1D Potts, Yang-Lee Edges and Chaos.

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Abstract

It is known that the (exact) renormalization transformations for the one-dimensional Ising model in field can be cast in the form of a logistic map $f(x) = \lambda x(1-x)$ with $\lambda = 4$ and x a function of the Ising couplings K and h. Remarkably, the line bounding the region of chaotic behaviour in x is precisely that defining the Yang-Lee edge singularity in the Ising model. The generalisation of this relation between the edge singularity and chaotic behaviour to other models is an open question.

In this paper we show that the one dimensional q-state Potts model for $q \geq 1$ also displays such behaviour. A suitable combination of couplings (which reduces to the Ising case for $q = 2$) can again be used to define an x satisfying $f(x) = 4x(1-x)$. The Yang-Lee zeroes no longer lie on the unit circle in the complex $z = exp(h)$ plane for $q \neq 2$, but their locus is still reproduced by the boundary of the chaotic region in the logistic map.

1 Introduction, Ising

Yang and Lee [2, 3], later followed by various other authors [4], provided an important paradigm for understanding the nature of phase transitions by looking at the behaviour of spin models in *complex* external fields They observed that the partition function of a system above its critical temperature T_c was non-zero throughout some neighbourhood of the real axis in the complex external field plane. As $T \to T_c+$ the endpoints of loci of zeroes moved in to pinch the real axis, signalling the transition. When such endpoints occur at non-physical (i.e. complex) external field values they can be considered as ordinary critical points with an associated edge critical exponent. This appealing picture was later extended by Fisher to temperature driven transitions [5].

On any finite graph G_n with n vertices the free energy of an Ising-like spin model can be written as

$$F(G_n, \beta, z) = -nh - \ln \prod_{k=1}^{n} (z - z_k(\beta)) \tag{1}$$

where the fugacity $z = \exp(h)$, and h is the (possibly complex) external field. The $z_k(\beta)$ are the Yang-Lee zeroes, which in the thermodynamic limit generally condense on curves in the complex z plane. In the infinite volume limit $n \to \infty$ the free energy per spin is

$$F(G_\infty, \beta, z) = -h - \int_{-\pi}^{\pi} d\theta \rho(\beta, \theta) \ln(z - e^{i\theta}) \tag{2}$$

where $\rho(\beta, \theta)$ is the density of the zeroes, which can be shown to appear on the unit circle in the complex z plane in the Ising case (the Yang-Lee circle theorem). For $T > T_c$ or, if one prefers $\beta < \beta_c$, there is a gap with $\rho(\beta, \theta) = 0$ for $|\theta| < \theta_0$, and at these edge singularities we have

$$\rho(\beta, \theta) \sim (\theta - \theta_0)^\sigma \tag{3}$$

which defines the Yang-Lee edge exponent σ. This also implies $M \sim (\theta - \theta_0)^\sigma$. Various finite size scaling relations relate the Yang-Lee exponent to the other critical exponents [6] and can be used in numerical determinations of critical behaviour [7].

At first sight there is no apparent reason why the Yang-Lee edge singularity should bear any relation to the onset of chaotic behaviour in non-linear maps. The relation between the two in the case of the 1D Ising model was exposed in [1] by considering the renormalization group flow as an example of just such a mapping. The partition function for the 1D Ising model is given by

$$Z_N(K, h) = \sum_{\{\sigma\}} exp \left[K \sum_{j=1}^{N} \sigma_j \sigma_{j+1} + h \sum_{j=1}^{N} \sigma_j \right] \tag{4}$$

where $K = \frac{J}{kT}$ and $h = \frac{H}{kT}$, with J the spin coupling and H the external magnetic field, and periodic boundary conditions require $\sigma_{N+1} \equiv \sigma_1$. The well-known solution to the 1D Ising model proceeds by expressing $Z_N(K, h)$ in terms of the transfer matrix V as $Z_N = TrV^N$, where

$$V(K, h) = \begin{pmatrix} V_{++} & V_{+-} \\ V_{-+} & V_{--} \end{pmatrix} = \begin{pmatrix} e^{K+h} & e^{-K} \\ e^{-K} & e^{K-h} \end{pmatrix} \tag{5}$$

