THE PHASE DIAGRAM OF A SPIN GLASS ON A TREE WITH FERROMAGNETIC INTERACTIONS

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Abstract. A spin glass problem on a Cayley tree with ferromagnetic interactions is solved rigorously. Using a level-I large deviation argument together with the martingale approach used by Buffet, Patrick and Pulé [1], explicit expressions for the free energy are derived in different regions of the phase diagram. It is found that there are four phases: a paramagnetic phase, a spin-glass phase, a ferromagnetic phase and a mixed phase. The nature of the phase diagram depends on the power with which the ferromagnetic term occurs in the Hamiltonian.

1. The Directed Polymer Problem and the Generalized Random Energy Model

The problem of a spin glass on a Cayley tree (or equivalently, directed polymers) is one of a handful of models in disordered systems that can be solved exactly. It is a simplification of the more realistic case where one considers a regular lattice in place of the Cayley tree. The problem has been treated for instance using the replica method [5], using the properties of Generalized Random Energy Model [4, 2] by reducing the problem to a reaction-diffusion system [8] and by a martingale approach [1]. The latter approach is particularly elegant and achieves a completely rigorous and transparent solution to the problem. Here we use a combination of the martingale approach of [1] and a level-1 large deviations argument [15, 11] to solve a spin glass model on a Cayley tree with an additional mean-field ferromagnetic interaction term in the Hamiltonian. We consider a one-parameter family of such models distinguished by the power $p \geq 2$ to which this term is raised and show that the phase diagram in the case $p > 2$ is qualitatively different from that in the case $p = 2$. The two phase diagrams are depicted in Figure 1.2(a) and 1.2(b). We derive completely rigorously a variational expression for the free energy of our model and then analyze carefully the various regions of the phase diagram. We note that the free energy of the Generalized Random Energy Model in a magnetic field has been computed by Derrida and Gardner [6] using the replica method, which is of course, not rigorous. Our result (§ 2) has a direct analogy with theirs (§ 4 in [6]) in this case.

The spin glass on a Cayley tree is in fact similar to Derrida's Generalized Random Energy Model, which has been used in various applications, notably information theory [13, 14] and neural networks [7, 6]. It follows that our results may have implications for applications in these areas. In particular, we have outlined the implications of our results for the optimal decoding problem as proposed by Sourlas [13, 14] in a separate paper [10]. Indeed, it turns out that the phase diagram is identical to that of the Random Energy Model with the same ferromagnetic interaction term as above. We claim that this model is relevant for Sourlas' decoding theory in the case of large p. Indeed, in [3, 4] Derrida already showed that the random energy model is the limit of the Sherrington-Kirkpatrick model with p-spin interaction. As explained in [10], Sourlas' coding scheme amounts to adding a p-spin Ising term, the ground state of which corresponds to the original message. Random noise in the transmission line then leads to a Random Energy Model with p-spin interaction in the limit $p \to \infty$.

Let us now define the model and the terminology we adopt in this paper: Consider a **Cayley tree** (cf. Fig 1.1) with *co-ordination number* 3- i.e. each node of the tree is connected to another two at the next level. Label the bonds of the tree by (j, k) where $j, k \in \mathbb{N}$ and j corresponds to the generation

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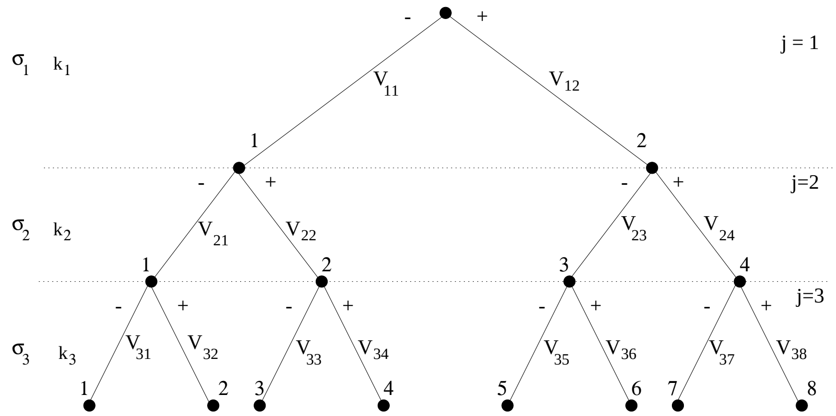

Figure 1.1: $k_{j+1} = 2k_j - (1-\sigma_{j+1})/2$. $k_i = 1, 2, \ldots$ and σ_i take ± 1.

and $k \in \{1, \ldots, 2^j\}$ labels the bonds from left to right within the j^{th} generation. To each bond of the tree attach i.i.d random variables $V_{j,k}$ with distribution depending on a parameter γ.

A **path** of length n starting at the top of the tree is defined as a finite sequence

$$\{(j, k_j); 1 \le j \le n\} \tag{1.1}$$

satisfying the relation

$$k_{j+1} = 2k_j - \frac{1}{2}(1 - \sigma_{j+1}) \tag{1.2}$$

where $\sigma_j \in \{-1, 1\}$ correspond to taking the left or right branch out of generation j. (see Figure 1.1). Denote $(\sigma)^j$ the sequence of Ising spins $\{\sigma_k\}_{k=1}^{j}$. Then the path is completely determined by $(\sigma)^n$. Define the Hamiltonian by

$$-\mathscr{H} = \sum_{j=1}^{n} V_{j,(\sigma)^j} + \frac{\lambda}{n^{p-1}} \left| \sum_{j=1}^{n} \sigma_j \right|^p \tag{1.3}$$

where $p \ge 2$ is an arbitrary parameter and $\lambda > 0$ is a coupling constant. The **partition function** is defined by

$$\mathscr{Z}_n = \sum_{\{\sigma_j\}_{j=1}^n} \mathrm{e}^{-\beta \mathscr{H}} \tag{1.4}$$

The **(specific) free energy** of the model is defined by

$$-\beta f(\beta, \lambda, \gamma) = \lim_{n \to \infty} \frac{1}{n} \log \mathscr{Z}_n(\beta). \tag{1.5}$$

In Section 4 we show that this limit exists almost surely with respect to the random variables V and we derive expressions for it in the cases $p = 2$ and $p > 2$ respectively. The phase diagram consists of four different phases: the paramagnetic phase (P) , a spin-glass phase (SG) , a ferromagnetic phase (F) and a mixed phase (M). The phase diagram in the case $p = 2$ is depicted in Figure 1.2(a); and that in the case $p > 2$ in Figure 1.2(b). With reference to Figure 1.2(a), in paramagnetic regime (P), where $\lambda < \mathcal{C}_1(\beta)$ and $\beta < \beta_0$, and also in region (SG) where $\beta > \beta_0$, $\lambda < \lambda_0$, the magnetization $m = 0$. In the latter phase the free-energy remains constant and in the absence of long-range order this is a spin glass or frozen phase. The region (F) where $\mathcal{C}_1(\beta) < \lambda < \mathcal{C}_2(\beta)$ and $m \neq 0$ is the ferromagnetic phase. (M) is a mixed phase where $\lambda > \mathcal{C}_2(\beta)$ or $\lambda > \lambda_0$ and the magnetization $m \neq 0$ depends only on λ.

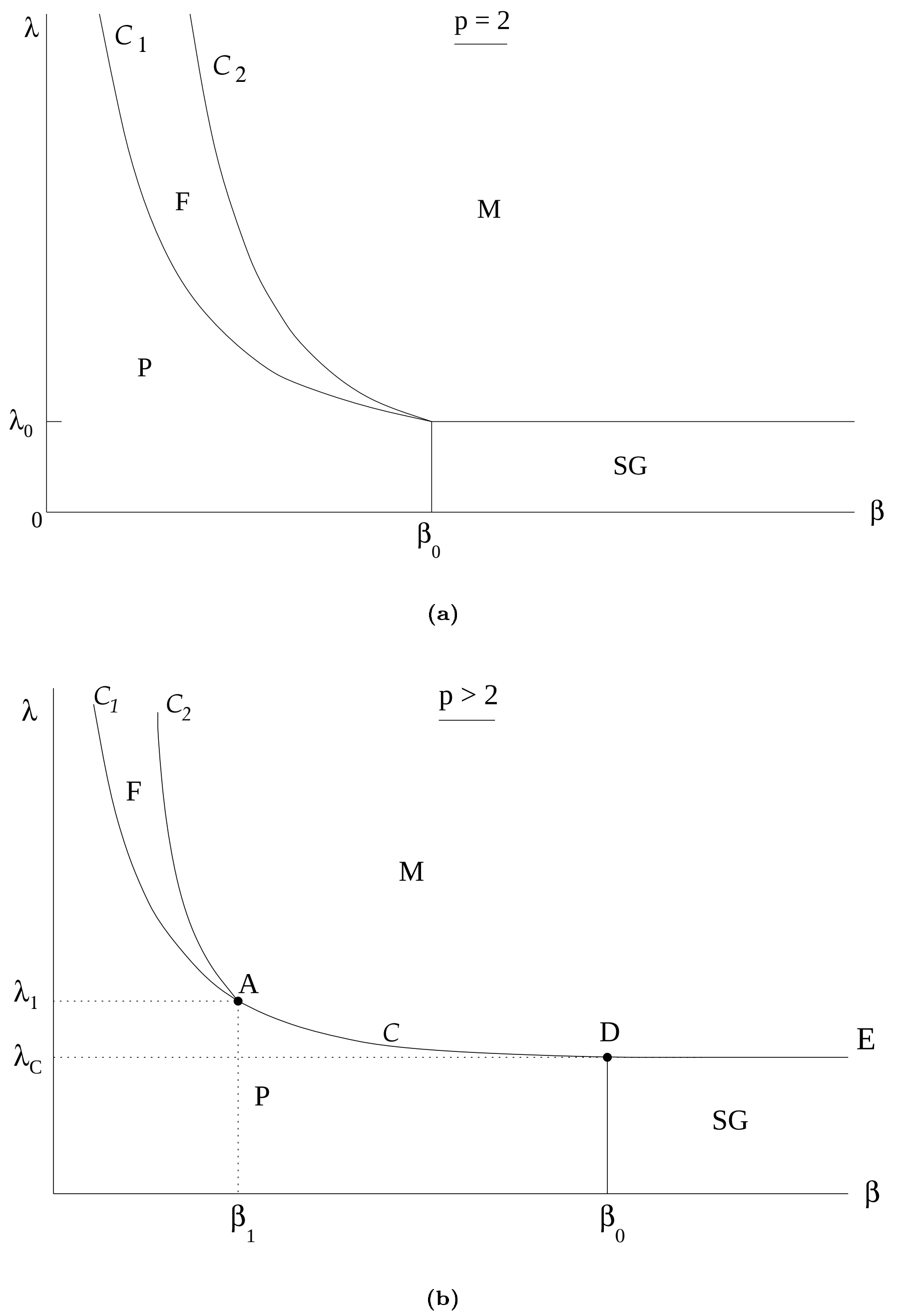

Figure 1.2: (a) The phase-diagram for p = 2: (b) The phase-diagram for p > 2:

In Figure1.2(b), the effect of the higher order ferromagnetic term in (1.3) is visible from the curve $\mathcal{C}$ in contrast to the case $p = 2$. Indeed, it will be shown that as p increases the points A and D in Figure 1.2(b) drift apart. We will also show that for $p > 2$ the magnetization is discontinuous across the lines $\mathcal{C}_1$, $\mathcal{C}$ and $\lambda = \lambda_c$ whereas it is continuous accross the curve $\mathcal{C}_2 A$. In the paramagnetic region (P), $m = 0$ but the free-energy depends on β and in the ferromagnetic region (F) $m(\beta, \lambda) \neq 0$ while in the Spin Glass phase (SG) $m = 0$ with the free-energy remaining constant. In the mixed phase (M), $m \neq 0$ and the free-energy depends only on λ.

