega\mu\tilde{\sigma}\mathbf{A} + \mu\tilde{\sigma}\nabla\phi = \mu\mathbf{J}^s} \tag{3}$$

- Setting up the second equation using the Ampère's law

$$\nabla \cdot \mathbf{J} = \begin{cases} -\nabla \cdot \mathbf{J}^s & \text{source location} \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

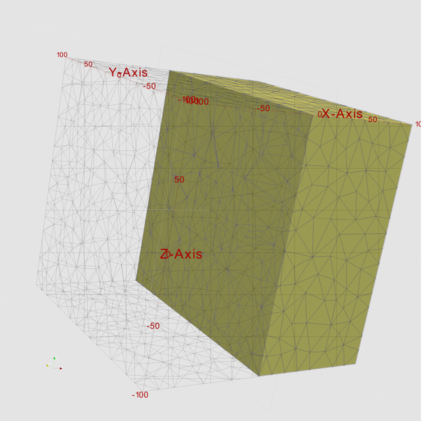
Because $\mathbf{J} = \tilde{\sigma}\mathbf{E}$,

$$\boxed{-i\omega\nabla \cdot (\tilde{\sigma}\mathbf{A}) - \nabla \cdot (\tilde{\sigma}\nabla\phi) = -\nabla \cdot \mathbf{J}^s} \tag{5}$$

- *Blocks2mesh* a code written by *Peter Lelievre <plelievre@mun.ca>*, is used to create a three dimensional grid of unstructured tetrahedrons
- The code makes a block poly file that get fed into **Tetgen**

- Auxiliary subroutines to set up the connectivity arrays:
  1. edge - to - node
  2. edge - to - element
  3. element - to - edge
  4. ...

- Figure is generated by *Kitware Paraview 3.6.2*



THEOREM

- $$\nabla \times \nabla \times \boldsymbol{A} + i\omega\mu\tilde{\sigma}\boldsymbol{A} + \mu\tilde{\sigma}\nabla\phi = \mu\boldsymbol{J}^s$$
- $$-i\omega\nabla \cdot (\tilde{\sigma}\boldsymbol{A}) - \nabla \cdot (\tilde{\sigma}\nabla\phi) = -\nabla \cdot \boldsymbol{J}^s$$

is achieved.

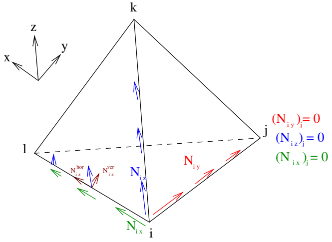
- Finite-element solution
- Approximation of the vector and scalar potentials

$$\tilde{\mathbf{A}} = \sum_{j=1}^{N_{edges}} \tilde{A}_j \mathbf{N_j} \quad (6)$$

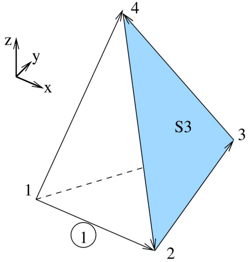
$$\tilde{\phi} = \sum_{k=1}^{N_{nodes}} \tilde{\phi_{k_1}} N_{k_1} \quad (7)$$

- $\mathbf{N}_j$ is the edge-element vector basis function
- and N_{k_1} is the nodal-element scalar basis function

- $N_k^e(x, y, z) = \frac{1}{6V^e}(a_k^e + b_k^e x + c_k^e y + d_k^e z)$
- The scalar basis function N_i is equal to unity at node i and decreases linearly in all three orthogonal directions.
- Vanishes linearly towards the other nodes and faces $\rightarrow$ guarantees the continuity of the tangential $\mathbf{E}$.
- The vertical component of the $\mathbf{E}$ is not necessarily continuous.



