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N=2 Supersymmetric Yang-Mills and the Quantum Hall Effect

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Abstract

It is argued that there are strong similarities between the infra-red physics of N=2 supersymmetric Yang-Mills and that of the quantum Hall effect, both systems exhibit a hierarchy of vacua with a subgroup of the modular group mapping between them. The β-function for pure $SU(2)$ $N=2$ supersymmetric Yang-Mills in 4-dimensions is re-examined and an earlier suggestion in the literature, that was singular at strong coupling, is modified to a form that is well behaved at both weak and strong coupling and describes the crossover in an analytic fashion. Similarities between the phase diagram and the flow of SUSY Yang-Mills and that of the quantum Hall effect are then described, with the Hall conductivity in the latter playing the role of the θ-parameter in the former. Hall plateaux, with odd denominator filling fractions, are analogous to fixed points at strong coupling in N=2 SUSY Yang-Mills, where the massless degrees of freedom carry an odd monopole charge.

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1 Introduction

In this paper arguments will be given that there there are strong similarities in the infra-red physics of $N = 2$ supersymmetric Yang-Mills and that of the quantum Hall effect. A common feature of these two systems is the emergence of modular symmetry (strictly a sub-group of the modular group) in the infra-red regime. That the modular group may be relevant to the quantum Hall effect was first suggested in [1], though the correct subgroup was only found later [2].

In the infra-red limit both theories can be parameterised by a complex parameter whose real part is the co-efficient of a topological term in the effective action describing the infra-red physics and whose imaginary part is essentially the co-efficient of the kinetic term, which must be positive. In $N = 2$ SUSY the complex parameter is $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$ with θ the QCD vacuum parameter and g^2 the Yang-Mills coupling (in units with $\hbar = c = 1$), in the quantum Hall effect the complex parameter is $\sigma = \sigma_{xy} + i\sigma_{xx}$ where σ_{xy} is the Hall conductivity and σ_{xx} the Ohmic conductivity. In both cases the complex parameter is constrained to lie in the upper-half complex plane for stability reasons.

In the low energy effective action for supersymmetric Yang-Mills θ is proportional to the co-efficient of the topological term $F \wedge F$ and $1/g^2$ is proportional to the co-efficient of the kinetic F^2 term [3]. In the low energy effective action for the electromagnetic field in quantum Hall effect σ_{xy} is proportional to the co-efficient of the Chern-Simons term and σ_{xx} is related to the co-efficient of the effective kinetic term [4] (in a conducting medium the dielectric constant is the co-efficient of $\vec{E}.\vec{E}$ in the long wavelength effective action and has a simple pole at zero frequency, the Ohmic conductivity is the residue of the pole).

Both systems have a hierarchy of phases: different strong coupling vacua of SUSY Yang-Mills on the one hand and different quantum Hall plateaux on the other. These different phases are mapped into each other by the action of a sub-group of the modular group.

There are also remarkable similarities between the scaling flow of the complex couplings in both systems and the main focus of this paper is to examine this relation. Mathematically the link is the modular group.

In section 2 β-functions for $N = 2$ SUSY Yang-Mills without matter are discussed and a flow of the effective complex coupling as the Higgs VEV is varied is constructed which is compatible with $\Gamma_0(2)$ symmetry and reduces to the Callan-Symanzik β-function near the fixed points. The flow is a modification of the flow described in [5], that had singularities at strong coupling, and these singularities are avoided in the flow proposed here. Striking similarities with the temperature flow of the conductivities of the quantum Hall effect are described in section 3, together with other parallels between these two systems.

2 $N = 2$ SUSY $SU(2)$ Yang-Mills

After the seminal results of Seiberg and Witten [3], in which the low energy effective action for $N = 2$ supersymmetric Yang-Mills theory in 4-dimensions was explicitly constructed, β-functions were derived explicitly for particular gauge groups and matter content in [5]. The case of pure $SU(2)$ Yang-Mills was taken further in [6]. The β-function can be constructed in terms the complex coupling $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$ and the vacuum expectation value of the Higgs field, or better the gauge invariant mass scale, $u = (1/2)tr < \varphi^2 >$, where φ is the Higgs field in the adjoint of $SU(2)$. A non-zero vacuum expectation value for φ gives the $W^{\pm}$-bosons a mass and breaks the gauge symmetry down to $U(1)$. The β-function introduced in [5] was essentially the logarithmic derivative of the Seiberg-Witten low energy effective coupling $\tau(u)$ with respect to u,

$$\beta_0(\tau) = -u\frac{d\tau}{du} = \frac{1}{2\pi i}\left(\frac{1}{\vartheta_3^4(\tau)} + \frac{1}{\vartheta_4^4(\tau)}\right), \tag{1}$$

where $\vartheta_3(\tau) = \sum_{n=-\infty}^{\infty} e^{i\pi n^2\tau}$ and $\vartheta_4(\tau) = \sum_{n=-\infty}^{\infty}(-1)^n e^{i\pi n^2\tau}$ are Jacobi ϑ-functions (the reason for the subscript 0 on β_0 will be evident momentarily). In the definitions of [7] the β-function (1) is a modular function of the group $\Gamma_0(2)$, of weight -2.[1]

Although (1) does take into account all non-perturbative effects, it is only valid in the weak-coupling regime and was criticized in [8, 9] since it is unphysical at strong coupling, giving a singularity at $\tau = 0$. The β-function (1) also has an attractive fixed point at $u = 0$ ($\tau = \frac{1+i}{2}$ and its images under $\Gamma_0(2)$) which arises because the flow is defined to be radially inwards towards the origin in the u-plane. We can change this and put the attractive fixed point anywhere in the u-plane, without affecting the behaviour at infinity, by defining

$$\beta_{u_0}(\tau) = -(u - u_0)\frac{d\tau}{du}. \tag{2}$$

[1] $\Gamma_0(2)$ is the sub-group of the full modular group $\Gamma(1)$ consisting of matrices $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$ with a, b, c and d integers, $ad - bc = 1$ and c restricted to be even.