The computation of the free energy involves large deviation theory. First we write the partition function as an integral with respect to measures defined in terms of the spin-glass on a Cayley tree with an external magnetic field. Then we show that these a priori measures satisfy the large deviation principle (LDP) by first calculating the cumulant generating function. This is done in Section 2, using an extension of the martingale approach of [1]. It is well-known that the existence of the cumulant generating function implies the LDP for level-I measures (see [15, 11]). We compute the corresponding rate function as a Legendre transform of this cumulant generating function in Section 3. By Varadhan's theorem we can then write a variational expression for the free energy density. This expression is analyzed in Section 4 for the cases $p = 2$ and $p > 2$ respectively. Exact expressions for the free energy in the various regions of phase diagram are derived.

2. The Cumulant Generating Function

2.1. **Definitions.** Let the configuration space be the set X^n of all sequences $\{\sigma_i\}_{i=1}^n$ with $\sigma_i \in X =: \{-1, 1\}$. Let $\mu(\sigma_i = +1) = \mu(\sigma_i = -1) = 1/2$ so that the a-priori probability of each configuration of spin variables is $\mu_n = 1/2^n$. Now the partition function (1.4) can be written as

$$\mathscr{Z}_n = \sum_{\{\sigma_j\}_{j=1}^n} \exp\left\{\beta n \left[\frac{1}{n}\sum_{j=1}^n V_{j,(\sigma)^j} + \lambda\left(\frac{1}{n}\sum_{j=1}^n \sigma_j\right)^p\right]\right\}. \tag{2.1}$$

Note that since $V_{j,(\sigma)^j}$ depends on all the previous σ_k, $k \le j$, performing the above summation over $\{\sigma_j\}$ is not straight-forward. So, we will exploit martingale properties [17] related to $\mathscr{Z}_n$.

We write the free energy (1.5) as

$$-\beta f(\beta, \lambda, \gamma) = \lim_{n\to\infty} \frac{1}{n} \log \int_{X_n} \mathrm{e}^{-\beta \mathscr{H}} \mu_n(d\sigma) + \log 2. \tag{2.2}$$

Also define the observables $V_n : X_n \to \mathbb{R}$, $m_n : X_n \to \mathbb{R}$ by

$$V_n(\sigma) = \frac{1}{n}\sum_{j=1}^n V_{j,(\sigma)^j}, \qquad m_n(\sigma) = \frac{1}{n}\sum_{j=1}^n \sigma_j. \tag{2.3}$$

Notice that the partition function (2.1) only depends on these two variables so that (2.2) can be rewritten as an integral with respect to the distribution N_n of $W_n = (V_n(\sigma), m_n(\sigma))$, i.e. the image measure [12] on $\mathbb{R}^2$ induced by the map W_n:

$$-\beta f(\beta, \lambda, \gamma) = \lim_{n\to\infty} \frac{1}{n} \log \int_{\mathbb{R}^2} \mathrm{e}^{n\beta(v+\lambda m^p)}\, N_n(dv, dm) + \log 2. \tag{2.4}$$

We wish to compute this limit and show that it converges almost surely with respect to the distribution of the random variables $V_{j,(\sigma)^j}$, which we also denote by ω. We do this by first computing the **cumulant generating function** $C(t_1, t_2)$ defined by

$$C(t_1, t_2) = \lim_{n\to\infty} \frac{1}{n} \log \int e^{t_1 v + t_2 m} N_n(dv, dm). \tag{2.5}$$

This will enable us to compute the corresponding rate function , i.e.

$$I(v, m) = \sup_{t_1, t_2} \{t_1 v + t_2 m - C(t_1, t_2)\} \tag{2.6}$$

and to apply Varadhan's Theorem [11, 15] to get

$$-\beta f(\beta, \lambda, \gamma) = \sup_{v,m} \{\beta(v + \lambda m^p) - I(v, m)\} + \log 2. \tag{2.7}$$

Denote

$$\tilde{\mathscr{Z}}_n(t_1,t_2) = \sum_{\{\sigma_j\}_{j=1}^n} \mathrm{e}^{t_1 \sum_{j=1}^n V_{j,(\sigma)^j} + t_2 \sum_{j=1}^n \sigma_j} \tag{2.8}$$

and define

$$\nu^n = \left\{ V_{j,(\sigma)^j}; 1 \le k \le 2^j, 1 \le j \le n \right\} \tag{2.9}$$

which denotes the set of all the random variables $V_{j,(\sigma)^j}$ between generation 1 and n. Notice that the cumulant generating function (2.5) can be written as

$$C(t_1,t_2) = \lim_{n\to\infty} \frac{1}{n} \log \tilde{\mathscr{Z}}_n(t_1,t_2) - \log 2. \tag{2.10}$$

Define

$$\Phi(t_1,t_2) = \cosh(t_2)\mathbb{E}\left[\mathrm{e}^{t_1 V}\right] \tag{2.11}$$

$$\text{and} \qquad M_n(t_1,t_2) = \frac{\tilde{\mathscr{Z}}_n(t_1,t_2)}{(2\Phi(t_1,t_2))^n}, \tag{2.12}$$

where $\mathbb{E}$ denotes the expectation with respect to the random variables V.

2.2. **Martingale Results.**

Proposition 2.1. $\{M_n\}_{n=1}^\infty$ *is a martingale with respect to the increasing family of random variables* $\{\nu^n\}_{n=1}^\infty$, *that is,*

$$\mathbb{E}(M_{n+1}|\nu^n) = M_n. \tag{2.13}$$

Proof. Write

$$V_{n+1,(\sigma_1,\dots,\sigma_n,\sigma_{n+1})} = \begin{cases} V_1 & ;\ \sigma_{n+1} = +1 \\ V_2 & ;\ \sigma_{n+1} = -1. \end{cases}$$

$$\tilde{\mathscr{Z}}_{n+1}(t_1,t_2) = \sum_{\{\sigma_j\}_{j=1}^n} \exp\left[t_1 \sum_{j=1}^n V_{j,(\sigma)^j} + t_2 \sum_{j=1}^n \sigma_j\right] \sum_{\sigma_{n+1}\in\{-1,1\}} \exp\left[t_1 V_{n+1,(\sigma)^{n+1}} + t_2\sigma_{n+1}\right] \tag{2.14}$$

Taking the expectation with respect to $V_{n+1,\sigma}$,

$$\begin{aligned} \mathbb{E}\big[\tilde{\mathscr{Z}}_{n+1}(t_1,t_2)|\nu^n\big] &= \tilde{\mathscr{Z}}_n(t_1,t_2)\mathbb{E}\left[\mathrm{e}^{t_1 V_1 + t_2} + \mathrm{e}^{t_1 V_2 - t_2}\right] \\ &= \tilde{\mathscr{Z}}_n(t_1,t_2) 2\cosh(t_2)\mathbb{E}\left(\mathrm{e}^{t_1 V}\right). \end{aligned} \tag{2.15}$$

Dividing by $(2\Phi(t_1,t_2))^{n+1}$ the result follows. ■

Remark 2.1*:*

1. $\mathbb{E}[M_n(t_1,t_2)] = 1$.
2. As in [1], if $M_\infty > 0$ with probability 1 then we have

$$\lim_{n\to\infty} \frac{1}{n} \log \tilde{\mathscr{Z}}_n(t_1,t_2) = \log 2\Phi(t_1,t_2). \qquad \text{a.s} \tag{2.16}$$

◊

Lemma 2.1. *For any fixed* t_1, t_2,

$$\mathbb{P}\left[M_\infty(t_1,t_2)=0\right]=0 \quad or \quad 1. \tag{2.17}$$

Proof. Let L_n(resp. R_n) denote the set of paths of length n which start with a branch in the left(resp. right) direction. Then we have

$$\begin{aligned} M_n(t_1,t_2) = (2\Phi(t_1,t_2))^{-n}\Bigg[& \mathrm{e}^{t_1 V_{11}+t_2\sigma_1} \sum_{(\sigma)^j\in L_n} \mathrm{e}^{t_1\sum_{j=1}^n V_{j,(\sigma)^j}+t_2\sum_{j=1}^n\sigma_j} \\ & + \mathrm{e}^{t_1 V_{12}+t_2\sigma_2} \sum_{(\sigma)^j\in R_n} \mathrm{e}^{t_1\sum_{j=1}^n V_{j,(\sigma)^j}+t_2\sum_{j=1}^n\sigma_j}\Bigg]. \end{aligned} \tag{2.18}$$

The event $\{\lim_{n\to\infty} M_n(t_1,t_2)=0\}$ is independent of V_{11} and V_{12}. Hence it is independent of ν^2. Similarly it is independent of ν^p for every p. Hence the result follows by Kolmogorov's 0, 1- Law. ■

Remark 2.2*:*
If $\mathbb{P}[M_\infty=0]=1$ then $\mathbb{E}[M_\infty]=0$. Therefore, if we know that $\mathbb{E}[M_\infty(t_1,t_2)]>0$ then $\mathbb{P}[M_\infty(t_1,t_2)=0]=1$ is impossible. As in [1], we prove that

$$\sup_{n\geq 1}\mathbb{E}\big[M_n^\alpha(t_1,t_2)\big]<\infty \qquad \text{for some } \alpha>1 \tag{2.19}$$

from which it follows that $\mathbb{E}[M_\infty(t_1,t_2)]=1$. ◊

Although the proof of the next lemma takes a similar reasoning as that of [1], we present the proof for completeness.

