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Chiral Fermions and Spin^c Structures on Matrix Approximations to Manifolds

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Abstract: The Atiyah-Singer index theorem is investigated on various compact manifolds which admit finite matrix approximations ("fuzzy spaces") with a view to applications in a modified Kaluza-Klein type approach in which the internal space consists of a finite number of points. Motivated by the chiral nature of the standard model spectrum we investigate manifolds that do not admit spinors but do admit *Spin*^{c} structures. It is shown that, by twisting with appropriate bundles, one generation of the electroweak sector of the standard model, including a right-handed neutrino, can be obtained in this way from the complex projective space $\mathbf{CP}^2$. The unitary Grassmannian $U(5)/\left(U(3)\times U(2)\right)$ yields a spectrum that contains the correct charges for the Fermions of the standard model, with varying multiplicities for the different particle states.

Keywords: Non-Commutative Geometry, Field Theories in Higher Dimensions, Differential and Algebraic Geometry.

Contents

1. Introduction

Non-commutative geometry has recently come to the fore as contender for a possible modification of physics with applications in attempts to unify gravity and gauge theories (for reviews see [1]). Long before the current surge of interest via superstrings it was suggested by Connes and Lott that the standard model of particle physics could be derived from non-commutative geometry, [2] [3]. A related concept is that of "matrix manifolds"—these are a version of non-commutative geometry in which continuous spaces with an infinite number of degrees of freedom are replaced with finite dimensional non-commutative matrix algebras approximating the continuum space. As the size of the matrices is taken to infinity the algebra becomes commutative and the continuum space is recovered. These algebras are often called "fuzzy spaces" but we shall refer to them as "matrix manifolds" in order to avoid the negative connotations of the word "fuzzy". The matrix manifold approach has much in common with generalised coherent states in quantum mechanics, [4] [5] and examples of manifolds which admit a finite matrix approximation are S^2 [6] [7], and more generally $\mathbf{CP}^n$, [9] as well as unitary Grassmannians

$$\frac{U(n)}{U(k) \times U(n-k)} \tag{1.1}$$

[10] (star products on continuous complex projective spaces and unitary Grassmannians were constructed in [11] [12]). One of the attractive features of matrix manifolds is that they have the same symmetries as the continuum space, so a matrix version $(G/H)_M$ of a coset space G/H has all the same symmetries of its continuous parent, despite being a finite approximation. Matrix manifolds are also closely related to harmonic expansions of functions on coset spaces, indeed the matrix algebras are nothing more than cunning rearrangements of the expansion coefficients into a matrix, and it is natural to ask if matrix manifolds might have a rôle to play in Kaluza–Klein theory. This question was investigated in [13] and is one of the motivations for the present work. Another motivation is the calculation of the spectrum of the Dirac operator on $\mathbf{CP}^2_M$ in [14] [15], the calculation in the latter reference being built on a construction which bears a remarkable resemblance to the electroweak sector of the standard model. This naturally leads one to ask if there might be a larger matrix manifold which could incorporate the whole standard model in its spectrum.

One of the problems with the Kaluza–Klein programme was the realisation that it was unlikely to generate a chiral gauge theory in 4-dimensions without some modification, [16]. To obtain a chiral gauge theory it seems necessary to introduce fundamental gauge fields and then one is faced with the difficulty of anomaly cancellation, which is more difficult in higher dimensions because there are more potentially anomalous graphs to worry about. The introduction of fundamental gauge fields also negates the whole Kaluza–Klein philosophy whose aim is to derive the gauge fields purely from a metric. If the internal space is a matrix manifold however fundamental gauge fields are more natural, as a matrix manifold has no simple definition of a metric, but it does have symmetries. To call a theory with a matrix manifold as an internal space a Kaluza–Klein theory is really a misnomer as all it has in common with the usual Kaluza–Klein approach is an 'internal' space with a symmetry—if a metric is not defined there are no induced gauge fields so they must be added by hand. Nevertheless there is a symmetry, the symmetry of the isometries and holonomy of the coset space are there even at the finite level— like the grin of the Cheshire cat the symmetries remain even though the metric has gone.[1] For this reason we shall continue to refer to matrix Kaluza–Klein theory because the concept has much in common with continuum Kaluza–Klein theory, though there are also strong differences.

[1] For brevity we shall refer to G and H as the isometry and the holonomy group, even when no metric or connection are defined.

Fundamental gauge fields are therefore natural in matrix Kaluza–Klein theories, but we must still worry about anomalies. To our knowledge the question of gauge anomalies on matrix manifolds has not yet been investigated, though chiral anomalies have been [7] [8]. If one takes a model consisting of 4-dimensional Minkowski space-time with a matrix internal space one can hope that it may be sufficient for the 4-dimensional gauge anomalies to cancel, without worrying about graphs with more external legs that would be important if the internal space were continuous. To a large extent this is a question of dynamics on the internal space, if the dynamics reproduces that of the continuum in the continuum limit (though it doesn't have to if we don't want to take that limit) then these other graphs would have to be important in the limit. But as long as the internal space consists of a (small) number of finite degrees of freedom it seems not unreasonable to assume that only the usual 4-dimensional graphs contribute to a potential anomaly. For example the matrix manifold representing the 2-sphere S^2_M has an approximation consisting of only 2 points. A matrix Kaluza–Klein theory based on S^2_M would look like two copies of Minkowski space with an $SU(2)$ action on the 2 points (much like the Higgs sector in Connes' version of the standard model). There seems no compelling reason to believe that such a model would exhibit a six-dimensional gauge anomaly.

For the reasons outlined above it seems worthwhile investigating the possibility of obtaining chiral gauge theories in 4-dimensions from a matrix Kaluza–Klein theory with internal space $(G/H)_M$ and fundamental gauge fields. The tool that we use will be standard differential geometry and the Atiyah–Singer index theorem for the Dirac operator on continuous manifolds. Though the aim is to apply the concepts to finite matrix geometries it is not unreasonable to expect that the usual index theorem applies since it makes statements about topology by counting finite data. Indeed a Dirac operator can be defined on matrix manifolds, even though they are finite dimensional.

The spectrum of the Dirac operator on some specific matrix manifolds has already been investigated, notably S^2_M [7] [8] and $\mathbf{CP}^2_M$, [14] [15]. The construction of the spectrum on $\mathbf{CP}^2$ in [15] is built on a 4-dimensional reducible representation of $SU(2)\times U(1)$ which is that of electroweak sector of the standard model of particle physics, including a state with the quantum numbers of a right-handed neutrino which is a chiral zero-mode. As is well known $\mathbf{CP}^2 \cong SU(3)/U(2)$ is not a spin manifold, it has an obstruction to the global definition of spinors, but coupling spinors to an appropriate background $U(1)$ gauge field allows spinors to be defined—a construction which is called a spin^c structure in the mathematical literature—and this gives rise to the right-handed neutrino in [15]. Since $\mathbf{CP}^2_M$ is a finite matrix algebra approximation to continuum $\mathbf{CP}^2$, which captures all the topological features of the continuum manifold, the topology is reflected at the matrix level and the emergence of chiral spinors on $\mathbf{CP}^2_M$ is a direct consequence of the Atiyah-Singer index theorem for spinors on $\mathbf{CP}^2$.

