DIAS-STP-00-09

Counting Rank-2 Matroids

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We enumerate the number of rank-2 matroids, non-isomorphic rank-2 matroids, connected rank-2 matroids and non-isomorphic connected rank-2 matroids on a ground set of size n. A surprising connection between these matroids, the Bell numbers and partitions of an integer is uncovered.

Key Words: Matroid, Enumeration, Rank-2, Integer partitions, 2-Connectedness, Bell numbers.

1. INTRODUCTION

A *matroid* is a pair $M(V_n, \mathcal{B})$ such that V_n is a finite set of size n (called the *ground* set) and $\mathcal{B}$ is a non-empty collection of subsets of V_n (called the *basis*) such that,

Basis Exchange Axiom: For $X, Y \in \mathcal{B}$ we have that for all $x \in X \backslash Y$ there exists $y_x \in Y \backslash X$ such that $X - x \cup y_x \in \mathcal{B}$.

The rank of a matroid denotes the cardinality of the sets in its basis and it is a consequence of the basis exchange axiom that all these sets have equal cardinality. In this paper we deal with rank-2 matroids, namely those matroids whose basis contains a collection of distinct pairs of ground set elements.

The enumeration of non-isomorphic matroids on a finite set of size n, denoted below by $f(n)$, is still an open problem. First investigated by Crapo[2], it led to a steady tightening of upper and lower bounds by Bollobás[1], Knuth[3], Piff[5] and Welsh[7]. Figure 1 shows the table in Oxley's book[6] giving these numbers for ground sets containing up to 8 elements.

The numbers $f_r(n)$ are the number of non-isomorphic matriods of rank-r on a ground set of size n. We count those matroids of rank-2, namely

r n	0	1	2	3	4	5	6	7	8
0	1	1	1	1	1	1	1	1	1
1		1	2	3	4	5	6	7	8
2			1	3	7	13	23	37	58
3				1	4	13	38	108	325
4					1	5	23	108	940
5						1	6	37	325
6							1	7	58
7								1	8
8									1
$f(n)$	1	2	4	8	17	38	98	306	1724

FIG. 1. Table showing the value of $f_r(n)$ for $0 \leqslant n \leqslant 8$.

those entries in the third row of figure 1 and connected rank-2 matroids. Alongside these structures we enumerate both in the non-isomorphic sense.

2. RANK-2 MATROIDS

Let $\mathbf{M_n}$ be the set of all rank-2 matroids on the ground set V_n. Similarly, let $\mathcal{M}_n$ be the set of all rank-2 matroids strictly on the ground set V_n (by strict we mean the union of the sets in the basis is V_n).

THEOREM 2.1. *The number of non-isomorphic rank-2 matroids on a ground set of size n is given by,*

$$f_2(n) = p(1) + p(2) + p(3) + \cdots + p(n) - n,$$

for all $n \geqslant 2$ and where $p(n)$ is the number of partitions of the integer n.

Our proof of this result relies mainly on graph theory. We give a brief introduction to those concepts which will be used.

Let $\mathbf{G_n}$ be the set of all simple graphs (no loops or multiple edges) with V_n as the set of vertices and $\mathcal{G}_n \subset \mathbf{G_n}$ be the set of all graphs with no isolated vertices. For $0 \leqslant j \leqslant 3$ let $\mathcal{G}_n^{(j)}$ be those graphs in $\mathcal{G}_n$ which contain no subgraph on three vertices isomorphic to the graph H_3^j.

For any $G \in \mathbf{G_n}$, write $V(G)$ for the vertex set of G and $E(G)$ for the corresponding set of edges. We define the *inverse* graph of G to be $G^{-1}(V(G), \overline{E(G)})$ where $\overline{E(G)} := \{\ (ab) \mid (ab) \notin E(G)\ \}$. Given

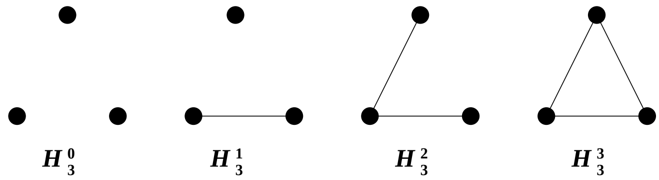

FIG. 2. The graphs H_3^j for $0 \leqslant j \leqslant 3$.