From (1) it is straightforward to express (2) as an explicit function of τ using elementary properties of ϑ-functions (see *e.g.* [10]). The effective coupling can be written in terms of elliptic integrals [11]

$$\tau = i\frac{K'(k)}{K(k)} + 2n \tag{3}$$

where $K(k) = \int_0^{2\pi} \frac{1}{\sqrt{1-k^2\cos^2\phi}} d\phi$ is the complete elliptic integral of the first kind, $k^2 := \frac{2\Lambda^2}{\Lambda^2+u}$ with Λ^2 the QCD scale, $K'(k) = K(k')$ with $k'^2 = 1 - k^2$ and n is an integer (τ is not uniquely determined by k, rather $e^{i\pi\tau} = e^{-\pi\frac{K'}{K}}$). In terms of $K(k)$ the θ-functions are

$$\vartheta_3(\tau) = \sqrt{\frac{2K(k)}{\pi}} \qquad \text{and} \qquad \vartheta_4(\tau) = \sqrt{\frac{2k'K(k)}{\pi}}. \tag{4}$$

Since $k^2 = 2\Lambda^2/(u + \Lambda^2)$ we obtain

$$\frac{\vartheta_4^4(\tau)}{\vartheta_3^4(\tau)} = 1 - k^2 = \frac{u - \Lambda^2}{u + \Lambda^2} \qquad \Rightarrow \qquad \frac{u}{\Lambda^2} = \frac{\vartheta_3^4 + \vartheta_4^4}{\vartheta_3^4 - \vartheta_4^4}. \tag{5}$$

Combining this with (1) gives

$$\beta_{\tilde{u}_0}(\tau) = -(u - u_0)\frac{d\tau}{du} = \frac{1}{2\pi i}\left(\frac{1 - \tilde{u}_0}{\vartheta_4^4} + \frac{1 + \tilde{u}_0}{\vartheta_3^4}\right) \tag{6}$$

with $\tilde{u}_0 = u_0/\Lambda^2$.

The shift $\tau \to \tau + 1$ is equivalent to $u_0 \to -u_0$ (this follows from the property of Jacobi ϑ-functions, $\vartheta_3(\tau + 1) = \vartheta_4(\tau)$) which in turn is a manifestation of the the Z_2 action on the u-plane familiar from [3]. Since (6) is not invariant under this shift when $\tilde{u}_0 \neq 0$ it is not a modular function of $\Gamma_0(2)$, rather it is a modular function of weight -2 for the smaller group $\Gamma(2)$.[2] The singularity at $\tau = 0$ present in β_0 can be avoided by choosing $\tilde{u}_0 = 1$, while that at $\tau = 1$ is avoided by using $\tilde{u}_0 = -1$. In either case the flow runs along the semi-circle spanning $\tau = 1$ to $\tau = 0$, this semi-circle is the straight-line segment in the u-plane running between $u = +\Lambda^2$ and $u = -\Lambda^2$ and all its images under $\Gamma(2)$ are also semi-circles in the upper-half τ-plane.

Consider $\tilde{u}_0 = +1$ (a similar analysis applies to $\tilde{u}_0 = -1$),

$$\beta_{+1}(\tau) = \frac{1}{\pi i}\frac{1}{\vartheta_3^4(\tau)}. \tag{7}$$

[2] $\Gamma(2)$ consists of matrices $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$ with a, b, c and d integers, $ad - bc = 1$ and both b and c even.

The resulting flow in the τ-plane shown in figure 1 and is symmetric under translations in τ by 2, $\tau \to \tau \pm 2$. Any starting value of τ at weak coupling $u = \infty$ with $-2\pi < \theta < 2\pi$ is driven to $\theta = 0$ at strong coupling so consider the positive imaginary axis in the τ-plane $\theta = 0$, where $\tau = i\frac{4\pi}{g^2}$. In figure 2 the function β_{+1} is plotted as a function of $Im(\tau) = \frac{4\pi}{g^2}$ along the positive imaginary axis, $\theta = 0$. Note the constant value $-1/\pi$ at small g, consistent with asymptotic freedom behaviour $u\frac{dg}{du} \approx -\frac{g^3}{8\pi^2}$.

For large g, $\beta_{+1} \approx \frac{i\tau^2}{\pi}$ is perfectly well behaved and in terms of the dual coupling g_D, defined by $\tau_D = -\frac{1}{\tau}$ so $\tau_D = i\frac{4\pi}{g_D^2}$ when $\theta = 0$, one finds $-(u - \Lambda^2)\frac{dg_D}{du} \approx -\frac{g_D^3}{8\pi^2}$ in agreement with the expectations for the dual theory [3]. β_{+1} for the theory and its dual are mapped onto each other using the identity $\vartheta_3(-1/\tau) = \sqrt{-i\tau}\vartheta_3(\tau)$ giving

$$\beta_{+1}(\tau_D) = \frac{i}{\pi}\frac{1}{\vartheta_3^4(\tau_D)}. \tag{8}$$

This therefore is a well behaved β-function which is defined for both the the theory and its dual and gives the crossover between weak and strong coupling. It was suggested in [8] that the zero of β_0 at $u = 0$ was spurious and that a better β-function would be

$$\beta(\tau) = G(\tau)\beta_0(\tau) \tag{9}$$

where $G(\tau)$ is a renormalisation factor with a pole at $u = 0$ which accounts for the unphysical zero in β_0. We see here that $\beta_{+1} = \left(\frac{u-\Lambda^2}{u}\right)\beta_0$ is indeed related to β_0 by a simple pole in the u variable at $u = 0$.