Diagonalising V gives the eigenvalues $\lambda_\pm = e^K \left\{ \cosh h \pm \sqrt{\sinh^2 h + e^{-4K}} \right\}$ and allows us to express the partition function as

$$Z_N = \lambda_+^N + \lambda_-^N. \tag{6}$$

The Yang-Lee zeroes of this partition function in the complex h plane are the N roots of $Z_N(K, h) = 0$, which are for real K the solutions of

$$Z_N = (\lambda_+)^N + (\lambda_-)^N = 0 \qquad \Leftrightarrow \qquad \lambda_+ = \exp(\frac{in\pi}{N})\lambda_- \tag{7}$$

where $-N < n \leq N$ is odd. This gives the N Yang-Lee zeroes $h_n = i\theta_n$,

$$\cos\left(\frac{n\pi}{2N}\right)\sqrt{e^{-4K} + \sinh^2(h_n)} = i\sin\left(\frac{n\pi}{2N}\right)\cosh(h_n). \tag{8}$$

which may also be rewritten as

$$\cos(\theta_n) = \sqrt{1 - e^{-4K}}\cos\left(\frac{n\pi}{2N}\right) \tag{9}$$

In particular we can see that when $K \to \infty$ (the zero temperature "transition point" for the 1D Ising model) the zeroes are uniformly distributed on the unit circle in the complex z plane, as demanded by the Yang-Lee theorem.

So far, so standard. Now note that the recursive renormalization group transformation for the 1D Ising model can be obtained by demanding that any renormalised couplings K' and h' satisfy

$$Z_{\frac{N}{2}}(K', h') = A^N Z_N(K, h) \tag{10}$$

where A is some renormalization factor. Thinking in terms of a decimation type renormalization scheme it is clear that we can satisfy this by taking

$$V(K', h') = A^2 V(K, h)^2, \tag{11}$$

where V is the transfer matrix given in equ.(5). Viewed geometrically, we are welding two line segments together and demanding a suitable rescaling of the couplings, so one copy of the rescaled transfer matrix $V(K', h')$ must serve in place of two copies of the original $V(K, h)$. This leads to the recursion relations

$$\begin{aligned} e^{2h'} &= e^{2h}\ \frac{\cosh(2K + h)}{\cosh(2K - h)} \\ e^{4K'} &= \frac{\cosh(4K) + \cosh(2h)}{2\cosh^2(h)}. \end{aligned} \tag{12}$$

The crucial observation of [1] was that these recursion relations could be recast by making use of the renormalization invariant $m = 1 + e^{4K}\sinh^2(h)$ to eliminate h and introducing the variable

$$x = -\frac{m}{(e^{4K} - 1)} \tag{13}$$

to transform equs.(12) into the logistic map $x' = 4x(1 - x)$. This will exhibit chaotic behaviour for $0 < x < 1$, i.e. if $m = 1 + e^{4K}\sinh^2(h) < 0$ which for imaginary external field, $h = i\theta$, will occur if $\sin^2(\theta) > e^{-4K}$.

What has this got to do with Yang-Lee edge singularities? Looking back at equ.(9) we can see that the lowest Yang-Lee zero will lie at $\sin^2(\theta_0) = e^{-4K}$, which is precisely the "boundary of chaos", $m < 0$, in x observed in the renormalization transformation above. One can also identify a gap exponent for the chaotic map which is identical to the Yang-Lee exponent $\sigma = -1/2$ for the 1D Ising model [1]. It is natural to ask whether the identification of the onset of chaos in an RG transformation and the Yang-Lee edge singularity is a peculiarity of the 1D Ising model, or whether other examples of the phenomenon exist. In the remainder of the paper we go to answer the question in the affirmative for the 1D Potts model where one can also obtain the Yang-Lee zeroes explicitly and construct an exact renormalization transformation along similar lines to the Ising model.

2 1D Potts, Yang-Lee

The partition function for the 1D Potts model is given by

$$Z_N(y, z) = \sum_{\{\sigma\}} exp\left[\tilde{K}\sum_{j=1}^{N}\delta(\sigma_j, \sigma_{j+1}) + \tilde{h}\sum_{j=1}^{N}\delta(\sigma_j, 1)\right] \tag{14}$$

where the $\delta()$s are Kronecker deltas and there are now q possible states for each spin σ. and we have defined $y = e^{\tilde{K}}$ and $z = e^{\tilde{h}}$ for later convenience. We can write down a transfer matrix for this as a $q \times q$ matrix $V(y,z)$ with $q-2$ diagonal elements $(y-1)/(yz)^{1/q}$ and a 2×2 sub-matrix $T(y,z)$ [8] [1]

$$T(y,z) = \frac{1}{(yz)^{1/q}} \begin{pmatrix} yz & z^{1/2}(q-1) \\ z^{1/2} & y+q-2 \end{pmatrix}. \tag{15}$$

For $q = 2$ we recover $V(K,h)$ from $T(y,z)$ providing we identify $\tilde{K} = 2K$, $\tilde{h} = 2h$.