$G(V_n, E) \in \mathcal{G}_n^{(1)}$, define the map $\pi : \mathcal{G}_n^{(1)} \to \mathcal{M}_n$ by $\pi(G) = M(V_n, E^*(G))$ where $E^*(G) = \{ \{a, b\} \mid (ab) \in E \}$.

LEMMA 2.1. *The function π is a bijection between the sets $\mathcal{G}_n^{(1)}$ and $\mathcal{M}_n$.*

Proof. We see that if we choose any graph $G(V_n, E) \in \mathcal{G}_n^{(1)}$, then E will not be the empty set since the graph may have no isolated vertices. If G has one edge then it must be on two vertices only since none may be isolated. This graph is then the complete graph on two vertices and maps to the matroid $M(V_2, \mathcal{B})$ where $\mathcal{B} = \{a, b\}$.

Without loss of generality let us examine any two distinct edges of G, say e_1 and e_2. If these edges have a vertex a in common, $e_1 = (ab)$ and $e_2 = (ac)$, then we see the basis exchange axiom holds for the sets $\{a, b\}$ and $\{a, c\}$. Otherwise the edges have no vertex in common, $e_1 = (ab)$ and $e_2 = (cd)$. Since this graph is in $\mathcal{G}_n^{(1)}$ it must be the case that for the graphs restricted to the points $\{a, b, c\}, \{a, b, d\}, \{a, c, d\}$ and $\{b, c, d\}$, none can have the configuration H_3^1.

All possible configurations of the subgraph of G restriced to the four vertices $\{a, b, c, d\}$ are shown in Figure 3 whose images under the map π are;

(1) $\{\{a, b\}, \{a, c\}, \{b, d\}, \{c, d\}\}$,
(2) $\{\{a, b\}, \{a, d\}, \{b, c\}, \{c, d\}\}$,
(3) $\{\{a, b\}, \{a, c\}, \{a, d\}, \{b, d\}, \{c, d\}\}$,
(4) $\{\{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}, \{c, d\}\}$,
(5) $\{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}\}$,

respectively. We see that each of these collections fulfill the requirements as specified by the basis exchange axiom to be sets in the basis of a matroid.

To show that this map is bijective we must firstly let us show that π is injective. Suppose that we have $G_1(V_n, E_1), G_2(V_n, E_2) \in \mathcal{G}_n^{(1)}$ such that $E_1 \neq E_2$. Then $\pi(G_1) = M_1(V_n, E^*(G_1))$ and $\pi(G_2) = M_2(V_n, E^*(G_2))$. It follows that $M_1 = M_2$ if and only if $E^*(G_1) = E^*(G_2)$ which happens if and only if $E_1 = E_2$. Clearly this cannot be the case for since $E_1 \neq E_2$, there must exist some edge $(ab) \in E_1$(resp. E_2) such that $(ab) \notin E_2$(resp. E_1) and hence $\{a,b\} \in E^*(G_1)$(resp. $E^*(G_2)$) but $\{a,b\} \notin E^*(G_2)$(resp. $E^*(G_1)$). Thus $M_1(V_n, E^*(G_1)) \neq M_2(V_n, E^*(G_2))$ and hence the map π is injective.

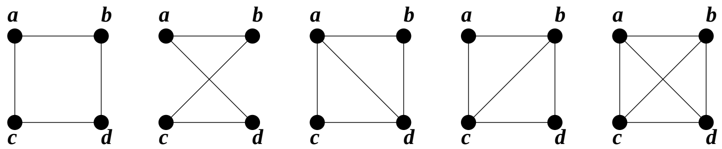

FIG. 3. The possible configurations for the subgraph of any $G \in \mathcal{G}_n^{(1)}$ with edges (ab) and (cd).