Near $u \approx \infty$ and $u \approx +\Lambda^2$ the function β_{+1} behaves like a Callan-Symanzik β-function and this is the physical difference from the β-function considered in [5]. At weak coupling, where $u \approx \infty$, the squarks have a mass M proportional to the the VEV of the Higgs, $< \varphi >= a$, and $u \approx a^2/2$ so β_{+1} gives

$$\beta_{+1} \approx -u\frac{d\tau}{du} \approx -a^2\frac{d\tau}{da^2} \approx -M^2\frac{d\tau}{dM^2}. \tag{10}$$

Near $\tau \approx 0$, where $u \approx +\Lambda^2$, the dual Higgs VEV, a_D, goes like $a_D \propto (u - \Lambda^2)/\Lambda$ and the monopole mass is $M_D = \sqrt{2}a_D$ so β_{+1} gives

$$\beta_{+1} \approx -(u - \Lambda^2)\frac{d\tau}{du} \approx -a_D\frac{d\tau}{da_D} \approx -\frac{M_D^2}{2}\frac{d\tau}{dM_D^2}. \tag{11}$$

Apart from a factor of 2, which can be viewed as arising from an anomalous dimension of φ^2 at strong coupling, β_{+1} can be interpreted as a Callan-Symanzik β-function, at least near the two dual points $\tau = i\infty$ and $\tau = 0$.

There is however still a pathology in β_{+1} since there is singularity at $u = -\Lambda^2$ where $\tau = 1$. This can be avoided by using β_{-1} to describe the crossover from $u = -\infty$ to $u = -\Lambda^2$ and the discussion exactly parallels that for β_{+1} except that τ is replaced with $\tau+1$ or equivalently ϑ_3 is replaced with ϑ_4. The flow is that of figure 1 with the horizontal axis displaced by one unit in either direction. Any starting value $0 < \theta < 4\pi$ at weak coupling is driven to 2π at strong coupling. There is no holomorphic β-function compatible with $\Gamma(2)$ symmetry which has no singularities at all, since any modular function of weight -2 must have at least one singularity somewhere within, or on the boundary of, the fundamental domain — at best the β-function is meromorphic.

In fact one can do better and define a β-function that gives the correct behaviour at all three singular points in the u-plane (from now on we shall set $\Lambda = 1$ so these singularities are at $u = \infty$, $u = +1$ and $u = -1$). We want to construct a β-function that is meromorphic in u and vanishes at both $u = \pm 1$, without disturbing the behaviour $\beta \approx -\frac{i}{\pi}$ at $u \approx i\infty$ and this can be done using the ideas of Ritz in [6]. If we wish to avoid extraneous poles or zeros the only possibility is

$$\beta = -\left(\frac{(u-1)^m(u+1)^n}{u^{m+n}}\right) u\frac{d\tau}{du} \tag{12}$$

with m and n positive integers.

Demanding the correct asymptotic behaviour at $u \approx \pm 1$ requires $m = n = 1$ so

$$\beta(\tau) = -\frac{(u^2-1)}{u}\frac{d\tau}{du} = \frac{2}{\pi i}\frac{1}{(\vartheta_3^4(\tau) + \vartheta_4^4(\tau))} \quad \underset{\tau\to i\infty}{\longrightarrow} \quad \frac{1}{\pi i}. \tag{13}$$

This flow is shown in figure 3. The behaviour near $\tau = 0$ is

$$\beta(\tau) \approx \frac{2i}{\pi}\tau^2 \tag{14}$$

or, in terms of the dual coupling τ_D,

$$\beta(\tau_D) \to \frac{2i}{\pi} \tag{15}$$

as $\tau_D \to i\infty$. This is twice $\beta_{+1}(\tau_D)$ as $\tau_D \to i\infty$, because of the pre-factor $u + 1 \to 2$ as $u \to 1$. It is therefore a factor of 4 greater than Callan-Symanzik β-function at this fixed point, the other factor of 2 coming from the anomalous dimension of φ^2 as before.

By construction (13) is symmetric under $\tau \to \tau + 1$ and so is a modular function for $\Gamma_0(2)$, just as β_0 was, but now there is a singularity at $\tau = \frac{1+i}{2}$ (where $\vartheta_3^4 = -\vartheta_4^4$) corresponding to a repulsive fixed point, rather than the attractive fixed point of β_0.

There is an infinite hierarchy of crossovers near the real axis in figure 3 as a consequence of $\Gamma_0(2)$ symmetry. Under the action of an element $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $\Gamma_0(2)$, $\tau \to \frac{a\tau+b}{c\tau+d}$ (with c even and $ad-bc=1$) the point $\tau = i\infty$ is mapped to $\gamma(\tau) = a/c$, $\tau = 0$ to $\gamma(\tau) = b/d$ and $\tau = 1$ to $\gamma(\tau) = (a+b)/(c+d)$. As observed in [3], the infinite hierarchy of vacua at strong coupling can thus be classified into three types, in terms of $\Gamma_0(2)$ these are: images of $\tau = i\infty$ with $\theta/2\pi = a/c$, a rational number with even denominator; images of $\tau = 0$ with $\theta/2\pi = b/d$, a rational number with odd denominator and images of $\tau = 0$ with $\theta/2\pi = (a+b)/(c+d)$ again a rational number with odd denominator. The fermionic degrees of freedom at $\tau = i\infty$ are squarks with electric charge $+1$ and magnetic charge 0, at $\tau = 0$ they are monopoles with electric charge zero and magnetic charge $+1$ while at $\tau = 1$ they are dyons with electric charge -1 and magnetic charge $+1$. In general, at a strong coupling fixed point, $\theta/2\pi = -q/m$ where q is the electric charge and m is the magnetic charge of the massless fermionic degrees of freedom, which are composite objects in terms of the relevant degrees of freedom at weak coupling.