Since the holonomy group of $\mathbf{CP}^2$ is $U(2)$ the spectrum of the Dirac operator can be decomposed into representations of $SU(2)\times U(1)$ and the representations in [15] are built on that of the electroweak sector of the standard model with the addition of a right-handed neutrino

$$\mathbf{1}_0=\nu_{\mathbf{R}} \qquad \mathbf{1}_{-2}=e_{\mathbf{R}} \qquad \mathbf{2}_{-1}=\begin{pmatrix}\nu_{\mathbf{L}}\\ e_{\mathbf{L}}\end{pmatrix}. \tag{1.2}$$

There is a zero-mode state in the construction of [15], the $\mathbf{1}_0$, and its existence requires a background Abelian 'monopole' field on $\mathbf{CP}^2$. In fact, as we shall see, coupling Fermions to monopole fields of higher charge and non-Abelian background fields as well allows every state in (1.2) to be realised as a zero-mode of the Dirac operator on $\mathbf{CP}^2$. The fact that a right-handed neutrino appears naturally in the construction is particularly appealing in view of the recent evidence for solar neutrino oscillations [17] [18] whose simplest interpretation requires a right-handed neutrino.

Another manifold which has holonomy group $U(2)$ and does not admit spinors is

$$\frac{Sp(2)}{U(2)}\cong\frac{SO(5)}{SO(3)\times SO(2)}. \tag{1.3}$$

This space has a finite matrix approximation and has been proposed as a matrix version of the cotangent bundle to S^3 [19]. One might wonder if the spectrum in (1.2) is generic for spin^c structures on manifolds with holonomy $U(2)$ and this space is a counter-example. We shall see that a spectrum emerges which contains the correct charges of the electroweak sector but the Dirac operator for electron-neutrino doublet has zero index. Nevertheless it is useful to include this as an example of a space of dimension 2 mod 4 which is not spin (the significance for chiral spinors of a distinction between spaces of dimension 0 mod 4 and dimension 2 mod 4 was emphasised in [16]).

The last space which we shall examine is the matrix version of the unitary Grassmannian

$$\frac{U(5)}{U(3)\times U(2)}. \tag{1.4}$$

This space has a finite matrix approximation and an explicit local formula for a star-product, in terms of a finite sum of derivatives, was derived in [10]. It is not a spin manifold but admits a spin^c structure, and so seems a good candidate for chiral spinors. Furthermore the holonomy group is exactly right for the standard model, since

$$\frac{U(5)}{U(3)\times U(2)}\cong\frac{SU(5)}{S[U(3)\times U(2)]} \tag{1.5}$$

and the particle spectrum of the standard model is really such that the Fermions fall into a representation whose true group is precisely $S[U(3)\times U(2)]$, [20]. For this space the spectrum contains one generation of the full standard model, including a right-handed neutrino, though the multiplicities are different for different states, some of them having zero index.

Section 2 contains an index theorem analysis of spinors on $\mathbf{CP}^2$ and reproduces the zero-mode spectrum (1.2). Section 3 contains a discussion of the 6-dimensional manifold $Sp(2)/U(2)$. Section 4 analyses spin^c structures, and their non-Abelian generalisations, on the unitary Grassmannian $U(5)/(U(3) \times U(2))$ and its relation to the standard model spectrum. Our results are summarised in section 5. The analysis relies on the index theorem for the Dirac operator for various bundles over these three spaces. The derivation of the relevant index for the cases under study is given in four appendices, where a general discussion of spin^c structures is also given as an aid to those who may not be familiar with the construction.

2. Chiral Fermions on $\mathbf{CP}^2$

The complex projective space $\mathbf{CP}^2 \cong SU(3)/U(2)$ was actively investigated in the 1980s as an interesting candidate for an Euclidean gravitational instanton [21]. The Euler characteristic of $\mathbf{CP}^2$ is $\chi = 3$ and the signature is $\tau = 1$. It is not a spin manifold, there is a global obstruction to putting spinors on this space, but one can put spinors on it provided fundamental gauge fields are introduced and an appropriate topologically non-trivial background gauge field is introduced. This fact was used in [21] to construct a "generalised spin structure", where spinors with an Abelian charge move in the field of the Kähler 2-form on $\mathbf{CP}^2$, which is somewhat analogous to a monopole field on $\mathbf{CP}^1 \cong S^2$.

The holonomy group of $\mathbf{CP}^2$ is

$$U(2) \subset SO(4) \cong \frac{SU(2) \times SU(2)}{\mathbf{Z}_2}. \tag{2.1}$$

If spinors could be defined this would be lifted to $SU(2) \times U(1) \subset Spin(4) \cong SU(2) \times SU(2)$, and the two different chiralities of Weyl spinors would transform under the different factors of $SU(2) \times U(1)$ as, for example,

$$\psi_+ = \mathbf{1}_1 + \mathbf{1}_{-1} \qquad \text{and} \qquad \psi_- = \mathbf{2}_0, \tag{2.2}$$

where the subscript denotes the $U(1)$ charge. But since spinors cannot be defined globally (cf. appendix B), the spinor bundle does not exist. This can be cured by introducing a $U(1)$ gauge field with non-trivial topology and correlating the charge with that of the $U(1)$ subgroup of $Spin(4)$. Mathematically, on a complex manifold X, we take the square root of the canonical line bundle K, as described in appendix A, and tensor it with the the spin bundle $S(X)$. Neither of these bundles exists separately but $S(X) \otimes K^{-1/2}$ does. In fact, if L is a *generating line bundle* (cf. the appendix) with $\int_{S^2} c_1(L) = -1$, where S^2 is a non-trivial two sphere embedded in X,[2] then $S(X) \otimes L^p$ is a well defined bundle for any half-integral p. For $\mathbf{CP}^2$ it is

[2]This is ambiguous if $H^2(X; \mathbf{Z})$ has dimension greater than one, but in all the examples we shall consider in this paper $H^2(X; \mathbf{Z})$ is one dimensional and this integral is uniquely defined.

shown in appendix B that

$$S(X) \otimes L^p = \wedge^{0,*}TX \otimes K^{1/2} \otimes L^p = \wedge^{0,*}TX \otimes L^{-q}, \quad X = \mathbf{CP}^2 \tag{2.3}$$

where $q = -p - \frac{3}{2}$, since the canonical line bundle for $\mathbf{CP}^2$ is given by $K = L^3$.

The net number of zero modes depends on q and for $\mathbf{CP}^2$ is given in (B.5) of appendix B as

$$\nu = \frac{1}{2}(q+1)(q+2). \tag{2.4}$$

In fact q can be interpreted as the effective $U(1)$ charge. The charge is not p because there is a contribution from the angular momentum associated with the spinor bundle $S(X)$. To evaluate the charge we use a general argument concerning spinor bundles over complex manifolds. We define the $U(1)$ charge, which will be identified later with the hypercharge Y, using the Chern character of the generating line bundle raised to the appropriate power, in this case L^{-q}, by taking a non-trivial S^2 embedded in the manifold X and defining

$$q = \int_{S^2} ch(L^{-q}) = -q\int_{S^2} ch(L) \qquad \text{since} \qquad \int_{S^2} c_1(L) = -1. \tag{2.5}$$

For comparison with the usual charge assignments of the standard model below, we rescale this by 2/3 to $Y = 2q/3$. For $q = 0$ for example $\nu = 1$ and, identifying positive chirality with right-handed spinors, this would appear as a neutral right-handed particle: a right-handed neutrino $\nu_{\mathbf{R}}$. A spinor with $q = -3$ also has $\nu = 1$, so would be right-handed, with $Y = -2$: the right-handed electron, $e_{\mathbf{R}}$.