Lemma 2.2.

$$\mathbb{E}\left[M_{n+1}^2(t_1,t_2)|\nu^n\right]=M_n^2(t_1,t_2)+\lambda(t_1,t_2)^n\left[\lambda(t_1,t_2)-\lambda(0,t_2)\right]M_n(2t_1,2t_2) \tag{2.20}$$

where

$$\lambda(t_1,t_2)=\frac{\Phi(2t_1,2t_2)}{2\Phi^2(t_1,t_2)}. \tag{2.21}$$

Proof.

$$\begin{aligned} \tilde{\mathscr{Z}}_{n+1}^2(t_1,t_2) &= \left\{\sum_{\{\sigma_j\}_{j=1}^n}\exp\left[t_1\sum_{j=1}^n V_{j,(\sigma)^j}+t_2\sum_{j=1}^n\sigma_j\right]\left(\mathrm{e}^{t_1V_1+t_2}+\mathrm{e}^{t_1V_2-t_2}\right)\right\}^2 \\ &= \sum_{\{\sigma_j\}_{j=1}^n}\mathrm{e}^{2t_1\sum_{j=1}^n V_{j,(\sigma)^j}+2t_2\sum_{j=1}^n\sigma_j}\left(\mathrm{e}^{2t_1V_2-2t_2}+2\mathrm{e}^{t_1V_2+t_1V_1}+\mathrm{e}^{2t_1V_1+2t_2}\right) \\ &\quad + \sum_{\substack{\{\sigma_j\},\{\sigma_j'\}\\ \exists j:\sigma_j\neq\sigma_j'}}\mathrm{e}^{t_1\sum_{j=1}^n V_{j,(\sigma)^j}+t_2\sum_{j=1}^n\sigma_j}\left(\mathrm{e}^{t_1V_2-t_2}+\mathrm{e}^{t_1V_1+t_2}\right) \\ &\qquad \times \mathrm{e}^{t_1\sum_{j=1}^n V_{j,(\sigma')^j}+t_2\sum_{j=1}^n\sigma_j'}\left(\mathrm{e}^{t_1V_2'-t_2}+\mathrm{e}^{t_1V_1'+t_2}\right) \end{aligned} \tag{2.22}$$

Taking the expectation

$$\begin{aligned}\mathbb{E}\left[\tilde{\mathscr{Z}}_{n+1}^2|\nu^n\right] &= \tilde{\mathscr{Z}}_n(2t_1,2t_2)\left(2\Phi(2t_1,2t_2)+2\mathbb{E}\left(\mathrm{e}^{t_1V}\right)^2\right)\\ &+\sum_{\substack{\{\sigma_j\},\{\sigma'_j\}\\ \exists j:\sigma_j\neq\sigma'_j}} \mathrm{e}^{t_1\sum_{j=1}^n V_{j,(\sigma)j}+t_2\sum_{j=1}^n\sigma_j}\mathrm{e}^{t_1\sum_{j=1}^n V_{j,(\sigma')_j}+t_2\sum_{j=1}^n\sigma'_j}4\Phi^2(t_1,t_2).\end{aligned}\tag{2.23}$$

A similar splitting of sums as in (2.22) shows that the double sum in the right hand side of (2.23) equals $\tilde{\mathscr{Z}}_n(t_1,t_2)^2-\tilde{\mathscr{Z}}_n(2t_1,2t_2)$. Hence

$$\begin{aligned}\mathbb{E}\left[\tilde{\mathscr{Z}}_{n+1}^2(t_1,t_2)|\nu^n\right] &= \tilde{\mathscr{Z}}_n(2t_1,2t_2)\left(2\Phi(2t_1,2t_2)+2\Phi^2(t_1,t_2)\mathrm{sech}^2(t_2)\right)\\ &+4\Phi^2(t_1,t_2)\left(\tilde{\mathscr{Z}}_n^2(t_1,t_2)-\tilde{\mathscr{Z}}_n(2t_1,2t_2)\right).\end{aligned}\tag{2.24}$$

where we have used the fact that $\mathbb{E}\left[\mathrm{e}^{t_1V}\right]^2=\Phi^2(t_1,t_2)\mathrm{sech}^2(t_2)$. Dividing by $(2\Phi(t_1,t_2))^{2n+2}$ we get

$$\begin{aligned}\mathbb{E}\left[M_{n+1}^2(t_1,t_2)|\nu^n\right] &= \frac{2\tilde{\mathscr{Z}}_n(2t_1,2t_2)}{(2\Phi(t_1,t_2))^{2n+2}}\left[\Phi(2t_1,2t_2)+\mathbb{E}\left(\mathrm{e}^{t_1V}\right)^2\right]\\ &+\frac{1}{(2\Phi(t_1,t_2))^{2n}}\left[\tilde{\mathscr{Z}}_n^2(t_1,t_2)-\tilde{\mathscr{Z}}_n(2t_1,2t_2)\right].\end{aligned}$$

■

Remark 2.3*:*
Taking expectations on both sides

$$\mathbb{E}\left(M_{n+1}^2\right)=\mathbb{E}\left(M_n^2\right)+\lambda(t_1,t_2)^n\left[\lambda(t_1,t_2)-\lambda(0,t_2)\right]\tag{2.25}$$

and iterating we find

$$\mathbb{E}\left(M_n^2\right)=\lambda(t_1,t_2)+\frac{1}{2}\mathrm{sech}^2(t_2)+\left[\lambda(t_1,t_2)-\lambda(0,t_2)\right]\sum_{k=1}^{n-1}\lambda(t_1,t_2)^k.\tag{2.26}$$

Hence we conclude that

$$\sup_{n\geq1}\mathbb{E}\left[M_n^2(t_1,t_2)\right]<\infty\qquad\text{whenever}\qquad\Phi(2t_1,2t_2)<2\Phi^2(t_1,t_2).\tag{2.27}$$

◊

The following lemma was proven in [1]:

Lemma 2.3. *For any finite set of real numbers $\{x_1,\dots,x_n\}$, the function*

$$g(\tau)=\frac{1}{\tau}\log\sum_{j=1}^n\mathrm{e}^{\tau x_j}\tag{2.28}$$

is decreasing and convex in τ.

Proposition 2.2. *Define*

$$F(t_1,t_2)=\log\left[2\Phi(t_1,t_2)\right].\tag{2.29}$$

If for any given t_1,t_2, there exists $\alpha>1$ such that

$$F(\alpha t_1,\alpha t_2)<\alpha F(t_1,t_2)\tag{2.30}$$

then

$$\sup_{n\geq 1}\mathbb{E}[M_n^\alpha(t_1,t_2)]<\infty. \tag{2.31}$$

Proof. Take $1<\alpha<2$. By Hölder's inequality

$$\begin{aligned}\mathbb{E}\left[M_{n+1}^\alpha|\nu^n\right] &\leq \left(\mathbb{E}\left[M_{n+1}^2|\nu^n\right]\right)^{\alpha/2}\\ &\leq M_n^\alpha + \left[\lambda(t_1,t_2)-\lambda(0,t_2)\right]^{\alpha/2}\lambda(t_1,t_2)^{n\alpha/2}M_n^{\alpha/2}(2t_1,2t_2).\end{aligned} \tag{2.32}$$

By Lemma 2.3 we have $\tilde{\mathscr{Z}}_n^{1/2}(2t_1,2t_2)\leq\tilde{\mathscr{Z}}_n^{1/\alpha}(\alpha t_2,\alpha t_2)$ for $0<\alpha<2$. Hence

$$\begin{aligned}M_n^{1/2}(2t_1,2t_2) &= \frac{\tilde{\mathscr{Z}}_n^{1/2}(2t_1,2t_2)}{(2\Phi(2t_1,2t_2))^{n/2}}\\ &\leq M_n^{1/\alpha}(\alpha t_1,\alpha t_2)\left[\frac{2\Phi(\alpha t_1,\alpha t_2)}{(2\Phi(2t_1,2t_2))^{\alpha/2}}\right]^{n/\alpha}.\end{aligned} \tag{2.33}$$

Inserting this in (2.32) we get

$$\begin{aligned}\mathbb{E}\left[M_{n+1}^\alpha|\nu^n\right] \leq M_n^\alpha(t_1,t_2) &+ \left[\lambda(t_1,t_2)-\lambda(0,t_2)\right]^{\alpha/2}\lambda(t_1,t_2)^{n\alpha/2}\\ &\times M_n(\alpha t_1,\alpha t_2)\left[\frac{2\Phi(\alpha t_1,\alpha t_2)}{(2\Phi(t_1,t_2))^\alpha}\right]^n.\end{aligned} \tag{2.34}$$

Taking expectations, iterating as in (2.26) and substituting for $\lambda(t_1,t_2)$ we find

$$\mathbb{E}\left[M_{n+1}^\alpha\right]\leq\mathbb{E}[M_1^\alpha]+\left[\lambda(t_1,t_2)-\lambda(0,t_2)\right]^{\alpha/2}\sum_{k=1}^n\left[\frac{2\Phi(\alpha t_1,\alpha t_2)}{(2\Phi(t_1,t_2))^\alpha}\right]^k. \tag{2.35}$$

This proves the proposition. ∎

Remark 2.4*:*

1. It follows from Hölder's inequality that $F(t_1,t_2)$ is a convex function. So, (2.30) implies that

   $$\left.\frac{d}{d\alpha}\frac{1}{\alpha}F(\alpha t_1,\alpha t_2)\right|_{\alpha=1}=\lim_{\alpha\to 1}\frac{(F(\alpha t_1,\alpha t_2)/\alpha)-F(t_1,t_2)}{\alpha-1}<0. \tag{2.36}$$

   On the other hand if (2.36) holds, that is if

   $$\left.\frac{d}{d\alpha}\frac{1}{\alpha}F(\alpha t_1,\alpha t_2)\right|_{\alpha=1}=-F(t_1,t_2)+\left.\frac{d}{d\alpha}F(\alpha t_1,\alpha t_2)\right|_{\alpha=1}<0 \tag{2.37}$$

   then there exists $\alpha>1$ such that (2.30) holds.
2. In the following we assume that V has a Gaussian distribution with zero mean and variance $1/\gamma$. In that case

   $$F(t_1,t_2)=\log\left[2\cosh(t_2)\right]+\frac{t_1^2}{2\gamma} \tag{2.38}$$

   and (2.37) reduces to

   $$|t_1|<\left\{2\gamma\left[\log(2\cosh(t_2))-t_2\tanh(t_2)\right]\right\}^{1/2}=:B(t_2). \tag{2.39}$$

   Also define $\beta_0:=B(0)$. We shall use these definitions throughout the paper hereafter. For future use we write $B^{-1}=T$. It will be convenient to keep the graph of $B(t_2)$ in mind as as proceed with proofs of future results (see Figure 2.1).