The solution proceeds as in the Ising case by writing $Z_N(y,z) = trV(y,z)^N$ and diagonalising V [8]. The dominant eigenvalues $\lambda_{0,1}$ come from $T(y,z)$

$$\lambda_{0,1} = \frac{1}{2}\left((y(1+z)+q-2) \pm \sqrt{(y(1-z)+q-2)^2 + (q-1)4z}\right)(yz)^{-\frac{1}{q}} \tag{16}$$

which can be rewritten as

$$\lambda_{0,1} = \frac{y}{2}\left(t_+t_- + z \pm \sqrt{(z-t_+^2)(z-t_-^2)}\right)(yz)^{-\frac{1}{q}} \tag{17}$$

with

$$t_\pm = \frac{1}{y}\left(\sqrt{(y-1)(y+q-1)} \pm \sqrt{1-q}\right). \tag{18}$$

The other $q-2$ eigenvalues given by $\lambda_2 = \lambda_3 = \ldots = (y-1)(yz)^{-\frac{1}{q}}$ play no role in the thermodynamic limit.

The Yang-Lee zeroes $z_n = e^{h_n}$, just as for the Ising model, appear as solutions of

$$Z_N = (\lambda_1)^N + (\lambda_0)^N = 0 \qquad \Leftrightarrow \qquad \lambda_1 = \exp(\frac{in\pi}{N})\lambda_0 \tag{19}$$

which, upon substituting in the values above for $\lambda_{0,1}$, gives

$$\cos\left(\frac{n\pi}{2N}\right)\sqrt{(z_n-t_+^2)(z_n-t_-^2)} = i\sin\left(\frac{n\pi}{2N}\right)(t_+t_- + z_n) \tag{20}$$

which is clearly of the same form as the Ising result in equ.(8) for general q and reproduces it exactly when $q = 2$ (and we set $\tilde{K} = 2K$, $\tilde{h} = 2h$), as it should. The resemblance runs deeper even for general q, as noted in [8]. If we define $\tilde{z} = z/(t_+t_-) = yz/(y+q-2)$ this may be rewritten as

$$\cos\left(\frac{n\pi}{2N}\right)\sqrt{\left(\tilde{z}_n - \frac{t_+}{t_-}\right)\left(\tilde{z}_n - \frac{t_-}{t_+}\right)} = i\sin\left(\frac{n\pi}{2N}\right)(1+\tilde{z}_n) \tag{21}$$

so in the complex $\tilde{z}$ plane the Yang-Lee zeroes are again uniformly distributed round the unit circle as $\tilde{K} \to \infty$ and $t_+ \to 1, t_- \to 1$.

We now try and pursue the same path with the renormalization group transformation in the Potts model as for the Ising model. We once more demand that renormalised couplings y' and z' satisfy

$$Z_{\frac{N}{2}}(y',z') = A^N Z_N(y,z) \tag{22}$$

and again attempt to solve this by taking $V(y',z') = A^2 V(y,z)^2$, where V is now the Potts transfer matrix. Since only $\lambda_{0,1}$ are playing any role in the thermodynamic limit we discard the remaining $q-2$ eigenvalues and concentrate our attentions on the sub-matrix T, by demanding $T(y',z') = A^2 T(y,z)^2$. It is always possible that an infelicitous choice of renormalization transformation could take us outside the space of couplings spanned by T, but we shall see that this is not the case here, at least for the symmetric choice of transfer matrix that we have made.

[1] We have chosen a slightly different form of the matrix $T(y,z)$ than [8] for convenience in formulating our renormalization group transformations. This simply corresponds to different definitions of the ground state energy.

We find the following recursion relations

$$\begin{aligned}
\frac{1}{(y'z')^{\frac{1}{q}}}(y'z') &= \frac{A^2}{(yz)^{\frac{2}{q}}}(y^2z^2+z(q-1)) \\
\frac{1}{(y'z')^{\frac{1}{q}}}(y'+q-2) &= \frac{A^2}{(yz)^{\frac{2}{q}}}(z(q-1)+(y+q-2)^2) \\
\frac{1}{(y'z')^{\frac{1}{q}}}(z')^{1/2} &= \frac{A^2z^{1/2}}{(yz)^{\frac{2}{q}}}(zy+y+q-2) \qquad (23)
\end{aligned}$$

which can be used to eliminate A giving

$$\begin{aligned}
\frac{y'z'}{y'+q-2} &= \frac{y^2z^2+z(q-1)}{(y+q-2)^2+z(q-1)} \\
\frac{(z')^{\frac{1}{2}}}{y'+q-2} &= \frac{z^{1/2}(yz+y+q-2)}{(y+q-2)^2+z(q-1)}. \qquad (24)
\end{aligned}$$

It is then straightforward to show that, as for the Ising model, an invariant exists — in this case

$$C = \frac{(y(1-z)+q-2)^2}{z} \qquad (25)$$

— so we can use y and C to reduce our recurrence relations to one for y alone. Eliminating z' from (24) leads to a single recursion relation which can be written as

$$y'(y'+q-2)-(q-1) = \frac{z\big(y(y+q-2)-(q-1)\big)^2}{\big(y(z+1)+q-2\big)^2}. \qquad (26)$$