If a fundamental $SU(2)$ gauge field is added with the spinors taken to be $SU(2)$ doublets then spinors can be obtained from the bundle $\wedge^{0,*}T\mathbf{CP}^2 \otimes F \otimes L^{-q}$, where F is the rank 2 vector bundle defined by $F \oplus L = I^3$ (I^3 denoting a trivial rank 3 bundle). The structure group of F is $U(2)$ corresponding to a $SU(2) \times U(1)$-gauge field. In fact F is associated to the principal $U(2)$ bundle induced by the coset construction

$$\begin{array}{ccc} U(2) & \longrightarrow & SU(3) \\ & & \downarrow \\ & & \mathbf{CP}^2\ . \end{array} \tag{2.6}$$

The Dirac index for $\wedge^{0,*}T\mathbf{CP}^2 \otimes F \otimes L^{-q}$ is derived in appendix B and is given by (B.11)

$$\nu = (q+1)(q+3). \tag{2.7}$$

Zero modes would give rise to chiral $SU(2)$ doublets.

The $U(1)$ charge is now calculated as the Chern character $ch(F \otimes L^{-q})$ evaluated on a topologically non-trivial S^2 embedded in $\mathbf{CP}^2$, the result is $2q+1$. As the Chern character involves tracing over a 2×2 matrix the $U(1)$ generator is $\frac{(2q+1)}{2}\mathbf{1}$, where $\mathbf{1}$ is the 2×2 identity matrix, so the individual charges are $q + \frac{1}{2}$. Re-scaling by 2/3, as above, gives $Y = \frac{2q+1}{3}$. In particular $q = -2$ yields $Y = -1$ with $\nu = -1$ and, identifying positive chirality with right-handed particles, we get a single generation of a left-handed doublet with charge -1, the electron-neutrino doublet.

So we can obtain a single generation of the electroweak sector of the standard model from $\mathbf{CP}^2$ by taking $SU(2)$ singlets with $q = 0$ and $q = -3$ ($\nu = +1$) and a single $SU(2)$ doublet with $q = -2$ ($\nu = -1$), that is

$$\mathbf{1}_0 = \nu_{\mathbf{R}} \qquad \mathbf{1}_{-2} = e_{\mathbf{R}} \qquad \mathbf{2}_{-1} = \begin{pmatrix} \nu_{\mathbf{L}} \\ e_{\mathbf{L}} \end{pmatrix}, \qquad \nu > 0 \quad \text{right-handed.} \tag{2.8}$$

3. Chiral Fermions on $Sp(2)/U(2)$

As an example of a six-dimensional space which does not admit a spin structure, but does admit a *Spin*^{c} structure, consider $Sp(2)/U(2)$. This space has Euler characteristic $\chi = 4$. In fact

$$\frac{Sp(2)}{U(2)} \cong \frac{SO(5)}{SO(3) \times SO(2)} \tag{3.1}$$

and this space admits a matrix approximation. The spinor bundle does not exist but a *Spin*^{c} structure can be defined using $S(X) \otimes L^p$, with L the generating line bundle and p half-integral. The canonical line bundle is related to the generating line bundle by $K = L^3$ (see appendix C) so that

$$S(X) \otimes L^p = \wedge^{0,*} TX \otimes L^{-q}, \quad X = \frac{Sp(2)}{U(2)} \tag{3.2}$$

where $q = -p - \frac{3}{2}$.

The Dirac index of this bundle is derived in appendix C and is given in (C.18):

$$\nu = \frac{1}{6}(2q+3)(q+1)(q+2). \tag{3.3}$$

The zero-modes will give rise to particles in 4-dimensions whose $U(1)$ charge is q, which we re-scale by $2/3$ to bring it line with the usual standard model conventions below. so, for example, $q = -3$ gives a single generation of negative chirality particles with charge -2 while $q = 0$ would give a single generation of positive chirality neutral particles.

As before we can also couple the Fermions to a fundamental $SU(2)$ gauge field by introducing a rank 2 vector bundle F associated to the principal bundle

$$\begin{array}{ccc} U(2) & \longrightarrow & Sp(2) \\ & & \downarrow \\ & & Sp(2)/U(2) \end{array} \tag{3.4}$$

with structure group $U(2)$. It is shown in appendix C that the index of $\wedge^{0,*} TX \otimes F \otimes L^{-q}$ is now

$$\nu = \frac{2}{3} q(q+1)(q+2). \tag{3.5}$$

The Chern character $ch(F \otimes L^{-q})$ evaluates to $2q+1$ on a non-trivial S^2. Again this is the trace of a 2×2 matrix and the individual states have charge $q + \frac{1}{2}$ which is rescaled by $2/3$ to give the $U(1)$ charge as $Y = \frac{2q+1}{3}$. For example $q = 1$ gives $Y = 1$ and $\nu = 4$ and thus four copies of positive chirality doublets while $q = -2$ gives $Y = -1$ and $\nu = 0$.

We can try to get the electroweak charges from this construction. For example interpreting positive chirality as left-handed the singlets would be the right-handed electron $e_{\mathbf{R}}$ and a left-handed anti-neutrino $(\overline{\nu})_{\mathbf{L}}$. But the doublets with $Y = 1$ would have to have negative chirality to fit with the standard model (the right-handed positron and anti-neutrino) and ν is positive. If we interpret positive chirality as right-handed, the doublet could the positron–anti-neutrino doublet $\begin{pmatrix} (\overline{\nu})_{\mathbf{R}} \\ (\overline{e})_{\mathbf{R}} \end{pmatrix}$, but then the singlet with $Y = -2$ has the wrong chirality to be the right-handed electron.

On the other hand choosing a doublet with $q = -2$ giving $Y = -1$, in addition to the singlets above, gives $\nu = 0$ for the doublet: in general the Dirac operator will have no zero modes for this doublet though it may have for specific choices of the $U(2)$ connection, but even then the zero modes will occur in pairs of opposite chirality. The spectrum contains one generation of the electroweak sector of the standard model, but there is an additional unwanted doublet of the wrong chirality.

4. Unitary Grassmannians

The final source of examples that we wish to discuss is the unitary Grassmannians

$$\frac{U(n)}{U(k) \times U(n-k)} \cong \frac{SU(n)}{S(U(n-k) \times U(k))} \tag{4.1}$$

of which the complex projective spaces, $k = 1$, are special cases. The first Chern class of the tangent bundle for these space evaluates to n, [23], and the second Stiefel-Whitney class is n mod 2—so these spaces admit a spin structure if and only if n is even. We shall focus on the particular case of $n = 5$ and $k = 2$, this is an interesting case because the holonomy group of $SU(5)/S(U(3) \times U(2))$ is precisely that of the standard model, [20]. This condition dictates that the Fermions actually sit in representations of $SU(3) \times SU(2) \times U(1)$ in which the generators are traceless—whence $S(U(3) \times U(2))$. As a matrix manifold $SU(5)/S(U(3) \times U(2))$ was studied in [10], where a star product was explicitly constructed in terms of derivatives.

The Grassmannian $SU(5)/S(U(3) \times U(2))$ has Euler characteristic $\chi = 10$ and signature $\tau = 2$. It is not a spin manifold but a $Spin^c$ structure exists. Taking the bundle $\wedge^{0,*}TX \otimes L^{-q}$, with X the Grassmannian and L the generating line bundle, the Dirac index is calculated in appendix D as (D.27),

$$\nu_{\{q,\mathbf{1},\mathbf{1}\}} = \frac{1}{144}(q+1)(q+2)^2(q+3)^2(q+4), \tag{4.2}$$

where the notation $\{q, \mathbf{1}, \mathbf{1}\}$ indicates the $U(1) \times SU(3) \times SU(2)$ structure of the vector bundle.