The map π is clearly surjective as well. Choose any $M(V_n, \mathcal{B}) \in \mathcal{M}_n$. Let us form the set $E := \{\ (ab) \mid \{a,b\} \in B\ \}$. Since $\mathcal{B}$ satisfies the basis exchange axiom and contains all elements of the ground set it resides upon, we have that for any three elements of the ground set, a, b and c say, there cannot exist the configuration $\{a,b\}$,$\{c,d\}$ for some element d without some interplay between the two sets. If so, appealing to the basis exchange axiom with $X = \{c,d\}$, $Y = \{a,b\}$, $x = d$ yields that $\mathcal{B}$ at least contains either $\{a,c\}$ or $\{b,c\}$. This shows that G must be in $\mathcal{G}_n^{(1)}$. The image of the graph $G(V_n, E)$ under π is then $M(V_n, E(G)) = M(V_n, \mathcal{B})$. ∎

LEMMA 2.2. *$G \in \mathcal{G}_n^{(1)}$ if and only if $G^{-1} \in \mathcal{G}_n^{(2)}$.*

Proof. Given any $G \in \mathcal{G}_n^{(1)}$ each subgraph of G on 3 vertices must be isomorphic to one of H_3^0, H_3^2 or H_3^3. The inverse G^{-1} will then contain the subgraph on those same 3 vertices isomorphic to H_3^3, H_3^1 or H_3^0 respectively. Thus G^{-1} will contain no H_3^2 subgraph and will be in $\mathcal{G}_n^{(2)}$. The converse follows in a similar fashion. ∎

LEMMA 2.3. *If $G \in \mathcal{G}_n^{(2)}$ and G has connected components $\Gamma_1, \Gamma_2, \cdots \Gamma_k$, where $k \geqslant 1$, then each of the graphs $\Gamma_1, \Gamma_2, \cdots \Gamma_k$ are complete graphs on their own vertices.*

Proof. Let us suppose that there is some connected component Γ_j (on more than one vertex) such that there are two vertices $a, b \in V(\Gamma_j)$ and $(a,b) \notin E(\Gamma_j)$. As Γ_j is connected we have that there exists a path $(a,a_1), (a_1,a_2), \cdots, (a_m, b)$ where $m \geqslant 1$. Then looking at the first two edges in this path, $(a,a_1), (a_1,a_2)$ we see that the subgraph of G restricted to the vertices a, a_1, a_2 is isomorphic to H_3^2. This is false since $G \in \mathcal{G}_n^{(2)}$ and hence $(a,a_2) \in E(\Gamma_j)$.

Let us assume that $(a,a_k) \in E(\Gamma_j)$ for some $1 < k \leqslant m$. Then as $(a,a_k) \in E(\Gamma_j)$ we apply the argument in the previous paragraph to yield

$(a, a_{k+1}) \in E(\Gamma_j)$. It follows by induction that $(a,b) \in E(\Gamma_j)$ and hence Γ_j is a complete subgraph. ▮

Proof (of Theorem 2.1). From lemma 2.1 we see that rather than dealing with rank-2 matroids we may enumerate the class of graphs in $\mathcal{G}_n^{(1)}$. It is not the cardinality of $\mathcal{G}_n^{(1)}$ we seek, but the number of non-isomorphic graphs in it. The map π preserves isomorphims because two graphs are isomorphic is there is a permutation of one vertex set to the other which is exactly how isomorphisms are characterised in matroids, a permutation of the ground set instead.

As the map π is a bijection and it preserves the isomorphisms counting the non-isomorphic graphs in $\mathcal{G}_n^{(1)}$ is equivalent to counting the inverse class of graphs, namely those in $\mathcal{G}_n^{(2)}$ as we see in lemma 2.2. We note that graphs in $\mathcal{G}_n^{(1)}$ are isomorphic if and only if their inverses are isomorphic in $\mathcal{G}_n^{(2)}$. Lemma 2.3 tells us about the structure of those graphs in $\mathcal{G}_n^{(2)}$ and from this it is easy to see that any graph $G \in \mathcal{G}_n^{(2)}$ may be characterised by a partition of its vertices. Two graphs in $\mathcal{G}_n^{(2)}$ are then isomorphic if they have the same partition configurations, i.e. if there is some permutation of the vertices of one graph which maps the vertices of one partition into the other. Thus the isomorphisms are relegated once we look at the number of partitions of the number of vertices. This is simply the number of partitions of the integer n less one (as the complete graph in $\mathcal{G}_n^{(2)}$ is not a matroid since it forces $\mathcal{B} = \emptyset$.) The number we seek is thus $p(n) - 1$ and so $f_2(n) = p(2) + p(3) + \cdots + p(n) - (n-1) = p(1) + p(2) + \cdots + p(n) - n$. ▮