- Edge-elements
- Combines three components into one vector
- tangential components are continuous while verticals are allowed to jump
- Linear Whitney 1-form functions
- $\mathbf{N_i^e} = l_i^e(N_{i1}^e \nabla \mathbf{N_{i2}^e} - N_{i2}^e \nabla \mathbf{N_{i1}^e})$



- Seeks the solution by weighting the residual of the differential equation

$$\mathbf{r} = \nabla \times \nabla \times \mathbf{A} + i\omega\mu\tilde{\sigma}\mathbf{A} + \mu\tilde{\sigma}\nabla\phi - \mu\mathbf{J}^s \tag{8}$$

$$R = \int_\Omega \mathbf{N}_i \cdot \mathbf{r} \, d\Omega = 0 \tag{9}$$

- Combining (8) and (9) and using the Green's theorem

$$\sum_{j=1}^{N_{edges}} \tilde{A}_j \int_\Omega (\nabla \times \mathbf{N_i}) \cdot \left(\nabla \times \mathbf{N_j}\right) \, d\Omega \, + i\omega\mu \sum_{j=1}^{N_{edges}} \tilde{A}_j \int_\Omega \tilde{\sigma}\mathbf{N_i} \cdot \mathbf{N_j} \, d\Omega \tag{10}$$

$$+\mu \sum_{k=1}^{N_{nodes}} \phi_k \int_\Omega \tilde{\sigma}\mathbf{N_i} \cdot \nabla N_{k_1} \, d\Omega = \mu \sum_{j=1}^{N_{edges}} \int_\Omega \mathbf{N_i} \cdot \mathbf{J_s} \, d\Omega$$

$k_1 = 1 \cdots N_{nodes}$
$i = 1 \cdots N_{edges}$
$j = 1 \cdots N_{edges}$

Galerkin's Method

$$r = i\omega\nabla\cdot(\tilde{\sigma}\mathbf{A}) - \nabla\cdot\tilde{\sigma}\nabla\phi$$

$$R_{scalar} = \int_{\Omega} N_{k_2} r \, d\Omega = 0$$

$$\cancelto{0}{i\omega\tilde{A}_j \sum_{j=1}^{N_{edges}} \int_S N_{k_2}\hat{n}\cdot(\tilde{\sigma}\mathbf{N_j})\, dS} - i\omega\tilde{A}_j \sum_{j=1}^{N_{edges}} \int_{\Omega} \nabla N_{k_2}\cdot(\tilde{\sigma}\mathbf{N_j})\, d\Omega$$

$$+ \cancelto{0}{\tilde{\phi}_{k_1} \sum_{k_1=1}^{N_{nodes}} \int_S N_{k_2}\hat{n}\cdot(\tilde{\sigma}\nabla N_{k_1})\, dS} - \tilde{\phi}_{k_1} \sum_{k_1=1}^{N_{nodes}} \int_{\Omega} \nabla N_{k_2}\cdot(\tilde{\sigma}\nabla N_{k_1})\, d\Omega \qquad (11)$$

$$= \int_{\Omega} N_{k_2}\nabla\cdot\mathbf{J_s}\, d\Omega$$

$k_1 = 1 \cdots N_{nodes}$
$k_2 = 1 \cdots N_{nodes}$
$j = 1 \cdots N_{edges}$

- Edge - Edge Products

$$\mathbf{T}_{ij} = \sum_{j=1}^{N_{edges}} \int_{\Omega} (\nabla \times \mathbf{N_i}) \cdot \left(\nabla \times \mathbf{N_j}\right) \, d\Omega$$

$$\mathbf{T}_{ij} = \sum_{j=1}^{N_{edges}} \frac{4\, l_i\, l_j}{(6)^4 (V^e)^3} \left[(b_{i1} c_{i2} - c_{i1} b_{i2})(b_{j1} c_{j2} - c_{j1} b_{j2}) \right. \tag{12}$$

$$\left. +(d_{i1} b_{i2} - b_{i1} d_{i2})(d_{j1} b_{j2} - b_{j1} d_{j2}) + (c_{i1} d_{i2} - d_{i1} c_{i2})(c_{j1} d_{j2} - d_{j1} c_{j2}) \right]$$