The Chern character $ch(L^{-q})$ evaluates to q on a 2-sphere so the $U(1)$ charge here is q which we shall rescale by a factor of 2 in order to reproduce the standard model quantum numbers later,

$$Y_{\{q,\mathbf{1},\mathbf{1}\}} = 2q. \tag{4.3}$$

In particular with $q = 0$ gives $\nu_{\{0,\mathbf{1},\mathbf{1}\}} = 1$, and interpreting positive chirality as right-handed gives a single generation of right-handed neutrino, while $q = -1$ gives $Y_{\{-1,\mathbf{1},\mathbf{1}\}} = -2$ which would be the right-handed electron but $\nu_{\{-1,\mathbf{1},\mathbf{1}\}} = 0$.

We can include fundamental $SU(2)$ gauge fields by taking the spinors to transform as a doublet and constructing the rank 2 vector bundle F with structure group $U(2)$ associated with the principal bundle

$$\begin{array}{ccc} U(2) & \longrightarrow & U(5)/U(3) \\ & & \downarrow \\ & & U(5)/(U(3) \times U(2)) \ . \end{array} \tag{4.4}$$

The Dirac index of the bundle $\wedge^{0,*}TX \otimes F \otimes L^{-q}$ is calculated in the appendix and shown to be (D.39)

$$\nu_{\{q,\mathbf{1},\mathbf{2}\}} = \frac{1}{72}(q+1)(q+2)(q+3)^2(q+4)(q+5). \tag{4.5}$$

The first Chern class $c_1(F)$ evaluates to 1 and the Chern character $ch(F \otimes L^{-q})$ to $2q+1$ on a non-trivial S^2 so, dividing by the rank of the bundle, the $U(1)$ charge here is $q + \frac{1}{2}$ which is rescaled by a factor of 2 as before to $Y_{\{q,\mathbf{1},\mathbf{2}\}} = 2q + 1$. In particular $q = 0$ gives $Y_{\{0,\mathbf{1},\mathbf{2}\}} = 1$ and $\nu_{\{0,\mathbf{1},\mathbf{2}\}} = 5$ which we interpret as copies of the positron–anti-neutrino doublet $\begin{pmatrix} (\overline{\nu})_{\mathbf{R}} \\ (\overline{e})_{\mathbf{R}} \end{pmatrix}$.

Fundamental $SU(3)$ gauge fields can be incorporated by a very similar procedure: take the spinors to transform as an $SU(3)$ triplet and construct the rank 3 vector bundle E with structure group $U(3)$ associated with the bundle

$$\begin{array}{ccc} U(3) & \longrightarrow & U(5)/U(2) \\ & & \downarrow \\ & & U(5)/[U(3) \times U(2)] \ . \end{array} \tag{4.6}$$

The index of the bundle $\wedge^{0,*}TX \otimes E \otimes L^{-q}$ is (D.37)

$$\nu_{\{q,\mathbf{3},\mathbf{1}\}} = \frac{1}{48}q(q+1)(q+2)(q+3)^2(q+4). \tag{4.7}$$

The bundles E and F are related by $E \oplus F = I^5$, so $c_1(E)$ of E evaluates to -1 so integrating $ch(E \otimes L^{-q})$ over an S^2 gives $3q - 1$ giving $U(1)$ charge $q - \frac{1}{3}$ which we rescale by 2 to give $Y_{\{q,\mathbf{3},\mathbf{1}\}} = 2q - \frac{2}{3}$.

In particular $q = 0$ gives the right-handed $d_{\mathbf{R}}$ but again

$$\nu_{\{0,\mathbf{3},\mathbf{1}\}} = 0. \tag{4.8}$$

The right-handed $u_{\mathbf{R}}$ quarks arises from $q = 1$ giving $Y_{\{1,\mathbf{3},\mathbf{1}\}} = 4/3$ with index

$$\nu_{\{1,\mathbf{3},\mathbf{1}\}} = 10. \tag{4.9}$$

The left-handed quarks of the standard model are both $SU(3)$ triplets and $SU(2)$ doublets so the bundle $\wedge^{0,*}TX \otimes E \otimes F \otimes L^{-q}$ is also considered in the appendix (D.41), leading to

$$\nu_{\{q,\mathbf{3},\mathbf{2}\}} = \frac{1}{24} q(q+2)^2(q+3)(q+4)(q+5). \tag{4.10}$$

The Chern character $ch(E \otimes F \otimes L^{-q})$ integrates to $6q+1$ on a non-trivial S^2, so the $U(1)$ charge is $q + \frac{1}{6}$ which is rescaled to $Y_{\{q,\mathbf{3},\mathbf{2}\}} = 2q + \frac{1}{3}$. The choice $q = 0$ leads to the quark doublet, $\begin{pmatrix} u_{\mathbf{L}} \\ d_{\mathbf{L}} \end{pmatrix}$, with $Y_{\{0,\mathbf{3},\mathbf{2}\}} = 1/3$, and

$$\nu_{\{0,\mathbf{3},\mathbf{2}\}} = 0. \tag{4.11}$$

To summarise we can find the standard model charge assignments with the unitary Grassmannian $U(5)/\left(U(3) \times U(2)\right)$, but the indices, and so the multiplicities, are wrong:

$$\nu_{\mathbf{R}} \quad \nu_{\{0,\mathbf{1},\mathbf{1}\}} = 1, \qquad e_{\mathbf{R}} \quad \nu_{\{-1,\mathbf{1},\mathbf{1}\}} = 0, \qquad d_{\mathbf{R}} \quad \nu_{\{0,\mathbf{3},\mathbf{1}\}} = 0, \qquad u_{\mathbf{R}} \quad \nu_{\{1,\mathbf{3},\mathbf{1}\}} = 10, \tag{4.12}$$

$$\begin{pmatrix} (\overline{\nu})_{\mathbf{R}} \\ (\overline{e})_{\mathbf{R}} \end{pmatrix} \quad \nu_{\{0,\mathbf{1},\mathbf{2}\}} = 5, \qquad \begin{pmatrix} u_{\mathbf{L}} \\ d_{\mathbf{L}} \end{pmatrix} \quad \nu_{\{0,\mathbf{3},\mathbf{2}\}} = 0. \tag{4.13}$$

Obviously this is unsatisfactory as it stands: the multiplets with zero index will not be zero-modes in general and even if they are they will be accompanied with zero-modes of the opposite handedness but the same hypercharge; also 5 weak doublets and 10 right-handed u-quark 'families' is clearly not compatible with the current experimental picture.

5. Conclusions

We have investigated zero modes of the Dirac operator on various manifolds which admit finite matrix approximations. Such spaces are candidates for finite internal spaces in non-conventional Kaluza-Klein theory, where the internal space consists of a finite number of points. In this paper we have focused on manifolds that do not admit a spin structure, as the inevitable twisting of bundles that enables spinors to be defined ($Spin^c$ structures) unavoidably leads to chiral Fermions. The electroweak sector of the standard model emerges naturally in this construction from $\mathbf{CP}^2 \cong SU(3)/U(2)$: the gauge group is $U(2)$ and the usual *Spin*^{c} structure gives rise to a neutral singlet which is identified with the right-handed neutrino while tensoring the standard *Spin*^{c} bundle with the inverse of the canonical line bundle gives another $SU(2)$ singlet with the quantum numbers of the right-handed electron. The electron-neutrino doublet arises by coupling spinors to a natural rank 2 bundle which is dual to the generating line bundle—the curvature associated with this bundle represents a $U(2)$ instanton on $\mathbf{CP}^2$.