Let us now look a little more closely at the flow structure generated by β near the real axis in figure 3. As u varies between $+\infty$ and $+1$, β generates a semi-circle between two states on the real line, $\theta_1/2\pi = -q_1/m_1$ and $\theta_2/2\pi = -q_2/m_2$ (we shall assume that q_1 and m_1 are mutually prime, similarly for q_2 and m_2). This semi-circle can be obtained from the the positive imaginary axis in the τ-plane, with end points $i\infty$ and 0, by the action of some $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2)$, with a and d odd and c even. Thus $q_1/m_1 = -a/c$ and $q_2/m_2 = -b/d$, so $q_1 = \pm a$ is odd and $m_1 = \mp c$ even, $q_2 = \pm b$ is of undetermined parity while $m_2 = \mp d$ is odd. Since $ad - bc = 1$ we see that

$$q_1 m_2 - q_2 m_1 = \pm 1 \tag{16}$$

and we have a selection rule for transitions between vacua in the strong coupling regime as u is varied.

The flow from $u = -1$ to $u = +1$, which is the semi-circular arch spanning $\tau = 1$ to $\tau = 0$ in the τ-plane, cannot be obtained from the flow along the imaginary axis in the τ-plane by using $\Gamma_0(2)$ so we consider this separately. This semi-circle is mapped to another semi-circle linking $(a+b)/(c+d)$ to b/d: that is linking $\theta_1/2\pi = -q_1/m_1 = (a+b)/(c+d)$ (so m_1 is odd) to $\theta_2/2\pi = -q_2/m_2 = b/d$ (so m_2 is odd). Again $q_1 m_2 - q_2 m_1 = \pm 1$.

This selection rule is related to the Schwinger-Zwanziger quantisation rule for a pair of Dyons with charges (Q_1, M_1) and (Q_2, M_2) [12]:

$$Q_1 M_2 - Q_2 M_1 = 4\pi n \tag{17}$$

with n an integer.[3] Writing $Q_i = q_i g$ and $M_i = m_i \frac{4\pi}{g}$, with q_i and m_i integers, this is

$$q_1 m_2 - q_2 m_1 = n \tag{18}$$

and the selection rule (16) dictates that flow always connects the nearest pairs allowed by the Schwinger-Zwanziger quantisation rule.

3 The Quantum Hall Effect

The interpretation of the infinite hierarchy of states for N=2 SUSY Yang-Mills presented in the previous section is very similar to the hierarchy of states observed in the quantum Hall effect (a connection between $N = 2$ SUSY Yang-Mills and the quantum Hall effect was suggested in [13]). For the quantum Hall effect the complex coupling τ is replaced with a complex conductivity, $\sigma = \sigma_{xy} + i\sigma_{xx}$, with σ_{xy} the Hall conductivity and σ_{xx} the Ohmic conductivity (for an isotropic layer with $\sigma_{xx} = \sigma_{yy}$). At Hall plateaux (where $\sigma_{xx} = 0$) the Hall conductivity is quantised, in units in which $e^2/h = 1$, as a rational number $\sigma_{xy} = q/m$ where m is odd. These plateau are attractive fixed points of a scaling flow, even denominators being repulsive, [15, 16]. The suggestion in [1] that the modular group might be relevant to the quantum Hall effect was further developed in [2, 13, 14, 18, 19, 20]. Indeed the group $\Gamma_0(2)$ has an action known as the "Law of Corresponding States" in the condensed matter literature [4, 21], which holds under the assumption of well separated Landau levels with completely spin polarised electrons (when spin effects are important it was argued in [22] that the $\Gamma_0(2)$ symmetry is broken to $\Gamma(2)$).

Constraints on the β-functions for the quantum Hall effect, as a result of modular symmetry, were discussed in [14]. If meromorphicity is assumed stronger statements can be made and a meromorphic flow diagram was presented in [13], which developed the original flow suggested in [15]. This meromorphic flow is given by[4]

$$\beta(\sigma) = \frac{2}{\pi i} \frac{1}{(\vartheta_3^4(\sigma) + \vartheta_4^4(\sigma))} \tag{19}$$

[3]More generally n could be half-integral, but in pure SUSY Yang-Mills we are always dealing with $U(1)$ charges coming from the adjoint representation of $SU(2)$ so it is integral in this case.

[4]Up to an undetermined constant which is chosen to agree with (13) here.

and is identical to figure 3 though the physical interpretation is different. For the quantum Hall effect the flow lines are in the direction of increasing effective system size (in practice this can be translated to decreasing temperature) [17] and different lines correspond to different values of the external magnetic field. As the temperature is lowered and the flow runs down from large σ_{xx}, σ is driven onto a semi-circular arch in the complex σ-plane, such as the semi-circle connecting $\sigma = 2$ to $\sigma = 1$ in figure 3 for example. On this semi-circle, there is a second order quantum phase transition, as $T \to 0$, at $\sigma = (3 + i)/2$. At or close to $T = 0$, σ becomes a function of a single scaling variable, $\sigma(\Delta B/T^{\mu})$, where $\Delta B = B_c$ is the deviation of the magnetic field from its critical value and μ a scaling exponent for the temperature [15, 16, 17]. In this regime of very low temperatures σ_{xy} runs between 2 and 1 as B is varied at fixed T.

The picture is the same at all copies of the semi-circle under the action of $\Gamma_0(2)$, the exponent μ is believed to be the same for every transition, a phenomenon known as 'super-universality', and there is good experimental evidence that this is indeed the case [23] (the experimental situation is a little murky on this point however, in some experiments scaling appears to be violated at low temperatures [24]). Under the assumption of $\Gamma_0(2)$ symmetry the critical points above the real axis are fixed points of $\Gamma_0(2)$[5] and so their positions can easily be calculated for the crossover between any two given plateaux.