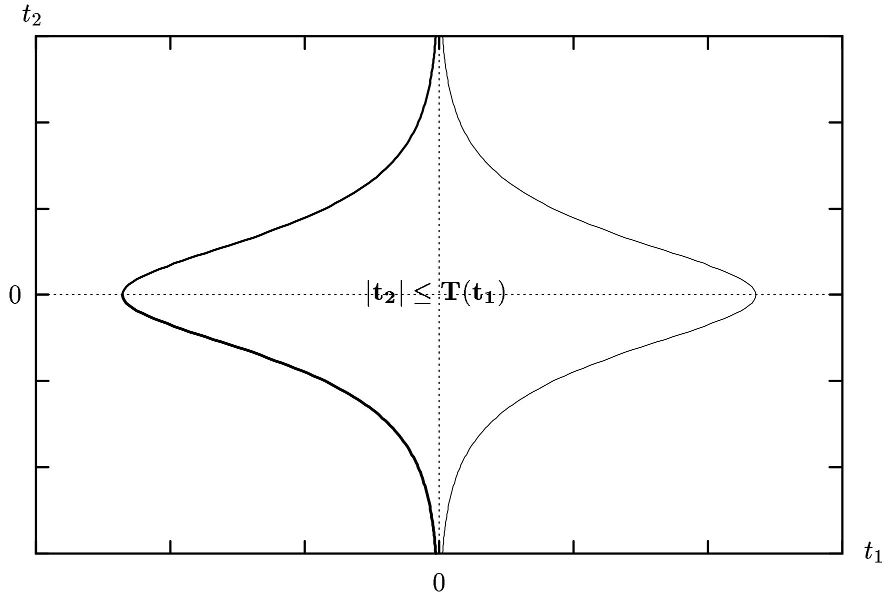

Figure 2.1: In the region $|t_1| < B(t_2)$, there is $\alpha > 1$ such that $\sup_{n \geq 1} \mathbb{E}\big[M_n^\alpha(t_1, t_2)\big] < \infty$.

◊

2.3. Existence of the Cumulant Generating Function.

Theorem 2.1. *Assume that V takes a Gaussian distribution with zero mean and variance $1/\gamma$ and define*

$$\Gamma = \big\{(t_1, t_2) \in \mathbb{R}^2 : |t_2| < T(t_1)\big\}.$$

Then the following limit holds almost surely:

$$\lim_{n\to\infty} \frac{1}{n} \log \tilde{\mathscr{Z}}_n(t_1, t_2) = \begin{cases} F(t_1, t_2) & ; (t_1, t_2) \in \Gamma \\ t_1 F(\bar{t}_1, \bar{t}_2)/\bar{t}_1 & ; (t_1, t_2) \in \Gamma^{\mathrm{C}} \end{cases} \tag{2.40}$$

where $\bar{t}_1$, $\bar{t}_2$ are the solutions of the equations

$$\frac{\bar{t}_2}{\bar{t}_1} = \frac{t_2}{t_1} \quad \textit{and} \quad \bar{t}_1 = B(\bar{t}_2). \tag{2.41}$$

Proof. (i) $(t_1, t_2) \in \Gamma$:

Now proposition (2.2) applies and we have $\mathbb{E}[M_\infty] = 1$ (see Remark 2.2). Hence using lemma (2.1), the result follows by (2.16). We restate this : Define

$$\Omega_{(t_1,t_2)} = \left\{\omega \in \Omega : \lim_{n\to\infty} \frac{1}{n} \log \tilde{\mathscr{Z}}_n(t_1, t_2) = F(t_1, t_2)\right\}. \tag{2.42}$$

Then

$$\mathbb{P}\left[\Omega_{(t_1,t_2)}\right] = 1 \qquad \text{for each } (t_1, t_2) \in \Gamma. \tag{2.43}$$

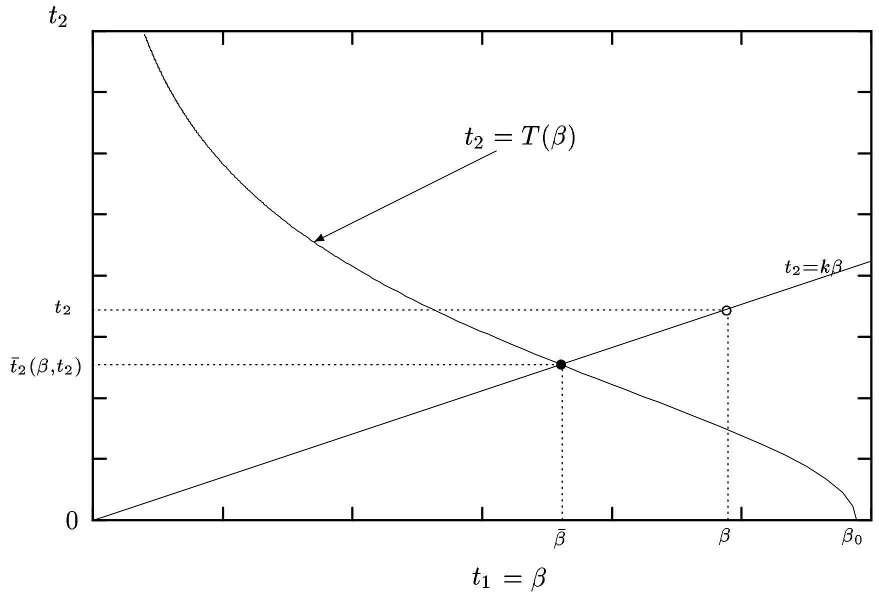

Figure 2.2: $t_2(\beta) = k\beta$ and $\bar{t}_2(\bar{\beta}) = k\bar{\beta}$. The points $(\bar{\beta}, \bar{t_2})$ lie on the boundary $t_2 = T(\beta)$.

But we need a stronger result, namely

$$\mathbb{P}\left[\bigcap_{(t_1,t_2)\in\Gamma} \Omega_{(t_1,t_2)}\right] = 1 \tag{2.44}$$

that is, the exceptional nullset is uniform in (t_1, t_2). This is proved in the Appendix.

(ii) $(t_1, t_2) \in \Gamma^{\mathrm{C}}$: Take any $(t_1, t_2) \in \Gamma^{\mathrm{C}}$ (cf. Figure 2.2). Let $(\bar{t}_1, \bar{t}_2)$ be the solution of (2.41) which is obviously unique. Put $\tau = t_1/\bar{t}_1 = t_2/\bar{t}_2 \geq 1$. By Lemma 2.3, $\log \tilde{\mathscr{Z}}_n(\tau t_1, \tau t_2)/\tau$ is decreasing and convex in τ. By the decrease

$$\limsup_{n\to\infty} \frac{1}{n\tau} \log \tilde{\mathscr{Z}}_n(\tau\bar{t}_1, \tau\bar{t}_2) \leq \limsup_{n\to\infty} \frac{1}{n(1-\epsilon)} \log \tilde{\mathscr{Z}}_n\left((1-\epsilon)\bar{t}_1, (1-\epsilon)\bar{t}_2\right). \tag{2.45}$$

Since $(1-\epsilon)(\bar{t}_1, \bar{t}_2) \in \Gamma$ we get by letting $\epsilon \to 0$

$$\limsup_{n\to\infty} \frac{1}{n\tau} \log \tilde{\mathscr{Z}}_n(\tau\bar{t}_1, \tau\bar{t}_2) \leq F\left(\bar{t}_1, \bar{t}_2\right). \tag{2.46}$$

On the other hand, by the convexity we find

$$\begin{aligned}\liminf_{n\to\infty} \frac{1}{n\tau} \log \tilde{\mathscr{Z}}_n(\tau\bar{t}_1, \tau\bar{t}_2) &\geq \liminf_{n\to\infty} \frac{1}{n(1-\epsilon)} \log \tilde{\mathscr{Z}}_n\left((1-\epsilon)\bar{t}_1, (1-\epsilon)\bar{t}_2\right) \\ &\quad + \liminf_{n\to\infty} \frac{d}{d\tau}\left[\frac{1}{n\tau} \log \tilde{\mathscr{Z}}_n(\tau\bar{t}_1, \tau\bar{t}_2)\right]_{\tau=1-\epsilon} (\tau - 1 + \epsilon).\end{aligned}$$

Since the sequence of convex functions

$$\frac{1}{n\tau} \log \tilde{\mathscr{Z}}_n(\tau t_1, \tau t_2)$$

converges to $F(\tau t_1, \tau t_2)/\tau$ a.s. for $\tau(t_1,t_2) \in \Gamma$ (by the proof in part (i) of the theorem), their derivatives converge to the limit

$$\frac{d}{d\tau}\left[\frac{1}{\tau}F(\tau t_1, \tau t_2)\right].$$

Hence

$$\liminf_{n\to\infty} \frac{1}{n}\frac{d}{d\tau}\left[\frac{1}{\tau}\log \tilde{\mathscr{Z}}_n(\tau\bar{t}_1, \tau\bar{t}_2)\right]_{\tau=1-\epsilon} = \frac{d}{d\tau}\left[\frac{1}{\tau}F(\tau\bar{t}_1, \tau\bar{t}_2)\right]_{\tau=1-\epsilon}. \tag{2.47}$$

Moreover since (2.37) is equivalent to (2.39),

$$\frac{d}{d\tau}\left[\frac{1}{\tau}F(\tau t_1, \tau t_2)\right]_{\tau=1} = 0 \qquad \text{at } (\bar{t}_1, \bar{t}_2). \tag{2.48}$$

It follows that

$$\liminf_{n\to\infty} \frac{1}{n\tau}\log \tilde{\mathscr{Z}}_n(\tau\bar{t}_1, \tau\bar{t}_2) \geq F(\bar{t}_1, \bar{t}_2) \qquad \text{a.s.} \tag{2.49}$$

as $\epsilon \to 0$. Hence we have

$$\lim_{n\to\infty} \frac{1}{n}\log \tilde{\mathscr{Z}}_n(\tau\bar{t}_1, \tau\bar{t}_2) = \tau F(\bar{t}_1, \bar{t}_2) \qquad \text{a.s.} \tag{2.50}$$

■

Corollary 2.1.1. *There exists a uniform null set $\mathscr{N}$ such that the cumulant generating function*

$$C(t_1,t_2)(\omega) = \begin{cases} F(t_1,t_2) - \log 2 & ;\ (t_1,t_2) \in \Gamma \\ t_1 F(\bar{t}_1, \bar{t}_2)/\bar{t}_1 - \log 2 & ;\ (t_1,t_2) \in \Gamma^{C} \end{cases} \tag{2.51}$$

is defined and exists for all (t_1,t_2) if $\omega \notin \mathscr{N}$.