The resulting spectrum represents one generation of the electroweak sector of the standard model. The smallest non-trivial matrix approximation to $\mathbf{CP}^2$ is the algebra of 3×3 matrices, [9], acting on a three dimensional complex vector space which carries the fundamental representation of the isometry group $SU(3)$. It may be that this could be interpreted as a horizontal symmetry giving rise to three generations. Note that the philosophy here is rather different to the usual Kaluza-Klein approach where the isometry group is identified with the gauge group—here the isometry group is being identified with a horizontal symmetry group and the holonomy group is the gauge group.

We have also investigated two other manifolds: a six dimensional manifold with holonomy group $U(2)$, $Sp(2)/U(2)$; and a twelve dimensional manifold with holonomy group $SU(3)\times SU(2)\times U(1)$, the unitary Grassmannian $U(5)/(U(3)\times U(2))$. These manifolds both admit finite matrix approximations and neither admits a spin structure. The former gives a spectrum containing the correct charges for the electroweak sector of the standard model, but the electron-neutrino doublet has zero index: generically the Dirac operator would have no zero modes corresponding to this doublet. There may exist particular connections for which the doublet is a zero mode but this would necessarily be accompanied by a doublet of the opposite chirality, unless some other mechanism could be invoked to eliminate it. The unitary Grassmannian gives the correct representations and charges for the whole Fermionic sector of the standard model, but again some multiplets have zero index and here the multiplicities are different for the multiplets with non-zero index.

A number of questions remain to be addressed. Obviously it is of interest to look further for other manifolds that might give a better fit to the standard model spectrum with this approach. One possibility is to consider manifolds that admit spinors directly, without the necessity of a *Spin*^{c} structure. After all it is only the electroweak sector of the standard model that requires different representations for right and left-handed particles and, as we have seen, this can be obtained from $\mathbf{CP}^2$. QCD does not require any such asymmetry and so could arise more directly, without the introduction of chirally asymmetric Fermions. Indeed we hope to show elsewhere that this is indeed the case [22].

There is also the question of the Higgs sector of the standard model, which we have not addressed here. In Connes' approach to the standard model, the Higgs field is associated with two 'copies' of space time, which could be viewed as coming from an internal space consisting of two points acted on by the $SU(2)$ symmetry of weak interactions. This looks very much like a two dimensional vector space acted on by a matrix approximation to the 2-sphere. Grand unified models could also be investigated with the techniques used here.

Lastly we have assumed that the usual differential-geometric analysis of the Atiyah-Singer index theorem on continuous manifolds will carry over to finite matrix approximations without change. While this seems reasonable to us there is certainly no proof that it is true in general, but this would require a much more involved investigation than is presented here.

A. Spin and spin^c structures on a manifold

In this appendix we provide details for the calculations presented in the main text; useful references for this material are Borel and Hirzebruch [23], Michelson and Lawson [24] and Bott and Tu [25].

This section describes, in brief, what is involved for a manifold X to admit spinors—X is then said to have a *spin structure* or to be a *spin manifold*—and failing that, we describe how X can have what is called a *spin*^{c} *structure*. A manifold X has a *spin*^{c} *structure* when it admits a certain pair consisting of a $U(1)$ connection and a 'local spinor'—X is then said be a *spin*^{c} *manifold*. Spin manifolds are automatically spin^c manifolds but the converse is false.

If an n dimensional manifold X (compact and closed in this discussion) is a spin manifold then its tangent bundle

$$TX \tag{A.1}$$

whose principal bundle we denote by

$$P_{SO(n)}(X) \tag{A.2}$$

has structure group $SO(n)$. Sections of TX are then vectors on X.

The fact that X is spin means that TX possesses a lifting of its structure group from the group $SO(n)$ to the group $Spin(n)$. Such a lifting, which need not be unique, constitutes a choice of spin structure on X. This lifting induces from $P_{SO(n)}(X)$ a $Spin(n)$ principal bundle

$$P_{Spin(n)}(X) \tag{A.3}$$

on X; also induced from TX, and associated to $P_{Spin(n)}(X)$, is the bundle of spinors

$$S(X) \tag{A.4}$$

over X. Finally sections of $S(X)$ are called spinors.

The existence of this lifting requires topological obstructions to vanish namely that the first two Stiefel–Whitney classes of TX should vanish i.e.

$$w_1(X) = 0, \qquad w_2(X) = 0. \tag{A.5}$$

The vanishing of $w_1(X)$ just guarantees that X is orientable and allows us to distinguish clockwise and anti-clockwise rotations; but the vanishing of $w_2(X)$ is needed to make the double covering of of $SO(n)$ by $Spin(n)$ work globally.

If X is orientable but

$$w_2(X) \neq 0 \tag{A.6}$$

then global spinors do not exist and X is not a spin manifold.

When X is not a spin manifold the situation can be saved if X admits a generalisation of a spin structure known as a spin^c structure; moreover this is quite a natural structure if X is a complex manifold, though X does not need to be complex. An orientable X admits a spin^c structure if $w_2(X)$ is the reduction mod 2 of an integral cohomology class in $H^2(X;\mathbf{Z})$. This is guaranteed for complex manifolds since their Chern classes determine their Stiefel–Whitney classes via the relation

$$w_2(X) = c_1(X) \bmod 2. \tag{A.7}$$

The underlying mechanism which makes a spin^c structure work is easy to expose when X is complex. Suppose then that X has a Kähler metric and is complex with complex dimension n; let the canonical line bundle of X be K so that

$$K = \wedge^n T^* X. \tag{A.8}$$

Recall also for later use that $c_1(K) = -c_1(X)$. Now suppose first that X is a spin manifold so that

$$w_2(X) = 0 \Rightarrow c_1(X) \text{ is even} \tag{A.9}$$

and that the spinor bundle $S(X)$ does exist; this in turn means that the canonical bundle K has square roots: a choice of square root is a spin structure. Now consider the bundle $\wedge^{0,*}TX$ of all forms of type $(0,s)$—i.e. anti-holomorphic s forms—so we have

$$\wedge^{0,*}TX = \bigoplus_s \wedge^{0,s}\ \overline{T^* X}\,. \tag{A.10}$$

The the spinor bundle $S(X)$ is obtained by just tensoring $\wedge^{0,*}TX$ with $K^{1/2}$ i.e.

$$S(X) = \wedge^{0,*}TX \otimes K^{1/2}. \tag{A.11}$$

Hence spinors are $K^{1/2}$-valued $(0,s)$ forms—which we denote by $\Omega^s(K^{1/2})$—and are sections of $S(X)$. The full self-adjoint Dirac $\not{D}$ operator is loosely $\bar\partial_{K^{1/2}} + \bar\partial^*_{K^{1/2}}$, the

$\bar\partial$ operator acting on sections of $S(X)$; the chiral Dirac operator is denoted by $\partial\!\!\!/$ and this setup now gives us

$$D\!\!\!\!/ = \begin{pmatrix} 0 & \partial\!\!\!/ \\ \partial\!\!\!/^* & 0 \end{pmatrix} \qquad D\!\!\!\!/ \equiv \sqrt{2}\left(\bar\partial_{K^{1/2}} + \bar\partial^*_{K^{1/2}}\right) \tag{A.12}$$

$$\partial\!\!\!/ \;:\; \bigoplus_p \Omega^{2p}(K^{1/2}) \longrightarrow \bigoplus_p \Omega^{2p+1}(K^{1/2}) \tag{A.13}$$

$$\partial\!\!\!/^* \;:\; \bigoplus_p \Omega^{2p+1}(K^{1/2}) \longrightarrow \bigoplus_p \Omega^{2p}(K^{1/2}). \tag{A.14}$$

The chirality of a spinor now corresponds to its parity as a form.