The derivation of the $\Gamma_0(2)$ flow in [13] was made under the following assumptions: i) in the long wavelength limit the flow should commute with the action of $\Gamma_0(2)$, in particular this implies that any point which is a fixed point of $\Gamma_0(2)$ should also be a fixed point of the flow; ii) there are no fixed points of the flow that are not fixed points of $\Gamma_0(2)$; iii) the β-functions are modular forms of weight -2, in the sense of [7] (in particular they are meromorphic); iv) due to the stability of the quantum Hall plateaux, the flow should approach the real axis at rational numbers with odd denominators (attractive fixed points of the flow) as fast as possible; v) the flow should be vertically downwards when the Ohmic conductivity is large.

Assumption iii) is an assumption about the analyticity properties of flow. Mapping $\sigma \to \gamma(\sigma) = (a\sigma + b)/(c\sigma + d)$ we have, under any variation $\delta\sigma$ of σ,

$$\delta(\gamma(\sigma)) = \frac{1}{(c\sigma + d)^2}\delta\sigma \tag{20}$$

since $ad - bc = 1$, so $\beta(\sigma)$ is automatically a modular function of weight -2 if it is meromorphic. The general form of the flow does not depend crucially

[5] i.e. there exists an element of $\Gamma_0(2)$ which leaves the point invariant.

on assumption iii), provided i) and ii) hold small deformations away from meromorphicity cannot change the topology of the flow or even the position of the fixed points — any such deformation would smoothly distort the lines of figure 3 but must leave the fixed points and the topology invariant. Experimental flow diagrams are in remarkable agreement with the $\Gamma_0(2)$ predictions: the flow diagram in [25] found for the integer quantum Hall effect is reproduced in figure 4 and that found in [26] for the fractional effect in figure 5.

Another consequence of $\Gamma_0(2)$ symmetry for the quantum Hall effect is the semi-circle law. Semi-circles in the upper-half complex plane, spanning certain pairs of rational points on the real axis, are mapped into one another by the action of the modular group and so are rather special curves. Experimentally the crossover between two plateaux, as the external magnetic field is varied at fixed low temperature, is often very close to a semi-circle [27]. In the condensed matter literature this is known as the "semi-circle law" and it can be interpreted as a consequence of $\Gamma_0(2)$ symmetry. Even without assuming any meromorphicity properties, just assuming that $\Gamma_0(2)$ commutes with the flow and there is a symmetry between the pseudo-particles and the accompanying holes, the semi-circle law for the crossover between quantum Hall plateaux can be derived [28].

$\Gamma_0(2)$ symmetry also implies a selection rule $q_1 m_2 - q_2 m_1 = \pm 1$ for transitions between quantum Hall plateaux as the magnetic field is varied, [29]. This rule is only expected to hold for two well formed plateaux with no hint of unresolved sub-structure between and is very well supported by experimental data on the quantum Hall effect when this is the case, at least for quantum Hall monolayers with the spins well split.[6]

The selection rule and the semi-circle law do not require any assumptions about the analytic properties of the flow — they only require that the action of $\Gamma_0(2)$ commutes with it.

Another similarity between $N = 2$ SUSY Yang-Mills and the quantum Hall effect is that, in the composite boson picture of the quantum Hall effect described in [4], the effective degrees of freedom in a state with $\sigma_{xy} = 1/m$ are electrons with m-units of magnetic flux attached with m odd, that is the quasi-particles are composite objects with an odd number of vortices attached to fundamental charged particles.

While, as a mathematical theory, $N = 2$ SUSY has an infinite hierarchy of vacuum phases, the quantum Hall effect, as a physical phenomenon, does

[6] In figure 4, which is interpreted as a transition from $1/1$ to $0/1$, the spins are degenerate, which doubles the degrees of freedom in the lowest Landau level and so doubles the conductivity [25].

not. Experimentally there are limitations to the applicability of $\Gamma_0(2)$ in any given sample and not all mathematically allowed phases will be seen. For very strong magnetic fields, with filling factors less than about 1/7, it is believed that the 2-dimensional electron gas will enter a new phase at $T = 0$, the Wigner crystal, which is not part of the modular hierarchy, though factors as low as 1/9 have been seen at finite temperature [30]. In addition only fractions up to a maximum denominator, depending on the sample, will be seen — other physical factors, such as impurities, limit how far into the hierarchy one can penetrate. Also if the temperature is too high the quantum Hall effect is destroyed so there is a limit as to how high up into the complex conductivity plane one can trust the flow derived by assuming $\Gamma_0(2)$ symmetry.

4 Conclusions

It has been argued that Callan-Symanzik β-functions can be defined for $N = 2$ supersymmetric Yang-Mills, without matter fields, which are modular forms of $\Gamma_0(2)$ and which reproduce the correct asymptotic flow, both at strong and at weak coupling. The flow is shown in figure 3 and there is one singular fixed point in the fundamental domain, at $u = 0$, where the classical theory would have full $SU(2)$ gauge symmetry restored but the $W^{\pm}$ bosons remain massive in the quantum theory. Not only does the quantum theory prevent the classical symmetry restoration, it actively avoids the point $u = 0$ in both flow directions. Massless dyons in the strong coupling regime have electric charge q and magnetic charge m with m odd for attractive fixed points and m even for repulsive fixed points (in the flow direction in which the Higgs mass is lowered) and $\theta = -q/m$ is a rational fraction as $g \to \infty$.

Exactly the same flow pattern has been predicted theoretically for the long wavelength physics of the quantum Hall effect, on the basis of $\Gamma_0(2)$ symmetry (the "Law of Corresponding States") and observed experimentally. In the quantum Hall effect the parameter governing the flow is temperature and odd denominator plateaux are attractive fixed points while even denominator plateaux are repulsive.[7] Attractive fixed points give rise to quantised Hall plateaux where the Hall conductivity $\sigma_{xy} = q/m$ is rational with odd denominator.