$$\mathbf{U}_{ij} = \sum_{j=1}^{N_{edges}} \int_{\Omega} \tilde{\sigma} \mathbf{N}_i \mathbf{N}_j \, d\Omega$$

$$\mathbf{U}_{ij} = \frac{l_i l_j}{36 (V^e)^2} \tilde{\sigma} \left[F_{i2j2} G_{i1j1} - F_{i2j1} G_{i1j2} - F_{i1j2} G_{i2j1} + F_{i1j1} G_{i2j2} \right] \tag{13}$$

$F_{ij} = a_i a_j + b_i b_j + c_i c_j$.

$$G_{ij} = \int_{\Omega} N_i(x, y, z) N_j(x, y, z) \, d\Omega$$

- Mapping each tetrahedral element into the **simplex (Normalized) coordinate system**

$$G_{ij} = \int_{\Gamma} N_i(u, v, w) N_j(u, v, w) \, |\mathbf{J}| \, du\, dv\, dw$$

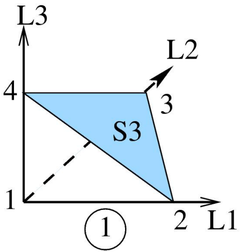
$$G_{ij} = \begin{cases} \frac{V^e}{10} & \text{i = j} \\ \frac{V^e}{20} & \text{i} \neq \text{j} \end{cases} \tag{14}$$

- Connectivity arrays used : Edge - to - edge
  1. Edge - to - cells
  2. Cell - to - edges

- Edge - node products

$$\mathbf{W_{ik}} = \int_\Omega \tilde{\sigma} \mathbf{N_i} \cdot \nabla N_k \, d\Omega$$

- Mapping into the normalize coordinate system



$$\int_\Omega N_i \, dxdydz = \int_\Gamma N_i(u, v, w)\, |\mathbf{J}| \, dudvdw \tag{15}$$

$$\int_\Omega N_i \, dxdydz = \frac{1}{24} |\mathbf{J}|$$

$$\mathbf{W_{ik}} = \sum_{i=1}^{N_{edges}} \frac{1}{144 \, V^e} l_i \tilde{\sigma} \left[(b_{i2} b_k + c_{i2} c_k + d_{i2} d_k) - (b_{i1} b_k + c_{i1} c_k + d_{i1} d_k) \right] \tag{16}$$

- Connectivity arrays used
  1. Edge - to - cells
  2. cell - to - nodes

- Node - node products

$$Z_{k_2k_1} = \int_\Omega \nabla N_{k_2} \cdot \tilde{\sigma} \nabla N_{k_1} \, d\Omega$$

$$Z_{k_2k_1} = \tilde{\sigma} (\frac{1}{6V^e})[b_{k2}b_{k1} + c_{k2}c_{k1} + d_{k2}d_{k1}] \quad (17)$$

- Connectivity array used: Cell - to - nodes

- **Source terms**

1. Edge - Source product

$$\mathbf{S} = \int_\Omega \mathbf{N_i} \cdot \mathbf{J_s} \, d\Omega \quad (18)$$

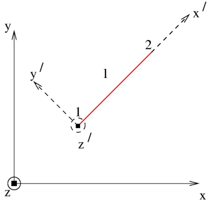
$$\mathbf{S} = \mu \sum_{i=1}^{N_{edges}} (\frac{1}{6V^e})^2 (a_{i1}b_{i2} - a_{i2}b_{i1}) j_s l \quad (19)$$

2. Node - Source product

$$\mathbf{SS} = \int_\Omega N_{k_2} \nabla \cdot \mathbf{J_s} \, d\Omega \quad (20)$$

$$\mathbf{SS} = \sum_{k=1}^{N_{nodes}} b_k j_s (\frac{1}{6V^e}) l \quad (21)$$

- A line source of current is chosen to be a delta function of finite length



$$\mathbf{J} = \delta(y') \, \delta(z') \, Box(x') \, j_s \, \hat{\mathbf{x}'} \quad (22)$$