All the above was for the case when X is spin. Now suppose that X is *not* spin then we see that

$$w_2(X) \neq 0 \Rightarrow c_1(X) = -c_1(K) \text{ is odd} \tag{A.15}$$

$$\Rightarrow\; K^{1/2} \text{ does not exist} \tag{A.16}$$

$$\Rightarrow\; S(X) \text{ does not exist.} \tag{A.17}$$

But, though $S(X) = \wedge^{0,*}TX \otimes K^{1/2}$ does not exist, the bundle $\wedge^{0,*}TX$ clearly does: this is the spin^c bundle which we denote by $S^c(X)$ so that

$$S^c(X) = \wedge^{0,*}TX. \tag{A.18}$$

Now if we abuse notation temporarily and write down the tensor product of the two non-existent bundles $S(X)$ and $K^{-1/2}$ we get the spin^c bundle $S^c(X)$ since we can write

$$S^c(X) = S(X) \otimes K^{-1/2} \text{ ('locally')} \tag{A.19}$$

and this bundle $S^c(X)$ does exist globally even though its factors do not. The point is that the factors do exist locally and the failure of one factor to behave properly (under, for example parallel transport round a closed loop) is compensated for by a failure of the other; this mechanism renders the 'product' well defined. This picture of $S^c(X)$ makes it clear at once that the generalised spinors of a spin^c structure are also coupled to a local $U(1)$ connection.

Finally, as we are interested in chiral Fermions, we want to point out that the Dirac operator exists for generalised spinors and it is natural to denote it by

$$\partial\!\!\!/_{K^{-1/2}}. \tag{A.20}$$

Just as a spin structure need not be unique nor need a spin^c structure: one can tensor the 'bundle' $K^{-1/2}$ by any other genuine line bundle: all our manifolds X will have one dimensional $H^2(X;\mathbf{Z})$ so that there is a 'smallest' line bundle L defined by requiring

$$c_1(L) = -1. \tag{A.21}$$

We shall call L the *generating line bundle* and, in each case, K will be some power of L—this power will be odd if X is not a spin manifold—so that

$$K = L^m, \ m \in \mathbf{Z}. \tag{A.22}$$

Hence a general spin^c structure will have the spin^c bundle

$$\wedge^{0,*}TX \otimes L^{-q} = S^c(X) \otimes L^{-q}, \quad q \in \mathbf{Z} \tag{A.23}$$

(the minus sign in the exponent is for later convenience). When $q = 0$ we have the *canonical* spin^c structure; there is also a dependence of the spin^c structure on an element of $H^1(X;\mathbf{Z}_2)$ but our examples have $H^1(X;\mathbf{Z}) = 0$ so we do not need to consider this.

If we use the fact that $K = L^m$ then the corresponding Dirac operator then becomes $\not{\partial}_{L^{-(q+m/2)}}$ which we shall neaten up slightly by writing it as

$$\not{\partial}_{L^p} \quad \text{where } p = -q - m/2. \tag{A.24}$$

There is also an index formula for the zero modes of $\not{\partial}_{L^p}$ which involves the usual $\hat{A}$ genus of X and the 'Chern class' of the line bundle L^p. Let $\not{\partial}_{L^p}$ denote the Dirac operator coupled to L^p then its index is given by [3]

$$\begin{aligned} \text{index}\,(\not{\partial}_{L^p}) &= \text{ch}\,(L^p)\hat{A}(X)[X] & \text{(A.25)}\\ &= \exp\,[pc_1(L)]\,\hat{A}(X)[X]. & \text{(A.26)}\end{aligned}$$

We will also need the case where the Dirac operator is further coupled to a second vector bundle E of rank possibly greater than one; in this case the requisite index formula is

$$\begin{aligned} \text{index}\,(\not{\partial}_{L^p\otimes E}) &= \text{ch}\,(L^p \otimes E)\hat{A}(X)[X] & \text{(A.27)}\\ &= \text{ch}\,(E)\exp\,[pc_1(L)]\,\hat{A}(X)[X],\ (p = -q - m/2). & \text{(A.28)}\end{aligned}$$

In the next section we treat an actual spin^c example in four dimensions.

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[3] We could equally have used instead the formula for index $(\bar{\partial}_{L^p})$ which would have involved $ch\,(L)$ and the Todd class $td(X)$. In fact this realisation of the Dirac operator as $\bar{\partial}_{L^p}$ enables one to easily understand why index $(\not{\partial}_{L^{-q-m/2}})$ is equal to unity for $q = 0$: it is because, when $q = 0$, index $(\bar{\partial}_{L^{-m/2}})$ gives the arithmetic genus $\sum(-1)^s h^{0,s}$ of the complex manifold X where the Hodge number $h^{r,s}$ denotes the dimension of the space of holomorphic forms of type (r,s). Now for the manifolds X we consider in this paper the only holomorphic forms are of type (s,s) a fact which reduces the arithmetic genus to $h^{0,0}$ which is trivially unity.

B. Cohomology and spin^c on $\mathbf{CP}^2$

On $\mathbf{CP}^2$ the Chern class is [25]

$$c(\mathbf{CP}^2) = 1 - 3c_1(L) + 3c_1^2(L) \tag{B.1}$$

where the generating line bundle L has $c(L) = 1 + c_1(L)$ with $-c_1(L)$ generated by the Kähler 2-form. The Euler characteristic is 3 so $c_1^2(L)[\mathbf{CP}^2] = 1$ and in this case $c_1(\mathbf{CP}^2) = -3c_1(L)$ so $m = 3$. Since the coefficient of $c_1(L)$ is odd $w_2 \neq 0$ and $\mathbf{CP}^2$ does not admit a spin structure. The index of the Dirac operator coupled to L^p is

$$\begin{aligned} \text{index}\,(\not{\partial}_{L^p}) &= \text{ch}\,(L^p)\hat{A}(X)[X] = \exp\left[pc_1(L)\right]\hat{A}(X)[X] && \text{(B.2)} \\ &= \left(1 + pc_1(L) + \frac{1}{2}p^2c_1^2(L)\right)\left(1 - \frac{p_1(X)}{24}\right)[X], \quad X = \mathbf{CP}^2 && \text{(B.3)} \\ &= \frac{1}{8}(4p^2 - 1), && \text{(B.4)} \end{aligned}$$

where p_1 is the Pontrjagin class and $p_1(X) = c_1^2(X) - 2c_2(X) = 3c_1^2(L)$ on $\mathbf{CP}^2$. This index is integral for half-integral p and, setting $p = -q - 3/2$, we obtain

$$\text{index}\,(\not{\partial}_{L^p}) = \frac{1}{2}(q+2)(q+1). \tag{B.5}$$

We can define a non-trivial rank 2 bundle F over $\mathbf{CP}^2$ with structure group $U(2)$ by $F \oplus L \cong I^3$ where I^3 is the trivial rank 3 bundle. Then $c(F)\,c(L) = 1$ so $c_1(F) = -c_1(L)$ and $c_2(F) = c_1^2(L)$; tensoring this with p copies of the generating line bundle L then gives, for the Chern character,

$$\begin{aligned} ch(L^p \otimes F) &= ch(L^p)ch(F) && \text{(B.6)} \\ &= \left(1 + pc_1(L) + \frac{1}{2}p^2c_1^2(L) + \cdots\right)\left(2 - c_1(L) - \frac{1}{2}c_1^2(L) + \cdots\right) && \text{(B.7)} \\ &= 2 + (2p-1)c_1(L) + \left(p^2 - p - \frac{1}{2}\right)c_1^2(L) + \cdots, && \text{(B.8)} \end{aligned}$$

leading to

$$\begin{aligned} \text{index}\,(\not{\partial}_{L^p\otimes F}) &= \text{ch}\,(F \otimes L^p)\hat{A}(X)[X] = \exp\left[pc_1(L)\right]\text{ch}\,(F)\hat{A}(X)[X] && \text{(B.9)} \\ &= \left(2 + (2p-1)c_1(L) + \left(p^2 - p - \frac{1}{2}\right)c_1^2(L)\right)\left(1 - \frac{p_1}{24}\right)[X] \\ &= \frac{1}{4}(2p-3)(2p+1), \quad X = \mathbf{CP}^2 && \text{(B.10)} \\ &= (q+1)(q+3), \qquad \text{again using } p = -q - 3/2. && \text{(B.11)} \end{aligned}$$