In both cases the effective degrees of freedom are composite objects in terms of the fundamental degrees of freedom — dyons in the case of SUSY

[7] There are exceptions to this at high Landau levels, the most famous being the 5/2 state. These states can be interpreted as being a due to Bose-Einstein condensation of pairs of composite Fermions and are not part of the $\Gamma_0(2)$ hierarchy.

Yang-Mills and composite fermions or composite bosons in the case of the quantum Hall effect. Composite bosons (odd denominator states) are charged particles carrying an odd number of magnetic vortices [4] and composite fermions (even denominator states) carry an even number of magnetic vortices [31] (in both cases the underlying particles are still fermionic).

It would seem that $N = 2$ supersymmetric Yang-Mills in $3+1$ dimensions has much in common with the quantum Hall effect in $2+1$ dimensions but exactly what the relation is, what are the essential features that are required to bring out modular symmetry in the effective action, is unclear at this stage, and there is scope for much work in the future to clarify this relation (the modular group was also found to play a role in in other models [32, 33] and was considered in $2 + 1$ dimensional abelian Chern-Simons theory in [34]). There is no suggestion here that supersymmetry is relevant to the quantum Hall effect, rather it would appear that some minimal set of criteria is necessary for modular symmetry to emerge at long wavelengths (general criteria for any system in $2 + 1$ dimensions were discussed in [20]). Supersymmetry is not essential, since nature has given us an experimental system which exhibits modular symmetry without it, and neither is Lorentz invariance. The number of dimensions can be either 3 or 4 (from the field theory point of view the quantum Hall effect is a 3-dimensional phenomenon, since it occurs at very low temperatures). Perhaps other dimensions are possible too. Certainly topological effects are essential and effective degrees of freedom that are composite objects in terms the fundamental degrees of freedom and the topologically non-trivial degrees of freedom are common to both systems.

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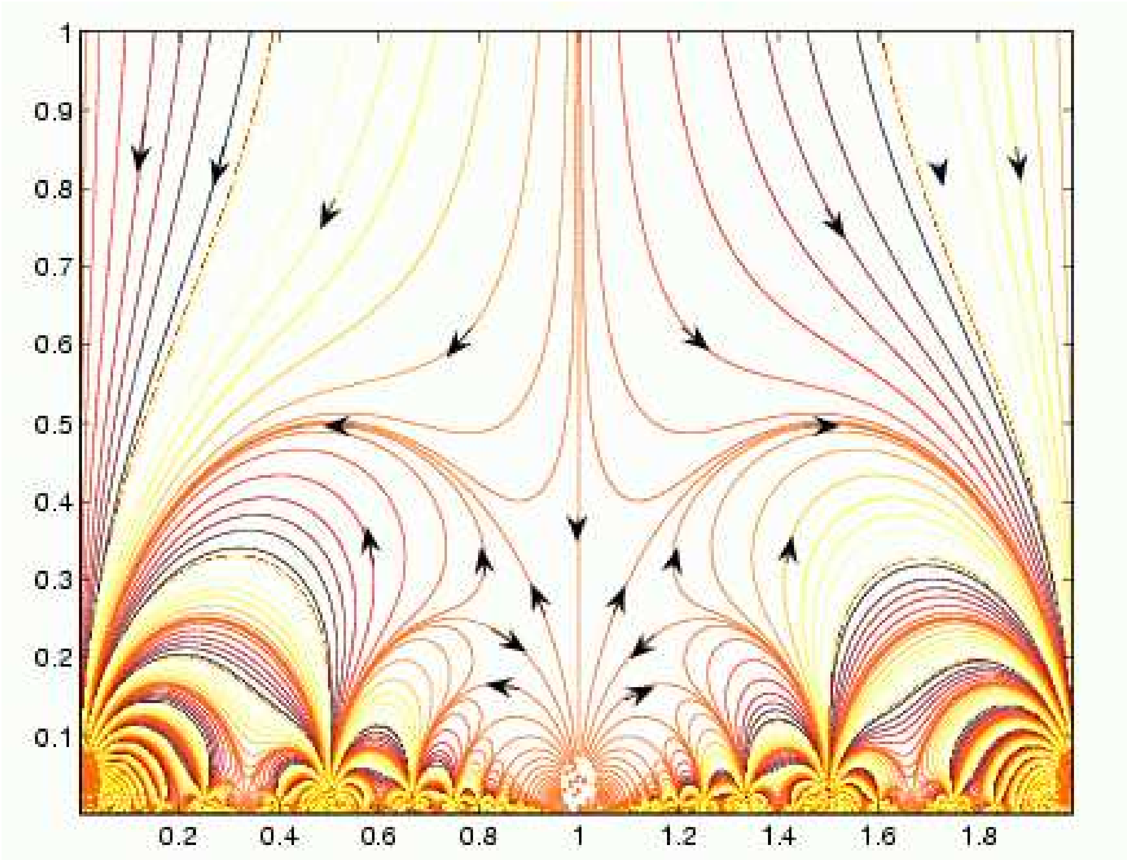

Fig. 1: Flow of effective coupling of $N = 2$ SUSY Yang-Mills, as the Higgs VEV is reduced, using (7). The pattern repeats under $\tau \to \tau + 2$. There is a singularity at $\tau = 1$ and its images under $\Gamma(2)$. Note that θ is driven to zero at strong coupling for any starting value between -2π and 2π at weak coupling.