N.B:
In the second case $\bar{t}_1$ and $\bar{t}_2$ has to be determined so that (2.41) is satisfied.

3. Variational Formula

3.1. The Rate Function.

Lemma 3.1. *Let*

$$I^{\beta}(m) := \sup_t \{tm - C(\beta, t)\}. \tag{3.1}$$

Then the free-energy expression (2.7) can be written as

$$-\beta f(\beta,\lambda) = \log 2 + \sup_m \left\{\lambda\beta m^p - I^{\beta}(m)\right\}. \tag{3.2}$$

Proof. Notice that the free-energy expression (2.7) can be written as

$$-\log 2 - \beta f(\beta,\lambda) = \sup_m \left\{\beta\lambda m^p + \sup_v\left[\beta v - I(v,m)\right]\right\} \tag{3.3}$$

where the second supremum is the Legendre transform of I with respect to the first variable. We can also write the rate function (2.6) as

$$I(v,m) = \sup_{t_1}\left\{t_1 v - \left(-I^{t_1}(m)\right)\right\}. \tag{3.4}$$

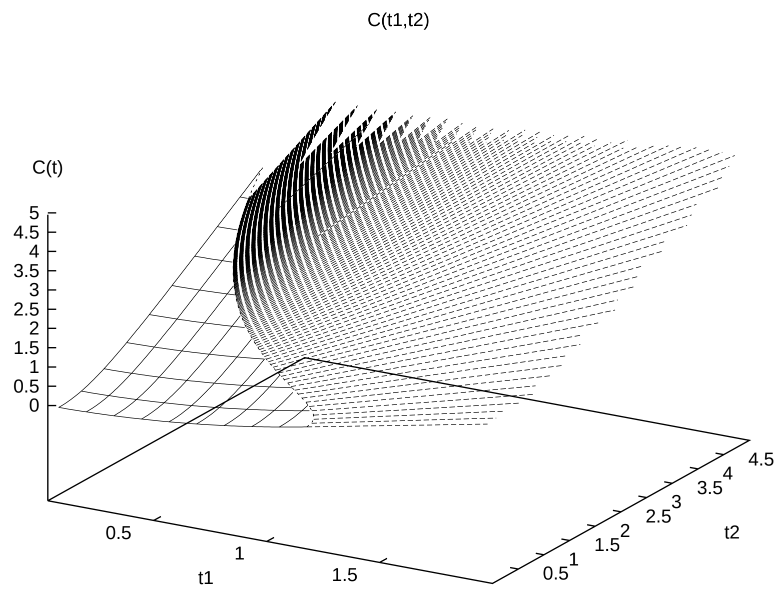

Figure 2.3: $C(t_1, t_2)$ at $\gamma = 1$ for given $k = t_2/t_1$. Notice it's linear behaviour beyond the region Γ along the radial lines.

By convexity of $-I^{t_1}(m)$ with respect to t_1 (since C is convex) we can invert (3.1) to get

$$-I^{t_1}(m) = \sup_v \{t_1 v - I(v, m)\}. \tag{3.5}$$

Inserting this in (3.3) yields the lemma. ■

Remark 3.1*:*

1. Notice that F is symmetric in both variables (provided the distribution of V is symmetric). In particular

$$C(-t_1, t_2) = C(t_1, t_2) = C(t_1, -t_2).$$

2. Since t_1 should take the same sign as v in the following, we have,

$$I(v, m) = \sup_{t_1 \in \mathbb{R}} \left\{t_1 v + I^{t_1}(m)\right\} = \sup_{t_1 > 0} \left\{t_1 |v| + I^{t_1}(m)\right\} = \sup_{t_1 \in \mathbb{R}} \left\{t_1 |v| + I^{t_1}(m)\right\} \tag{3.6}$$

and hence $I(v, m) = I(|v|, m)$. Also, it follows by a similar reasoning considering

$$-I^{t_1}(m) = \sup_{t_2} \{t_2 m - C(t_1, t_2)\}, \tag{3.7}$$

that $I(v, m) = I(v, -m)$ and hence we have

$$I(v, m) = I(|v|, |m|). \tag{3.8}$$

Therefore it suffices to consider only $t_1 = \beta > 0$ and $m > 0$, in all the derivations that follow. We find $t_1 = \beta$ and therefore it is convenient to write t instead of t_2 hereafter.

◊

Proposition 3.1. *Let* $m(\beta) = \tanh[T(\beta)]$, $\bar{\beta}(m) = \{2\gamma \log 2 - I_0(m)\}^{1/2}$ *and* $I_0(m) = [(1+m)\log(1+m) + (1-m)\log(1-m)]/2$. *Then*

$$I^\beta(m) = \begin{cases} I_1^\beta(m) := I_0(m) - \frac{\beta^2}{2\gamma} & ;\ 0 \le |m| \le m(\beta) \\ I_2^\beta(m) := -\frac{\beta\bar{\beta}}{\gamma} + \log 2 & ;\ m(\beta) < |m| < 1 \\ \infty & ;\ otherwise \end{cases} \tag{3.9}$$

Proof. Notice that since $I^\beta(m) = \max\{I_1^\beta, I_2^\beta\}$ one has to determine which one of I_1^β and I_2^β dominates and when that happens. This is done by solving the one-dimensional variational problem where one simply differentiates the two forms of $C(\beta,t)$ in the regions Γ and Γ^C (cf. (2.51)) with respect to t. ■

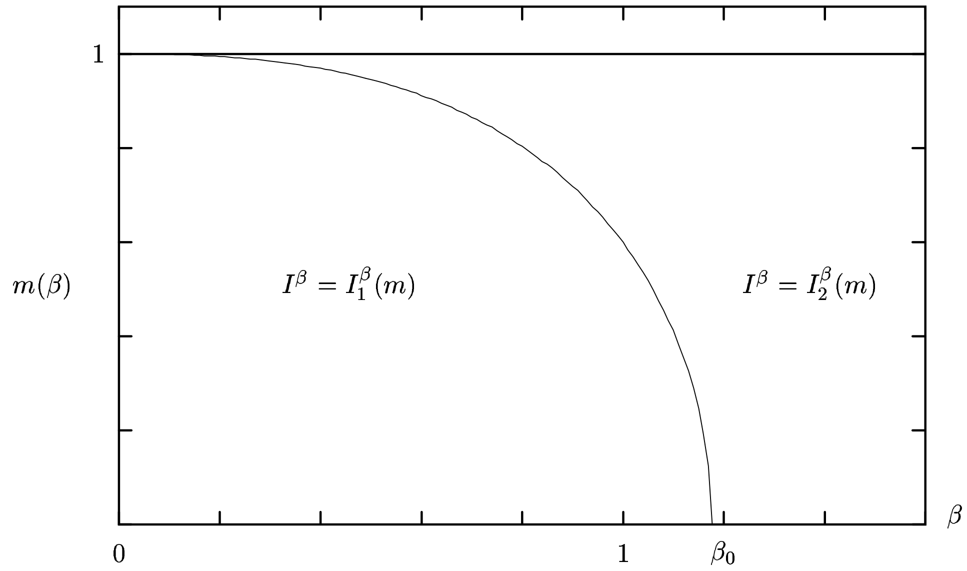

Figure 3.1: The curve and the line show the boundary of the domain of validity of $I^\beta(m)$

Remark 3.2*:*
The boundary of the domain of validity of $I_1^\beta(m)$ is shown in the Figure 3.1 which is obtained by solving the explicit equation $B(t) = \beta$ for t and then taking $m(\beta) = \tanh(t)$. ◊

4. Free Energy and the Phase Diagram

We now discuss the phase diagram as depicted in Figure 1.2(a) i.e. for the case $\mathbf{p} = \mathbf{2}$. We define the graphs $\lambda = \mathcal{C}_1(\beta)$, $\lambda = \mathcal{C}_2(\beta)$ by

$$\mathcal{C}_1(\beta) = 1/(2\beta), \qquad \mathcal{C}_2(\beta) = \frac{T(\beta)}{2\beta m(\beta)} \tag{4.1}$$

where $m(\beta) = \tanh[T(\beta)]$.

Theorem 4.1. *Let p=2. Then the free energy is given by*

(4.2)
$$f(\beta,\lambda)=\begin{cases} f_P(\beta)\equiv -\frac{\beta}{2\gamma}-\frac{1}{\beta}\log 2 & ;\lambda\le C_1(\beta),\ 0<\beta<\beta_0\\ f_F(\beta,\lambda)\equiv -\lambda\bar{m}(\beta,\lambda)^2+\frac{1}{\beta}I_0(\bar{m}(\beta,\lambda))-f_P(\beta) & ;C_1(\beta)\le\lambda\le C_2(\beta),\quad 0<\beta<\beta_0\\ f_M(\lambda)\equiv -\lambda m_\lambda^2-\frac{\bar{\beta}(m_\lambda)}{\gamma} & ;\lambda\ge C_2(\beta),\beta\le\beta_0\ \text{or}\ \lambda\ge\lambda_0,\ \beta>0\\ f_{SG}\equiv -\frac{\beta_0}{\gamma} & ;\lambda\le\lambda_0,\ \beta\ge\beta_0\end{cases}$$

and the corresponding phase diagram is given by Figure 1.2(a). Here

$$\bar{m}(\beta,\lambda)=\tanh[2\beta\lambda\bar{m}(\beta,\lambda)], \tag{4.3}$$

$$m_\lambda=\tanh[2\bar{\beta}(m_\lambda)\lambda m_\lambda], \tag{4.4}$$

$\lambda_0=C_1(\beta_0)=1/(2\beta_0)$ *and* $\bar{\beta}(m)=\sqrt{2\gamma[\log 2-I_0(m)]}$.