C. Cohomology and spin^c in six dimensions

In this section X is the complex manifold given by

$$X = \frac{Sp(2)}{U(2)} \tag{C.1}$$

whose real dimension is 6. The cohomology ring of X is generated by the even dimensional classes $\sigma_1 \in H^2(X;\mathbf{Z})$ and $\sigma_2 \in H^4(X;\mathbf{Z})$ subject to the single relation

$$\sigma_1^2 = 2\sigma_2. \tag{C.2}$$

Now X is not a spin^c manifold because we can compute that

$$\begin{aligned} c(X) &= 1 + c_1(X) + c_2(X) + c_3(X) && \text{(C.3)}\\ &= 1 + 3\sigma_1 + 8\sigma_2 + 4\sigma_1\sigma_2 && \text{(C.4)}\\ \Rightarrow c_1(X) &= 3\sigma_1 = -3c_1(L). && \text{(C.5)} \end{aligned}$$

We note that σ_1 generates $H^2(X;\mathbf{Z})$ and so deduce that $c_1(X)$ is odd and so

$$w_2(X) \neq 0 \Rightarrow X \text{ is not spin.} \tag{C.6}$$

We also see that

$$K = L^3 \tag{C.7}$$

so that the integer m of appendix A is equal to 3.

The index of the Dirac operator $\partial\!\!\!/_{L^p}$ can now be computed from the expansions of $ch\,(L^p)$ and $\hat{A}(X)$ giving us the formula

$$\begin{aligned} \text{index}\,(\partial\!\!\!/_{L^p}) &= \left(1 + pc_1(L) + \frac{1}{2}p^2c_1^2(L) + \cdots\right)\left(1 - \frac{p_1(X)}{24} + \cdots\right)[X] && \text{(C.8)}\\ &= \left(-pc_1(L)\frac{p_1(X)}{24} + \frac{1}{3!}p^3c_1^3(L)\right)[X]. && \text{(C.9)} \end{aligned}$$

But we can calculate that

$$\begin{aligned} p_1(X) &= c_1^2(X) - 2c_2(X) && \text{(C.10)}\\ &= 9\sigma_1^2 - 16\sigma_2, && \text{(C.11)} \end{aligned}$$

with $\sigma_1 = -c_1(L)$. Hence we find that

$$\begin{aligned} \text{index}\,(\partial\!\!\!/_{L^p}) &= \left(\frac{p}{24}\sigma_1(9\sigma_1^2 - 16\sigma_2) - \frac{1}{3!}p^3\sigma_1^3\right)[X] && \text{(C.12)}\\ &= -(4p^3 - p)\frac{\sigma_1^3}{24}[X] && \text{(C.13)}\\ &= -\frac{1}{12}(4p^3 - p) = -\frac{1}{12}p(2p-1)(2p+1), && \text{(C.14)} \end{aligned}$$

where we have used the Gauss–Bonnet theorem which says that

$$c_3(X)[X] = \chi(X) \tag{C.15}$$
$$= 4 = 2\sigma_1^3[X] \tag{C.16}$$

to deduce that $\sigma_1^3[X] = 2$. Before finishing we should check that the index is integral. Recall that

$$p = -q - m/2, \quad q \in \mathbf{Z},\ m = 3 \tag{C.17}$$

This fact immediately gives us the formula

$$\text{index } (\partial\!\!\!/_{L^p}) = \frac{1}{6}(2q+3)(q+1)(q+2),\ q \in \mathbf{Z} \tag{C.18}$$

and this is easily checked to give an integer index for integral q as it should.

If we tensor product with a further rank 2 bundle F, with $c_1(F) = \sigma_1$ then we find that

$$\text{index } (\partial\!\!\!/_{L^p \otimes F}) = ch(L^p \otimes F)\hat{A}(X)[X] \tag{C.19}$$
$$= ch(L^p)\, ch(F)\hat{A}(X)[X] \tag{C.20}$$
$$= -\frac{1}{12}(2p+3)(2p-1)(2p+1) \tag{C.21}$$
$$= \frac{2}{3}q(q+1)(q+2), \quad p = -q - 3/2,\ q \in \mathbf{Z} \tag{C.22}$$

and again this gives an integral index.

D. Cohomology and generalised spinors for a 12 dimensional Grassmannian.

In this section X is the 12 dimensional Grassmannian given by

$$X = \frac{U(5)}{U(3) \times U(2)}. \tag{D.1}$$

X is a perfectly standard complex manifold (of complex dimension 6) and its cohomology ring $H^*(X; \mathbf{Z})$ has 3 generators

$$\sigma_i \in H^{2i}(X; \mathbf{Z}),\ i = 1, 2, 3 \tag{D.2}$$

which obey the single relation

$$\sigma_3 = 2\sigma_1\sigma_2 - \sigma_1^3. \tag{D.3}$$

Its Chern class is given by

$$c(X) = (1 + c_1(X) + c_2(X) + c_3(X) + c_4(X) + c_5(X) + c_6(X)) \tag{D.4}$$
$$= (1 - 5\sigma_1 + 12\sigma_1^2 - 15\sigma_1^3 + 8\sigma_1^4 + 2\sigma_1^2\sigma_2 + 7\sigma_2^2 + 4\sigma_1^5 - 25\sigma_1\sigma_2^2 \tag{D.5}$$
$$-29\sigma_1^6 + 7\sigma_1^2\sigma_2^2 + 56\sigma_1^4\sigma_2 - 27\sigma_2^3) \tag{D.6}$$

from which we see that

$$c_1(X) = -5\sigma_1 \tag{D.7}$$

and hence we deduce, as we did in the previous section, that

$$w_2(X) \neq 0 \tag{D.8}$$

and so X is not spin.