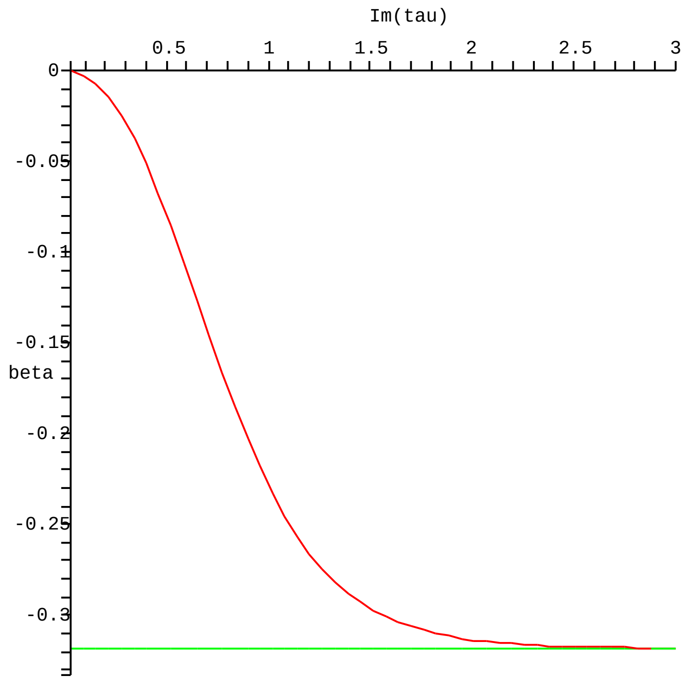

Fig. 2: Crossover of Imτ from weak to strong coupling along the imaginary axis $\theta = 0$.

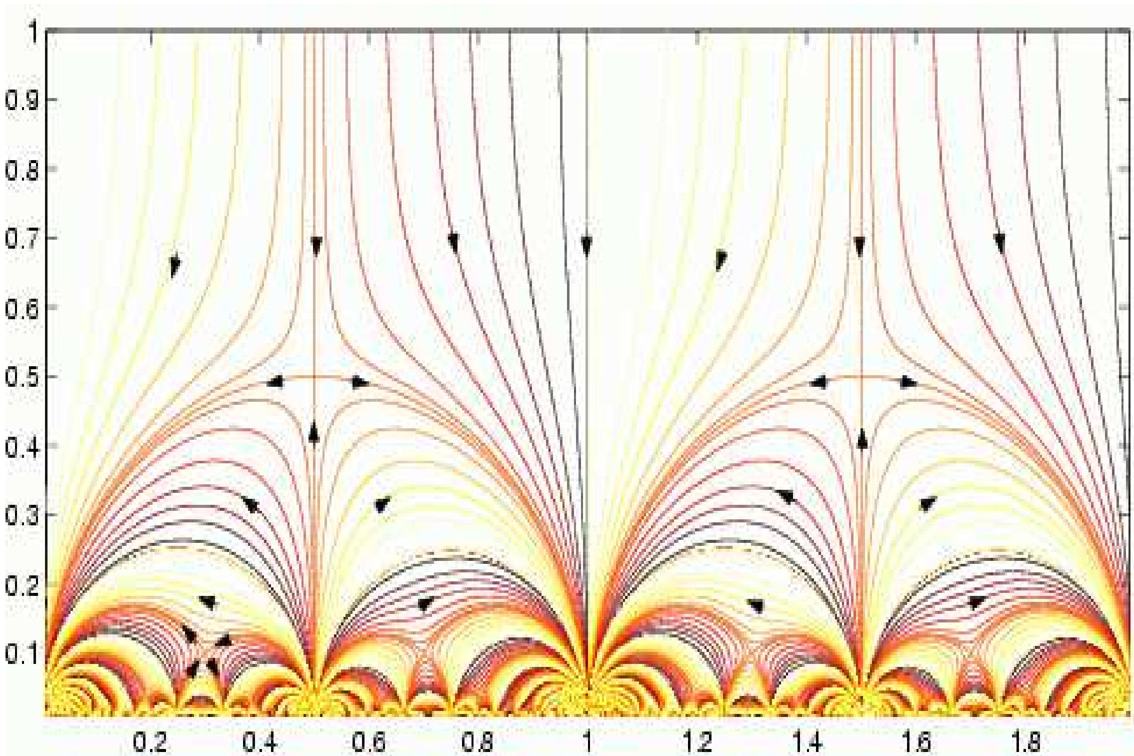

Fig. 3: Flow of effective coupling for $N = 2$ SUSY Yang-Mills, as the Higgs VEV is reduced, using (13). The same flow applies to the complex conductivity of the quantum Hall effect as the temperature is lowered. The pattern repeats under $\tau \to \tau + 1$. Note the repulsive fixed points at $(1+i)/2$ and its images under $\Gamma_0(2)$.