Proof. Define $g_i(\beta,\lambda;m)=\beta\lambda m^2-I_i^\beta(m)$, $i=1,2$, where I_i^β are the two forms of $I^\beta(m)$ as defined in (3.9). Clearly,

$$\partial g_1/\partial m=0\Leftrightarrow m=\tanh(2\beta\lambda m) \tag{4.5}$$

and, using

$$\bar{\beta}'(m)=-\frac{\gamma\tanh^{-1}(m)}{\bar{\beta}(m)}, \tag{4.6}$$

$$\partial g_2/\partial m=0\Leftrightarrow m=\tanh\big(2\bar{\beta}(m)\lambda m\big). \tag{4.7}$$

These are just (4.3) and (4.4) and it remains to determine which case applies in various regions of the β,λ-plane.

First suppose that $0<\beta<\beta_0$. If $\lambda\le C_1(\beta)$ then (4.3) has only the zero solution and the maximizer is attained at $m=0$. If $C_1(\beta)\le\lambda\le C_2(\beta)$ then the maximum is attained at the positive solution $m=\bar{m}(\beta,\lambda)$ of (4.5), that is (4.3) holds and $I^\beta(m)=I_1^\beta(m)$. Indeed, $\bar{m}(\beta,\lambda)\le m(\beta)$ since $\lambda\le C_2(\beta)$ and $\bar{m}$ increases with λ. In the case $\lambda>C_2(\beta)$ we need the following lemma, the proof of which we omit (see [16]):

Lemma 4.1. *If $\lambda>\lambda_0$ the equation* (4.4) *has exactly one positive solution on [0,1] which increases as λ increases, and if $\lambda\le\lambda_0$ the only solution is $m=0$.*

If $\lambda\ge C_2(\beta)$ then $g_1(\beta,\lambda;m)$ is increasing in m for $m\le m(\beta)$ so that its maximum is attained at $m=m(\beta)$. At that point $g_1=g_2$ so that the maximum is always attained for $m\ge m(\beta)$ and $f(\beta,\lambda)=g_2(\beta,\lambda;m)$.

By Lemma 4.1 equation (4.7) has a positive solution $m_\lambda\ge m(\beta)$ which corresponds to the maximum. The free energy follows by insertion: $f(\beta,\lambda)=-\lambda m_\lambda^2+I_2^\beta(m_\lambda)-\log 2=f_M(\lambda)$.

Next consider the case $\beta>\beta_0$. Then $I_1^\beta(m)$ does not apply so that $f(\beta,\lambda)=g_2(\beta,\lambda;m)$, where m is the maximizer. Clearly, if $\lambda\le\lambda_0$ then $m=0$ by Lemma 4.1 and $f(\beta,\lambda)=\beta^{-1}(I_2^\beta(0)-\log 2)=f_{\rm SG}$, whereas if $\lambda\ge\lambda_0$ the maximizer is given by the unique positive solution of (4.4) and $f(\beta,\lambda)$ is again given by $f_M(\lambda)$. ∎

When $\mathbf{p}>\mathbf{2}$, let $\beta<\beta_0$ and define

$$F_1(\beta,\lambda)=\sup_{0\le m\le m(\beta)}\left\{\beta\lambda m^p-I_1^\beta(m)\right\} \tag{4.8}$$

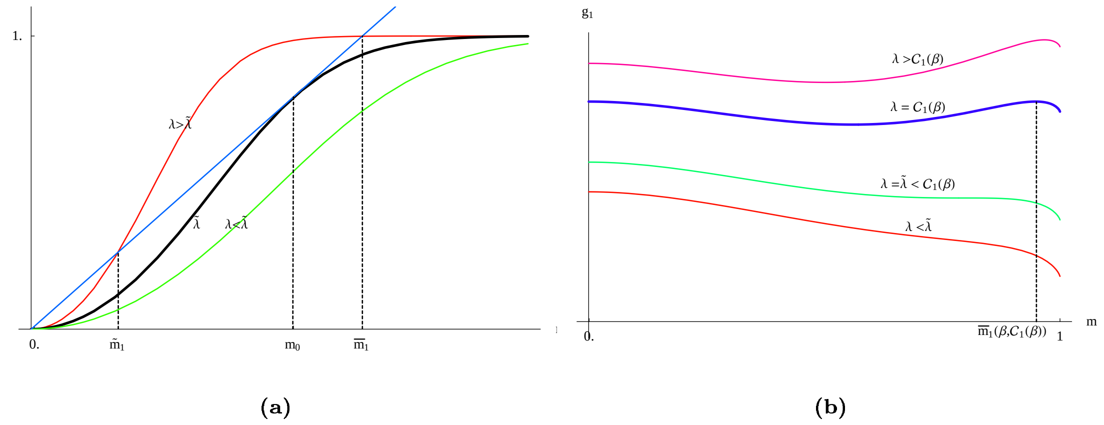

Figure 4.1: **(a)**$\tanh[p\beta\lambda m^{p-1}]$ at $\lambda \gtreqless \tilde\lambda$ **(b)** $\lambda = \mathcal{C}_1(\beta)$ defines the critical value

where $m(\beta) = \tanh[T(\beta)]$. Clearly the supremum of the above is attained when m satisfies

$$m = \tanh\left[p\beta\lambda m^{p-1}\right] \tag{4.9}$$

provided the solution $\bar m \le m(\beta)$. In fact, for fixed β, there is a critical value $\tilde\lambda(\beta)$ below which (4.9) has only the zero solution. Put $g_i(\beta,\lambda;m) = \beta\lambda m^p - I_i^\beta(m)$, $i = 1,2$. Then for $\lambda \le \tilde\lambda(\beta)$, $\partial g_1/\partial m \le 0$ and hence

$$F_1(\beta,\lambda) = -I_1^\beta(0).$$

For $\lambda > \tilde\lambda(\beta)$, (4.9) has non-zero solutions $\tilde m_1(\beta,\lambda)$ and $\bar m_1(\beta,\lambda) > \tilde m_1(\beta,\lambda)$ (see Figure 4.1(a)).

Clearly g_1 has a local minimum at $\tilde m_1$ and a local maximum at $\bar m_1$ (see Figure 4.1(a), 4.1(b)) but it is incorrect to assume that therefore $\tilde\lambda$ is the critical value of λ. Indeed, $g_1(\beta,\tilde\lambda;\bar m_1) < g_1(\beta,\tilde\lambda;0)$ and by continuity this remains the case if λ is increased slightly. The magnetization does become nonzero when λ reaches the value where

$$g_1(\beta,\lambda;\bar m_1) = g_1(\beta,\lambda;0) = -I_1^\beta(0). \tag{4.10}$$

This defines the critical line $\lambda = \mathcal{C}_1(\beta)$ (see Figure 4.1(b)). Across the line $\lambda = \mathcal{C}_1(\beta)$ the magnetization *jumps* from zero to the value m_p given by the positive solution of (4.9) such that

$$-I_1^\beta(0) = \beta\lambda m_p^p - I_1^\beta(m_p) \quad\Longleftrightarrow\quad \beta\lambda m_p^p = I_0(m_p). \tag{4.11}$$

Combined with (4.9) this yields $G_1(m_p) = 0$, where

$$G_1(m) = \left[1 + \left(1-\frac{1}{p}\right)m\right]\log(1+m) + \left[1-\left(1-\frac{1}{p}\right)m\right]\log(1-m). \tag{4.12}$$

Notice that m_p is independent of β,λ.

To state the main theorem of this section we need a number of lemmas. We state the first lemma without proof (see [16]).

Lemma 4.2. *For any $p > 2$, the equation $G_1(m) = 0$ has a unique solution $m_p \in (0,1)$. Moreover, for large p, $m_p \sim 1 - 2^{-(2p-1)}$.*

Inserting m_p into (4.9) it follows that

$$\mathcal{C}_1(\beta) = \frac{\tanh^{-1}(m_p)}{p\beta m_p^{p-1}}. \tag{4.13}$$

For $\lambda > \mathcal{C}_1(\beta)$, the maximum is attained at $m = \bar{m}_1 \geq m_p$ provided $\bar{m}_1 \leq m(\beta)$. Thus we obtain another critical value of λ above which $m = m(\beta)$. This line is given by $\lambda = \mathcal{C}_2(\beta)$ where

$$\mathcal{C}_2(\beta) = \frac{\tanh^{-1}(m(\beta))}{p\beta(m(\beta))^{p-1}}. \tag{4.14}$$

Notice that $m(\beta) \to 1$ as $\beta \to 0$ so $\mathcal{C}_2(\beta) > \mathcal{C}_1(\beta)$ for small β, but as β increases $m(\beta)$ decreases until it reaches m_p.

Define β_1 by

$$m(\beta_1) = m_p. \tag{4.15}$$

Then $\mathcal{C}_1(\beta_1) = \mathcal{C}_2(\beta_1)$. There is a minimum value $\lambda_{\min}$ below which

$$m = m(\beta) \qquad \text{and} \qquad m = \bar{m}_1(\beta, \lambda) \tag{4.16}$$

has no solution. Indeed, choose $\tilde{\lambda}(\beta)$ as above and let $m_0 > 0$ be the value where $m_0 = \tanh[\beta\tilde{\lambda}pm_0^{p-1}]$ (see Figure 4.1(a)). Now let $\beta_{\max}$ be such that $m(\beta_{\max}) = m_0$. For $\lambda < \tilde{\lambda}(\beta_{\max}) = \lambda_{\min}$ we must increase β in order that (4.9) has a solution but then $m(\beta)$ decreases below m_0 and is invalid. For $\beta < \beta_{\max}$, $m(\beta) > m_0$ so there is a unique $\lambda = \mathcal{C}_2(\beta)$ for which $\bar{m}_1(\beta, \lambda) = m(\beta)$. Only this part of the curve $\lambda = \mathcal{C}_2(\beta)$ is relevant.