Now we can use (25) to write

$$C+4y(y+q-2) = \frac{\big(y(z+1)+q-2\big)^2}{z}. \qquad (27)$$

So we define

$$x = -\frac{[(C/4)+q-1]}{[(y-1)(y+q-1)]} \qquad (28)$$

and the relation (26) is again reduced to the logistic map with the pre-factor 4,

$$x' = 4x(1-x). \qquad (29)$$

For C real and positive x is real and negative and so is outside the domain of chaos, but for $C < -4(q-1)$ x is positive we have chaos for $0 < x < 1$. On the critical line itself, $C = -4(q-1)$, we allow ourselves the possibility of complex $z = |z|e^{i\theta}$ and find from equ.(25) that

$$z = \frac{(y+q-2)}{y}e^{i\theta} = (t_+t_-)e^{i\theta} \qquad (30)$$

where

$$\cos(\theta) = 1-2\frac{(q-1)}{y(y+q-2)}. \qquad (31)$$

Given our earlier discussion it is no surprise to find that these are precisely the equations defining the Yang-Lee edge singularity in the 1D Potts model.

We have thus seen that defining a decimation type renormalization transformation for the 1D Potts model gives rise to a set of recursion relations which may be reduced using the renormalization invariant of equ.(25) to a single equation. This may in turn be mapped on to the logistic equation. The boundary of the chaotic region for this logistic map is identical to the critical line of the Yang-Lee edge singularity. This behaviour is entirely analogous to that seen in the 1D Ising model in [1].

Note that for complex temperatures y, and so x, is complex and defining $w = -4(x-\frac{1}{2})$ turns (29) into the Mandelbrot map on the complex plane,

$$w' = w^2 - 2. \qquad (32)$$

3 Discussion

The similarity of the Yang-Lee edge singularity for the general q state Potts models in 1D and for the Ising model was already remarked in [8]. The exponent $\sigma = -1/2$ is identical for all $q > 1$, and in suitably rescaled variables the Yang-Lee zeroes lie on the unit circle at $T = 0$ for all $q > 1$. Given this, it is perhaps not so surprising that the relation between the renormalization transformations and the Yang-Lee edge singularity also survive to general q from the Ising model. Nonetheless, it is intriguing that an exactly soluble model has again demonstrated a close relation between the onset of chaos in a renormalization map and the Yang-Lee edge singularity.

We have not discussed the case $0 \le q < 1$ in this paper, since [8] makes it clear that the correspondence with the Ising model is rather less direct for this. For $0 \le q < 1$ the Yang-Lee edges z_+, z_-, lie on the positive real z axis for all temperatures. Inside this interval points exist where a third eigenvalue λ_2 comes into play, which would involve the extension of our renormalization transformation to a 3×3 matrix. For $q = 1$ on the other hand, the discussion in the previous section suggests that the critical line for the logistic map is defined by $C = 0$ and equ.(25) then shows that this translates to $z = (y-1)/y$ and $\theta = 0$ which defines a circle of radius $(y-1)/y$ in the z plane with a "gap angle" $\theta = 0$. Direct consideration in [8] of equs. (30,31) gives the same result for the Yang-Lee zeroes. This demonstrates that the boundary of chaos in the renormalization map and the Yang-Lee edge singularity are one and the same for $q = 1$ also.

In a wider context pathologies of approximate real space renormalization transformations *within* phases in higher dimensional models have been discussed in a rigorous manner in [9]. Other cases exist where recursive non-linear maps are used in the definition of exact partition functions, notably for spin models on Bethe lattices (trees). There has been discussion of chaotic effects in such models when $q < 1$ [10] and the logistic equation has even been observed for a $q = 1$ state Potts model (related to percolation) on a Bethe lattice with co-ordination number 3 [11]. Similarly, there have been extensive investigations of the effects of frustration in inducing chaotic behaviour in such maps for spin models on trees [12, 14, 14, 15, 16] but no discussion of any Yang-Lee/chaos link in this context. Since results are available for the Yang-Lee edge singularity in the Ising model on ϕ^3 random graphs [17] [2], it would be worth investigating the correspondence, if any, between chaos and the Yang-Lee edge singularity in this case also. This would make it clearer whether the phenomenon reported in [1] and here was merely a 1D quirk, or something more general.

As a final remark we note that the 1D Potts models possess a temperature/field (i.e. y, z) duality [8]

$$
\begin{aligned}
y^D &= \frac{z+q-1}{z-1} \\
z^D &= \frac{y+q-1}{y-1}
\end{aligned}
\tag{33}
$$

so the relation between the chaotic behaviour discussed here and the endpoints of Yang-Lee (field) zeroes can also be couched in terms of Fisher (temperature) zeroes in the dual y^D, z^D variables.

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