We now pass to the spin^c bundle $S^c(X)$ and to the calculation of the index of its Dirac operator $\partial\!\!\!/_{L^p}$ where L is the generating line bundle as it was in the previous section. But this time we need the fact that σ_1 is actually a *negative generator* of $H^2(X, \mathbf{Z})$ with our orientation conventions and so we have

$$c_1(X) = -5\sigma_1, \quad \sigma_1 \text{ negative}, \quad \sigma_1 = c_1(L) \tag{D.9}$$

$$\Rightarrow K = L^5 \quad (m = 5) \tag{D.10}$$

$$\hat{A}(X) = \left(1 - \frac{p_1(X)}{24} + \frac{1}{5760}(7p_1^2(X) - 4p_2(X))\right. \tag{D.11}$$

$$\left. - \frac{1}{2^{10} \cdot 945}(16p_3(X) - 44p_1(X)p_2(X) + 31p_1^3(X)) + \cdots \right) \tag{D.12}$$

as well as

$$p_1(X) = c_1^2(X) - 2c_2(X) = \sigma_1^2 + 2\sigma_2 \tag{D.13}$$

$$p_2(X) = -2c_1(X)c_3(X) + c_2^2(X) + 2c_4(X) = 10\sigma_1^4 - 20\sigma_1^2\sigma_2 + 15\sigma_2^2 \tag{D.14}$$

$$p_3(X) = 2c_1(X)c_5(X) - 2c_2(X)c_4(X) + c_3^2(X) - 2c_6(X) \tag{D.15}$$

$$= 51\sigma_1^6 + 72\sigma_1^2\sigma_2^2 - 144\sigma_1^4\sigma_2 + 68\sigma_2^3. \tag{D.16}$$

This information allows to compute that

$$\text{index}\,(\partial\!\!\!/_{L^p}) = \exp\left[pc_1(L)\right] \hat{A}(X)[X] \tag{D.17}$$

$$= -\frac{1}{60480}\sigma_2^3[X] - \left(\frac{41}{15120} + \frac{1}{360}p^2\right)\sigma_1^2\sigma_2^2[X] \tag{D.18}$$

$$+ \left(\frac{353}{161280} + \frac{3}{320}p^2 - \frac{1}{288}p^4\right)\sigma_1^4\sigma_2[X] \tag{D.19}$$

$$+ \left(-\frac{407}{967680} - \frac{11}{3840}p^2 - \frac{1}{576}p^4 + \frac{1}{720}p^6\right)\sigma_1^6[X]. \tag{D.20}$$

Now use the cohomology generators and the fact that X clearly has Euler characteristic 10 we discover that

$$\sigma_2^3[X] = 1 \tag{D.21}$$

$$\sigma_1^2\sigma_2^2[X] = 2 \tag{D.22}$$

$$\sigma_1^4\sigma_2[X] = 3 \tag{D.23}$$

$$\sigma_1^6[X] = 5. \tag{D.24}$$

This all gives the formulae

$$\text{index }(\not{\partial}_{L^p}) = -\frac{1}{1024} + \frac{19}{2304}p^2 - \frac{11}{576}p^4 + \frac{1}{144}p^6 \tag{D.25}$$

$$= \frac{1}{9.2^{10}}(4p^2 - 9)(4p^2 - 1)^2 \tag{D.26}$$

$$= \frac{1}{144}(q+1)(q+2)^2(q+3)^2(q+4), \quad \text{using } p = -q - 5/2, \tag{D.27}$$

and this index is an integer for integral q as required.

We shall finish by calculating the index when we couple the Dirac operator to some higher rank bundles. We shall give the results for two bundles E and F which are naturally associated to X and also for the tensor product $E \otimes F$.

Let E be the rank 3 vector bundle over

$$X = \frac{U(5)}{U(3) \times U(2)} \tag{D.28}$$

whose fibre over a point $x \in X$ is the 3-plane x itself. This describes the bundle E. Now consider the product rank 5 bundle $X \times \mathbf{C^5}$ then F is the rank 2 bundle created by forming the quotient

$$\frac{X \times \mathbf{C^5}}{E}. \tag{D.29}$$

The bundles E and F satisfy

$$E \oplus F \cong I^5 \tag{D.30}$$

where I^5 is a trivial rank 5 bundle and it is not difficult to work out that

$$c(E)c(F) = 1 \tag{D.31}$$

$$\text{i.e.} \quad (1 + c_1(E) + c_2(E) + c_3(E))(1 + c_1(F) + c_2(F)) = 1 \tag{D.32}$$

$$ch(E) + ch(F) = 5. \tag{D.33}$$

In fact equation (D.33) can be used to derive the relation (D.3) since the classes σ_i are just the classes $c_i(E)$ and so this allows all of $c(E)$ and $c(F)$ to be expressed in terms of the σ_i.

The Chern characters of E and F are given by

$$\begin{aligned} ch(E) = 3 &+ c_1(E) + \frac{1}{2}\left(c_1^2(E) - 2c_2(E)\right) + \frac{1}{3!}\left(c_1^3(E) - 3c_1(E)c_2(E) + 3c_3(E)\right) + \frac{1}{4!}\left(c_1^4(E)\right.\\ &\left.-4c_1^2(E)c_2(E) + 4c_1(E)c_3(E) + 2c_2^2(E)\right) + \frac{1}{5!}\left(c_1^5(E) - 5c_1^3(E)c_2(E) + 5c_1^2(E)c_3(E)\right.\\ &\left.+5c_1(E)c_2^2(E) - 5c_2(E)c_3(E)\right) + \frac{1}{6!}\left(c_1^6(E) - 6c_1^4(E)c_2(E) + 6c_1^3(E)c_3(E)\right.\\ &\left.+9c_1^2(E)c_2^2(E) - 12c_1(E)c_2(E)c_3(E) - 2c_2^3(E) + 3c_3^2(E)\right)\\ ch(F) = 5 &- ch(E). \end{aligned}$$

Now we can calculate the index of the appropriate Dirac operators: Forming the product $L^p \otimes E$ we have

$$\text{index}\,(\partial\!\!\!/_{L^p\otimes E}) = \text{ch}\,(L^p \otimes E)\hat{A}(X)[X] \tag{D.34}$$

$$= \text{ch}\,(E)\exp\,[pc_1(L)]\,\hat{A}(X)[X], \tag{D.35}$$

and we find that

$$\text{index}\,(\partial\!\!\!/_{L^p\otimes E}) = -\frac{15}{1024} + \frac{3}{128}p + \frac{59}{768}p^2 - \frac{5}{48}p^3 - \frac{5}{64}p^4 + \frac{1}{24}p^5 + \frac{1}{48}p^6$$

$$= \frac{1}{3.2^{10}}(2p+5)(2p-1)(4p^2-9)(4p^2-1) \tag{D.36}$$

$$= \frac{1}{48}q(q+1)(q+2)(q+3)^2(q+4),\ (p=-q-5/2) \tag{D.37}$$

and for the product $L^p \otimes F$

$$\text{index}\,(\partial\!\!\!/_{L^p\otimes F}) = \frac{5}{512} - \frac{3}{128}p - \frac{41}{1152}p^2 + \frac{5}{48}p^3 - \frac{5}{288}p^4 - \frac{1}{24}p^5 + \frac{1}{72}p^6$$

$$= \frac{1}{9.2^9}(2p-5)(2p-1)(4p^2-9)(4p^2-1) \tag{D.38}$$

$$= \frac{1}{72}(q+1)(q+2)(q+3)^2(q+4)(q+5),\ (p=-q-\frac{5}{2}). \tag{D.39}$$

Finally for the bundle $L^p \otimes E \otimes F$ we have

$$\text{index}\,(\partial\!\!\!/_{L^p\otimes E\otimes F}) = -\frac{25}{512} - \frac{25}{384}p + \frac{103}{384}p^2 + \frac{13}{48}p^3 - \frac{29}{96}p^4 - \frac{1}{24}p^5 + \frac{1}{24}p^6$$

$$= \frac{1}{3.2^9}(4p^2-25)(2p-3)(4p^2-1)(2p+1) \tag{D.40}$$

$$= \frac{1}{24}q(q+2)^2(q+3)(q+4)(q+5),\ (p=-q-5/2) \tag{D.41}$$

and in each case one can verify that the index is an integer.

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