Remark 4.1*:*
Notice that $\tilde{\lambda}(\beta) = c_0/\beta$ since $\bar{m}_1$ depends only on $\beta\lambda$. For $\beta\lambda < c_0$, (4.9) has no solution, whereas if $c > c_0$, then the hyperbola $\beta\lambda = c$ intersects the curve $\lambda = \mathcal{C}_2(\beta)$ in two points, one with $\beta < \beta_{\max}$ corresponding to $m(\beta) = \bar{m}_1(\beta, \lambda)$ and one with $\beta > \beta_{\max}$ and $m(\beta) = \tilde{m}_1(\beta, \lambda)$. ◊

It now follows that for $\beta > \beta_1$, $\bar{m}_1(\beta, \lambda) > m(\beta)$ so we must compare $-I_1^\beta(0)$ and $\beta\lambda m(\beta)^p - I_1^\beta(m(\beta))$. This defines the critical line $\mathcal{C}_3(\beta)$:

$$\lambda = \frac{I_1^\beta(m(\beta)) - I_1^\beta(0)}{\beta m(\beta)^p} = \frac{I_0(m(\beta))}{\beta m(\beta)^p} \equiv \mathcal{C}_3(\beta). \tag{4.17}$$

It is easy to derive that

$$\mathcal{C}_3(\beta) = \frac{\beta_0^2 - \beta^2}{2\gamma\beta m(\beta)^p} \tag{4.18}$$

from which it follows that $\mathcal{C}_3(\beta) \to +\infty$ as $\beta \to \beta_0$ because

$$m(\beta) \sim \left\{ \frac{4\log 2}{\beta_0}(\beta_0 - \beta) \right\}^{1/2}. \tag{4.19}$$

The line $\lambda = \mathcal{C}_3(\beta)$ is in fact not relevant for $f(\beta, \lambda)$ as F_1 does not apply in this region. We now have

Lemma 4.3. *Let β_1 be the point of intersection of $\mathcal{C}_1(\beta)$ and $\mathcal{C}_2(\beta)$. If $0 < \beta \leq \beta_1$ then*

$$F_1(\beta, \lambda) = \begin{cases} -I_1^\beta(0) & \text{if } \lambda \leq \mathcal{C}_1(\beta) \\ \beta\lambda\bar{m}_1^p - I_1^\beta(\bar{m}_1) & \text{if } \mathcal{C}_1(\beta) \leq \lambda \leq \mathcal{C}_2(\beta) \\ \beta\lambda m(\beta)^p - I_1^\beta(m(\beta)) & \text{if } \lambda > \mathcal{C}_2(\beta). \end{cases} \tag{4.20}$$

If $\beta_1 < \beta < \beta_0$ *then*

$$F_1(\beta,\lambda) = \begin{cases} -I_1^\beta(0) & \text{if } \lambda \le \mathcal{C}_3(\beta) \\ \beta\lambda m(\beta)^p - I_1^\beta(m(\beta)) & \text{if } \lambda > \mathcal{C}_3(\beta). \end{cases} \tag{4.21}$$

Proof. See [16]. ■

Next consider

$$F_2(\beta,\lambda) = \sup_{m \ge m(\beta)} \left\{ \beta\lambda m^p - I_2^\beta(m) \right\}. \tag{4.22}$$

Analogous to (4.9), the maximization condition is given by

$$m = \tanh\big(p\bar{\beta}(m)\lambda m^{p-1}\big). \tag{4.23}$$

It is easy to see [16] that (4.23) can have at most two non-zero solutions. In fact there is a critical value $\lambda_2 > 0$ such that for $\lambda < \lambda_2$ (4.23) has no non-zero solutions, whereas for $\lambda > \lambda_2$ there are two solutions $\tilde{m}_2(\lambda)$ and $\bar{m}_2(\lambda) > \tilde{m}_2(\lambda)$. $g_2(\beta,\lambda;m)$ has a local maximum at the greater of these two: $m = \bar{m}_2(\lambda)$. Again, this can only be a global maximum if λ is large enough, i.e.

$$\beta\lambda\bar{m}_2(\lambda)^p - I_2^\beta\left(\bar{m}_2(\lambda)\right) \ge \beta\lambda m(\beta)^p - I_2^\beta(m(\beta)). \tag{4.24}$$

Put

$$\beta_2 = \bar{\beta}(\bar{m}_2(\lambda)) \quad \text{i.e.} \quad \bar{m}_2 = m(\beta_2). \tag{4.25}$$

We now claim:

Lemma 4.4. *For* $\beta \le \beta_2$, (4.24) *is equivalent to* $\lambda \ge \mathcal{C}_2(\beta)$.

Proof. See [16]. ■

For $\beta_2 < \beta < \beta_0$, (4.24) defines a curve $\lambda = \mathcal{C}_4(\beta)$ which does not coincide with $\lambda = \mathcal{C}_2(\beta)$. Before discussing this case, we consider first the case $\beta > \beta_0$:

If $\lambda < \lambda_2$, $g_2(\beta,\lambda;m)$ is decreasing so

$$F_2(\beta,\lambda) = -I_2^\beta(0). \tag{4.26}$$

For $\lambda > \lambda_2$ we must compare $-I_2^\beta(0)$ and $\beta\lambda\bar{m}_2(\lambda)^p - I_2^\beta(\bar{m}_2(\lambda))$. The critical value λ_c of λ is given by

$$-I_2^\beta(0) = \beta\lambda\bar{m}_2(\lambda_c)^p - I_2^\beta\left(\bar{m}_2(\lambda_c)\right). \tag{4.27}$$

Notice that this is independent of $\beta > \beta_0$:

$$\lambda_c = \frac{\beta_0 - \bar{\beta}(m_c)}{\gamma m_c^p}, \tag{4.28}$$

where $m_c = \bar{m}_2(\lambda_c)$. Inserting (4.28) into (4.23) with $\lambda = \lambda_c$ we have $G_2(m_c) = 0$ where

$$G_2(m) = \frac{p}{\gamma}\bar{\beta}(m)\left(\beta_0 - \bar{\beta}(m)\right) - m\tanh^{-1}(m). \tag{4.29}$$

Analogous to Lemma 4.2 we have

Lemma 4.5. *Let $p > 2$. The equation $G_2(m) = 0$ has a unique solution $m = m_c \in (0,1)$. For large p,*

$$1 - m_c \sim \frac{1}{8p^2 \log 2} \log(8p^2 \log 2). \tag{4.30}$$

Proof. See [16]. ∎

Lemma 4.6. *Let β_2 be the minimum of the curve $\mathcal{C}_2(\beta)$ given by (4.14). Then, for $0 < \beta \le \beta_2$,*

$$F_2(\beta,\lambda) = \begin{cases} \beta\lambda m(\beta)^p - I_2^\beta(m(\beta)) & \text{if } \lambda \le \mathcal{C}_2(\beta) \\ \beta\lambda \bar{m}_2(\lambda)^p - I_2^\beta(\bar{m}_2(\lambda)) & \text{if } \lambda > \mathcal{C}_2(\beta) \end{cases}$$

and for $\beta \ge \beta_0$,

$$F_2(\beta,\lambda) = \begin{cases} -I_2^\beta(0) & \text{if } \lambda \le \lambda_c \\ \beta\lambda \bar{m}_2(\lambda)^p - I_2^\beta(\bar{m}_2(\lambda)) & \text{if } \lambda \ge \lambda_c. \end{cases}$$

Finally, for $\beta_2 \le \beta \le \beta_0$ there exists a curve $\lambda = \mathcal{C}_4(\beta)$ given by (4.24) which lies below $\lambda = \mathcal{C}_2(\beta)$ and satisfies $\mathcal{C}_4(\beta_2) = \lambda_2$ and $\mathcal{C}_4(\beta_0) = \lambda_c$, such that

$$F_2(\beta,\lambda) = \begin{cases} \beta\lambda m(\beta)^p - I_2^\beta(m(\beta)) & \text{if } \lambda \le \mathcal{C}_4(\beta) \\ \beta\lambda \bar{m}_2(\lambda)^p - I_2^\beta(\bar{m}_2(\lambda)) & \text{if } \lambda > \mathcal{C}_4(\beta). \end{cases}$$

Proof. See [16]. ∎

Finally, we consider $f(\beta,\lambda)$. Clearly,

$$-\beta f(\beta,\lambda) - \log 2 = \max\{F_1(\beta,\lambda), F_2(\beta,\lambda)\}. \tag{4.31}$$

The complete phase diagram is depicted in Figure 1.2(b) and is described in the following:

Theorem 4.2. *Let the curves $\mathcal{C}_1(\beta)$ and $\mathcal{C}_2(\beta)$ be defined by (4.13) and (4.14) respectively. Let m_p be the unique positive solution of $G_1(m) = 0$ and put $\beta_1 = \bar{\beta}(m_p)$. If $0 < \beta \le \beta_1$ then*

$$-\beta f(\beta,\lambda) - \log 2 = \begin{cases} -I_1^\beta(0) = \frac{\beta^2}{2\gamma} & \text{if } \lambda \le \mathcal{C}_1(\beta) \\ \beta\lambda \bar{m}_1(\beta,\lambda) - I_1^\beta(\bar{m}_1(\beta,\lambda)) & \text{if } \mathcal{C}_1(\beta) \le \lambda \le \mathcal{C}_2(\beta) \\ \beta\lambda \bar{m}_2(\lambda) - I_2^\beta(\bar{m}_2(\lambda)) & \text{if } \lambda > \mathcal{C}_2(\beta). \end{cases} \tag{4.32}$$

The magnetization jumps from 0 to m_p across $\mathcal{C}_1(\beta)$ but is continuous across $\mathcal{C}_2(\beta)$.

If $\beta \ge \beta_0$ then

$$-\beta f(\beta,\lambda) - \log 2 = \begin{cases} -I_2^\beta(0) = \frac{\beta\beta_0}{\gamma} - \log 2 & \text{if } \lambda \le \lambda_c \\ \beta\lambda \bar{m}(\lambda)^p - I_2^\beta(\bar{m}(\lambda)) & \text{if } \lambda \ge \lambda_c. \end{cases} \tag{4.33}$$

The magnetization jumps from 0 to m_c at $\lambda = \lambda_c$.