From this proof it becomes apparent that all strict rank-2 matroids may be uniquely characterised by a partition of their ground sets. To count the number of strict rank-2 matroids $m_2(n)$ (i.e. including all those matroids which are isomorphic to one another) then it is the number of partitions of a set of size n less one. These partition numbers are known as the Bell numbers, $b(n)$, the exponential generating function(e.g.f.) is given in Wilf[9] to be,

$$\sum_{n \geqslant 0} \frac{b(n)}{n!} x^n = e^{e^x - 1}$$

and satisfies the recurrence;

$$b(n+1) = \sum_{0 \leqslant i \leqslant n} \binom{n}{i} b(i), \qquad n \geqslant 0,\ b(0) = 1.$$

Thus the e.g.f. of $m_2(n)$ will be,

$$\begin{aligned}\sum_{n\geqslant 2}\frac{m_2(n)}{n!}x^n &= \sum_{n\geqslant 2}\frac{b(n)-1}{n!}x^n\\ &= e^{e^x-1}-(1+x)-\sum_{n\geqslant 2}\frac{x^n}{n!}\\ &= e^{e^x-1}-(1+x)-(e^x-1-x)\\ &= e^{e^x-1}-e^x.\end{aligned}$$

Let $m(n)$ be the number of rank-2 matroids on a ground set of size n.

COROLLARY 2.2. *The number of rank-2 matroids strictly on a ground set of size n has the e.g.f.,*

$$m(n)=\sum_{i=2}^{n}m_2(i),\quad \textit{where}\quad \sum_{n\geqslant 2}\frac{m_2(n)}{n!}x^n=e^{e^x-1}-e^x.$$

3. CONNECTED RANK-2 MATROIDS

We now investigate the number of connected rank-2 matroids on V_n and denote this number by $c(n)$. The corresponding number of non-isomorphic connected rank-2 matroids on V_n will be denoted by $d(n)$. The condition for connectedness for rank-2 matroids simplifies a great deal when looked at graphically.

We say that a matroid is *2-connected* (herein *connected*) if it can be expressed as the sum of two matroids. More formally, given a matroid $M(S_n,\mathcal{B})$, we say that this matroid is *connected* if there do not exist matroids $M_1(X,\mathcal{B}_1)$ and $M_2(S_n\backslash X,\mathcal{B}_2)$ such that $\mathcal{B}=\{\ T_1\cup T_2\mid T_1\in\mathcal{B}_1 \text{ and } T_2\in\mathcal{B}_2\ \}$. If a matroid is not connected then it is *disconnected* and we write $M=M_1\oplus M_2$.

Let $\mathcal{C}_n$ be the set of connected rank-2 matroids strictly on the ground set V_n and denote its cardinality by $c_2(n)$. When connectedness is looked at as regards rank-2 matroids, the condition reduces to $M(V_n,\mathcal{B})$ is connected if we cannot find a proper partition of V_n such that $\mathcal{B}=\{\ \{x,y\}\mid x\in X \text{ and } y\in V_n\backslash X\ \}$.