- j_s is an arbitrary scalar quantity; $Box(x')$ is the boxcar function.

$$Box(x') = \begin{cases} 0 & x' > l, x < 0 \\ 1 & 0 < x' < l \end{cases} \quad (23)$$

- **Boundary Conditions**
- BCs are applied to the potentials on the truncation boundaries

$$\hat{n} \times \mathbf{A}|_{\partial\Omega} = \mathbf{0} \tag{24}$$

$$\phi|_{\partial\Omega} = 0 \tag{25}$$

- **System of Equations**
- A large system of equations is constructed

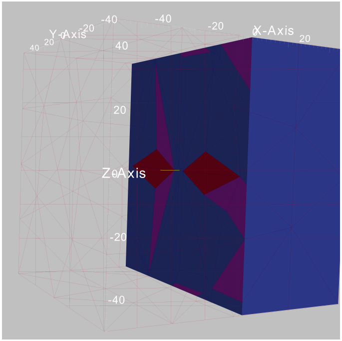
$$\mathbf{L} \cdot \mathbf{u} = \mathbf{F}$$

$$\begin{pmatrix} \mathbf{T_{ij}} & -\omega\mu\mathbf{U_{ij}} & \mu\mathbf{W_{ik_1}} & 0 \\ \omega\mu\mathbf{U_{ij}} & \mathbf{T_{ij}} & 0 & \mu\mathbf{W_{ik_1}} \\ 0 & \omega\mathbf{W_{k_2j}} & -\mathbf{Z_{k_2k_1}} & 0 \\ -\omega\mathbf{W_{k_2j}} & 0 & 0 & -\mathbf{Z_{k_2k_1}} \end{pmatrix} \begin{pmatrix} \mathbf{A_j^r} \\ \mathbf{A_j^I} \\ \phi_k^r \\ \phi_k^I \end{pmatrix} = \begin{pmatrix} \mathbf{S} \\ \mathbf{0} \\ \mathbf{SS} \\ 0 \end{pmatrix} \tag{26}$$

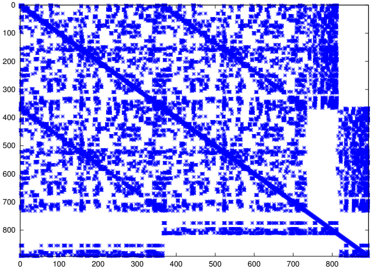
- Dimensions: $\mathbf{L}$ $(2(N_{edges} + N_{nodes}), 2(N_{edges} + N_{nodes}))$
- $\mathbf{u}$ $(2(N_{edges} + N_{nodes}))$
- $\mathbf{S}$ $(2(N_{edges} + N_{nodes}))$

Example for a Coarse Mesh

- $N_{cells} = 216, N_{nodes} = 80, N_{edges} = 366$



- Sparse structure of the coefficient matrix



$$\begin{pmatrix} \mathbf{T_{ij}} & -\omega\mu\mathbf{U_{ij}} & \mu\mathbf{W_{ik_1}} & 0 \\ \omega\mu\mathbf{U_{ij}} & \mathbf{T_{ij}} & 0 & \mu\mathbf{W_{ik_1}} \\ 0 & \omega\mathbf{W_{k_2j}} & -\mathbf{Z_{k_2k_1}} & 0 \\ -\omega\mathbf{W_{k_2j}} & 0 & 0 & -\mathbf{Z_{k_2k_1}} \end{pmatrix}$$

- The code is under development. We hope to apply it to simple Earth models
- The non-symmetric system of equations will be solved using **BICGSTAB** solver with an **LU** preconditioner
- The performance of the above approach will be compared with a modification in which the **Lorentz gauge condition** is used. Also, the relative contributions to the electric field from the **inductive and galvanic** terms will be investigated in different situations.