For $\beta_1 \le \beta \le \beta_0$ there is a curve $\mathcal{C}(\beta)$ given by

$$\lambda = \mathcal{C}(\beta) \Leftrightarrow -I_1^\beta(0) = \beta\lambda \bar{m}_2(\lambda)^p - I_2^\beta(\bar{m}_2(\lambda)) \tag{4.34}$$

such that $\mathcal{C}(\beta_1) = \mathcal{C}_2(\beta_1)$, $\mathcal{C}(\beta_0) = \lambda_c$ and

$$-\beta f(\beta,\lambda) - \log 2 = \begin{cases} -I_1^\beta(0) = \frac{\beta^2}{2\gamma} & \text{if } \lambda \le \mathcal{C}(\beta) \\ \beta\lambda \bar{m}(\lambda)^p - I_2^\beta(\bar{m}(\lambda)) & \text{if } \lambda \ge \mathcal{C}(\beta). \end{cases} \tag{4.35}$$

Proof. See [16]. ■

We finally state a result about the character of the transition line $\mathcal{C}$:

Proposition 4.1. *The curve* $\lambda = \mathcal{C}(\beta)$ *is decreasing and satisfies* $\mathcal{C}(\beta_1) = \lambda_1$, $\mathcal{C}(\beta_0) = \lambda_c$. *Moreover* $\mathcal{C}'(\beta_1) = \mathcal{C}_1'(\beta_0)$ *and* $\mathcal{C}'(\beta_0) = 0$.

These completely determine the phase diagram for the case $p > 2$.

Appendix A. Proof of Equation (2.44) in Theorem 2.1

Consider countable dense subsets $D_1 \subset (0, \beta_0)$ and $D_2 \subset (0, \infty)$ and denote $I = \Gamma \cap (D_1 \times D_2)$. Now (2.43) means that for each $(t_1, t_2) \in I$, $\exists$ a null set $\mathcal{N}_{(t_1,t_2)}$ such that, if $\omega \notin \mathcal{N}_{(t_1,t_2)}$ then

$$\lim_{n\to\infty} \frac{1}{n} \log \tilde{\mathscr{Z}}_n(t_1, t_2) = F(t_1, t_2). \tag{A.1}$$

Put $\mathcal{N} = \bigcup_{(t_1,t_2)\in I} \mathcal{N}_{(t_1,t_2)}$. Then $\mathcal{N}$ is also a null set since it is a countable union of null sets. We prove (2.44) in two steps: First we fix a $t_1 \in D_1$ and take $t_2 \notin D_2$ and show that if $\omega \notin \mathcal{N}$ then (A.1) holds for the given (t_1, t_2) following a similar argument as in [9]. Our strategy then follows the same lines as in [1]: considering an arbitrary $t_1 \in (0, \beta_0)$ we prove that the same limit (A.1) holds by approximating (t_1, t_2) by a sequence of points in $D_1 \times \mathbb{R}$ along the radial lines $t_2 = at_1$ with fixed $a \geq 0$. We have

$$\begin{aligned} \left|\frac{\partial}{\partial t_2} \log \tilde{\mathscr{Z}}_n(t_1, t_2)\right| &= \left|\frac{\partial}{\partial t_2} \log \sum_{\{\sigma\}} \mathrm{e}^{t_1 \sum_{j=1}^n V_{j,(\sigma)^j} + t_2 \sum_{j=1}^n \sigma_j}\right| \\ &= \left|\frac{1}{\tilde{\mathscr{Z}}_n(t_1, t_2)} \sum_{\{\sigma\}} \left[\mathrm{e}^{t_1 \sum_{j=1}^n V_{j,(\sigma)^j} + t_2 \sum_{j=1}^n \sigma_j} \sum_{j=1}^n \sigma_i\right]\right| \\ &\leq \sup_{\{\sigma\}} \left|\sum_{j=1}^n \sigma_j\right| \leq n. \end{aligned} \tag{A.2}$$

Therefore

$$\frac{1}{n} \frac{\partial}{\partial t_2} \log \tilde{\mathscr{Z}}_n(t_1, t_2) \leq 1, \qquad \text{independent of } t_1 \text{ and } t_2. \tag{A.3}$$

Write $F_n(t_1, t_2) = \log \tilde{\mathscr{Z}}_n(t_1, t_2)/n$ and take $t_1 \in D_1$, $t_2 \notin D_2$. Fix a given $\epsilon > 0$. Then $\exists t \in D_2$ such that $|t - t_2| < \epsilon/3$ since D_2 is dense. The continuity of $F(t_1, t_2)$ and $|\partial F(t_1, t_2)/\partial t_2| < 1$ yields that $|F(t_1, t) - F(t_1, t_2)| < \epsilon/3$. Moreover by (A.3) we have for n sufficiently large that $|F_n(t_1, t_2) - F_n(t_1, t)| < \epsilon/3$ and hence it follows that

$$\begin{aligned} |F_n(t_1, t_2) - F(t_1, t_2)| &\leq |F_n(t_1, t_2) - F_n(t_1, t)| + |F_n(t_1, t) - F(t_1, t)| \\ &\qquad + |F(t_1, t) - F(t_1, t_2)| \\ &\leq \epsilon. \end{aligned} \tag{A.4}$$

This shows that for $\omega \notin \mathcal{N}$

$$\lim_{n\to\infty} \frac{1}{n} \log \tilde{\mathscr{Z}}_n(t_1, t_2)(\omega) = F(t_1, t_2). \tag{A.5}$$

This completes the first step . To prove the second step pick a general point $(t_1, t_2) \in \Gamma$ and construct the sequences (τ_k^-) , (τ_k^+) for $\tau_k^-, \tau_k^+ \in \mathbb{R}$ so that

$$\tau_k^- \nearrow 1 \qquad \text{and} \qquad \tau_k^+ \searrow 1 \tag{A.6}$$

and $t_{1k}^- := \tau_k^- t_1$, $t_{1k}^+ := \tau_k^+ t_1 \in D_1$. This is possible since D_1 is dense in $(0, \beta_0)$ and $t_1 \in (0, \beta_0) \setminus D_1$. Next we define $t_{2k}^- = \tau_k^- t_2$, $t_{2k}^+ = \tau_k^+ t_2$. Then

$$t_{1k}^- \nearrow t_1, \quad t_{1k}^+ \searrow t_1, \quad t_{2k}^- \nearrow t_2, \quad t_{2k}^+ \searrow t_2.$$

Now we use the fact that

$$\frac{1}{\tau} \log \tilde{\mathscr{Z}}_n \left(\tau t_1, \tau t_2\right)$$

is convex and decreasing in τ: By the decrease we have (since $\tau_k^- < 1$)

$$\tag{A.7} \begin{aligned} \limsup_{n\to\infty} \frac{1}{n} \log \tilde{\mathscr{Z}}_n \left(t_1, t_2\right) &\le \limsup_{n\to\infty} \frac{1}{n\tau_k^-} \log \tilde{\mathscr{Z}}_n \left(\tau_k^- t_1, \tau_k^- t_2\right) \\ &= \frac{1}{\tau_k^-} \lim_{n\to\infty} \frac{1}{n} \log \tilde{\mathscr{Z}}_n \left(t_{1k}^-, t_{2k}^-\right) \\ &= \frac{1}{\tau_k^-} F\left(t_{1k}^-, t_{2k}^-\right) \qquad \text{if } \omega \notin \mathscr{N}. \end{aligned}$$

Taking the limit $k \to \infty$ we have by the continuity of $F(t_1, t_2)$

$$\tag{A.8} \limsup_{n\to\infty} \frac{1}{n} \log \tilde{\mathscr{Z}}_n \left(t_1, t_2\right) \le F(t_1, t_2).$$

Similarly since $\tau_k^+ > 1$

$$\tag{A.9} \begin{aligned} \liminf_{n\to\infty} \frac{1}{n} \log \tilde{\mathscr{Z}}_n \left(t_1, t_2\right) &\ge \liminf_{n\to\infty} \frac{1}{n} \frac{1}{\tau_k^+} \tilde{\mathscr{Z}}_n \left(t_1, t_2\right) \\ &= \frac{1}{\tau_k^+} F\left(t_{ik}^+, t_{2k}^+\right) \\ &= F\left(t_1, t_2\right), \qquad \text{as } k \to \infty. \end{aligned}$$

From (A.8) and (A.9)

$$\tag{A.10} \lim_{n\to\infty} \frac{1}{n} \log \tilde{\mathscr{Z}}_n \left(t_1, t_2\right) = F(t_1, t_2) \qquad \text{for all } (t_1, t_2) \in \Gamma \text{ if } \omega \notin \mathscr{N}$$

and this proves the required result.

References

1. E. Buffet, A. Patrick, and J. Pulé, J. Phys. A **26** (1993), 1823.
2. J. Cook and B. Derrida, J. Stat. Phys **63** (1991), no. 3/4, 505.
3. B. Derrida, Phys. Rev. Lett. **45** (1980), 79.
4. ———, Phys. Review **B24** (1981), no. 5, 2613.
5. ———, Physica **163A** (1990), 71.
6. B. Derrida and E. Gardner, J. Phys. C **19** (1986), 5783.
7. ———, J. Phys. C **19** (1986), 2253.
8. B. Derrida and H. Spohn, J. Stat. Phys **51** (1988), no. 5/6, 817.
9. T.C. Dorlas and J.R. Wedagedera, *Rigorous Solution of a Mean Field Spin Glass Model*, Accepted for publication in the Journal of App.Math. & Stochastic Analysis (1998).
10. ———, *Phase Diagram of the Random Energy Model with Higher-Order Ferro Magnetic Term and Error Correcting Codes Due to Sourlas*, Phys. Rev. Lett. **83** (1999), no. 21, 4441–4444.
11. R.S. Ellis, *Entropy, Large Deviations and Statistical Mechanics*, Springer-Verlag, 1985.
12. D. Ruelle, Thrmodynamic Formalism - The Mathematical Structures of Classical Equilibrium Statistical Mechanics, vol. 5, Addison-Wesley, 1978.
13. N. Sourlas, Nature **339** (1989), 693.

14. ______, in Statistical Mechanics of Neural Networks (L. Garrido, ed.), Lecture Notes in Physics, vol. 368, Springer-Verlag, 1991, p. 317.

15. S.R.S Varadhan, *Large Deviations and Applications*, SIAM, 1984.

16. J. R. Wedagedera, *Mathematical aspects of some mean field spin glass models*, Ph.D. thesis, Department of Mathematics, University of Wales Swansea., 1998.

17. D. Williams, *Probability with Martingales*, Cambridge University Press, 1991.

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