THEOREM 3.1. *For all $n\geqslant 2$,*

$$c(n)\ =\ m(n)+n-2^n-\frac{1}{2}\sum_{i=1}^{\lfloor n/2\rfloor}\binom{2i}{i}.$$

Proof. For any matroid $M(V_n, \mathcal{B}) \in \mathcal{M}_n \backslash \mathcal{C}_n$ we see that $M(V_n, \mathcal{B}) = M_1(V_k, \mathcal{B}_1) \oplus M_2(V_{n-k}, \mathcal{B}_2)$ for some integer $0 < k < n$. The collections $\mathcal{B}_1$ and $\mathcal{B}_2$ contain one element subsets of V_k and V_{n-k} respectively. We note that the elements in V_k and V_{n-k} are disjoint. Since the union of the sets in $\mathcal{B}_1$ must be V_k, we have that $\mathcal{B}_1$ is just the set of single elements of V_k and similarly for $\mathcal{B}_2$. There are thus $\binom{n}{k}$ ways of choosing $\mathcal{B}_1$, and once $\mathcal{B}_1$ has been chosen the collection $\mathcal{B}_2$ is given for $\mathcal{B}_2$ is the set of all those elements in V_n which are not in V_k. Hence the number of these is,

$$\begin{aligned} m_2(n) - c_2(n) &= \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{\lfloor \frac{n}{2} \rfloor} \\ &= \begin{cases} 2^{n-1} - 1, & \text{n odd,} \\ 2^{n-1} - 1 - \frac{1}{2}\binom{n}{n/2}, & \text{n even.} \end{cases} \end{aligned}$$

Rearranging this expression and using the fact $c(n) := c_2(2) + c_2(3) + \cdots + c_2(n)$ gives the result as stated above. ∎

THEOREM 3.2. *For all $n \geqslant 2$,*

$$d(n) = f(n) - \lfloor n/2 \rfloor^2 - \begin{cases} 0, & n \textit{ odd,} \\ \lfloor n/2 \rfloor, & n \textit{ even.} \end{cases}$$

Proof. The proof follows directly from the argument of the previous proof. The number of non-isomorphic connected matroids of rank-2 strictly on a ground set of size n is the number of non-isomorphic matroids of the same rank strictly on a ground set of size n, i.e. $f_2(n)$, less the number of non-isomorphic disconnected matroids strictly on a ground set of size n. The point at which the latter are isomorphic is exactly when they may be decomposed into matroids of rank-1 on ground sets of size k, $n-k$ respectively. Thus the number of such disconnected matroids will be $\lfloor n/2 \rfloor$.

$$\begin{aligned} d_2(n) &= p(n) - 1 - \lfloor n/2 \rfloor, & \text{for all } n \geqslant 2. \\ \Rightarrow \quad d(n) &= \textstyle\sum_{i=2}^{n} d_2(i) \\ &= \textstyle\sum_{i=2}^{n} p(n) - 1 - \lfloor n/2 \rfloor \\ &= f_2(n) - \textstyle\sum_{i=2}^{n} \lfloor n/2 \rfloor \\ &= f_2(n) - \lfloor n/2 \rfloor^2, & n \text{ odd,} \\ &= f_2(n) - \lfloor n/2 \rfloor^2 - \lfloor n/2 \rfloor, & n \text{ even.} \end{aligned}$$

∎

4. OTHER RESULTS

It is interesting to see that there is a natural bijection between the class of strict matroids of rank-2 on V_n and the class of finite set algebras on the set V_n excluding the trivial set algebra. From Proposition I-2-1 of Neveu[4] the collection of atoms in a finite set algebra is simply a partition of the set V_n. This leads to the correspondence between any strict rank-2 matroid $M(V_n, \mathcal{B})$ and the atoms of such an algebra in the following way. The set $\{a, b\} \in \mathcal{B}$ if a and b reside in different atoms of the set algebra. Conversely, if any two elements $a, b \in V_n$ are in the same atom of the set algebra then $\{a, b\} \notin \mathcal{B}$.

From Welsh[8] we find that by duality, the number of non-isomorphic rank-k matroids on a ground set of size n is equal to the number of non-isomorphic rank-2 matroids on the same ground set. Thus $f_{n-2}(n) = f_2(n)$ for all $n \geqslant 2$.

ACKNOWLEDGMENT

Since writing this paper I have become aware of Acketa's 1978 paper "On the enumeration of rank-2 matroids" which approaches the problem in much the same graph theoretic way. His paper enumerates the number of non-isomorphic rank-2 matroids but I feel the expression he obtains is less intuitive than the one presented here.

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