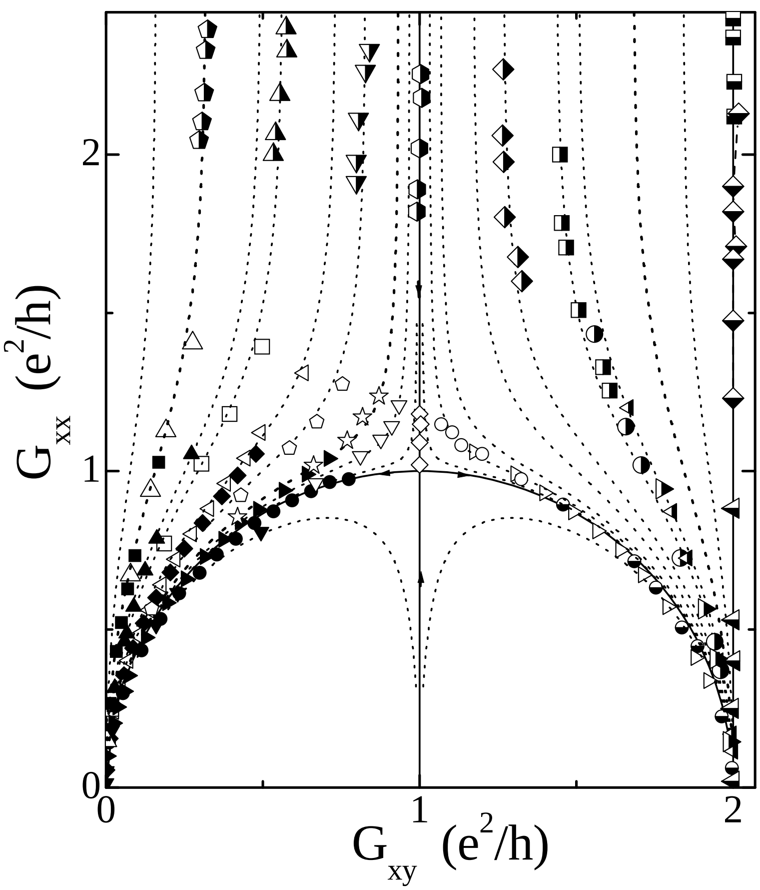

Fig. 4: As the temperature is lowered the conductivity flows down from $\sigma_{xx} = \infty$, different flow lines correspond to different magnetic fields. The dotted lines are the flow following from $\Gamma_0(2)$ symmetry and meromorphicity (equation (19)) and the symbols are experimental data. (The conductivities are twice those in the text due to spin degeneracy — the Landau levels in the sample used here are spin degenerate, so conductivities are multiplied by 2 and $\Gamma_0(2)$ acts on $\sigma/2$ rather than on σ.) Figure reproduced from [25].

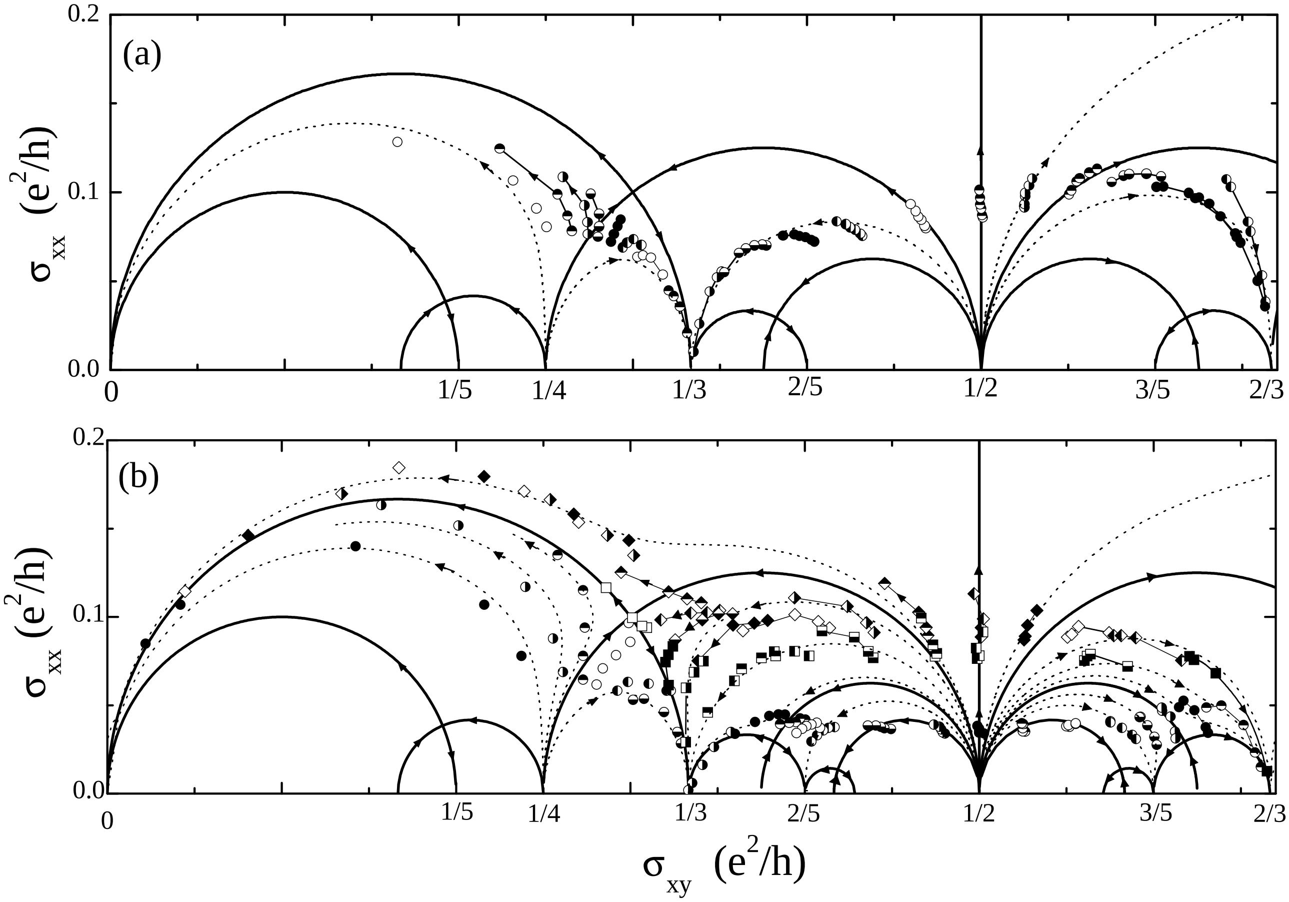

Fig. 5: Temperature flow for the fractional quantum Hall effect. The upper and lower figures, (a) and (b), represent two different samples. Dashed and solid lines are the flow following from $\Gamma_0(2)$ symmetry and meromorphicity (equation (19)) and the symbols are experimental data. Note the repulsive fixed point at $\sigma = 1/2$ (figures taken from [26]).