Zero-Field Hall Effect in (2+1)-dimensional QED

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Abstract

In three-dimensional QED, a quantum Hall effect occurs in the absence of any magnetic field. We give a simple and transparent explanation. In solid-state physics, the Hall conductivity for rational magnetic flux is expected to be given by a Chern number. In our field-free situation, however, the conductivity is $\pm 1/2$ in natural units. We explain why the integrality of the conductivity breaks down and explain its quantization geometrically. For quasi-periodic boundary conditions, we calculate the finite size correction to the Hall conductivity. Our paper establishes an explicit connection between quantum fied theory and solid-state physics.

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I. INTRODUCTION

The quantum Hall effect (QHE) in graphene (graphite monolayers) is a promising candidate for applications in nanoelectronics. Its quasiparticle excitations can be described approximately by a two-dimensional relativistic electron system [1]. Some modifications of graphene with suitable internal magnetic fields might make the application of strong external fields unnecessary [2].
In this paper we investigate a Hall effect of three-dimensional quantum electrodynamics (QED) which occurs in the absence of any magnetic field. The corresponding off-diagonal conductivity is $\pm 1/2$ in natural units. We will derive and discuss the result for the Dirac vacuum state of a non-interacting electron system.
The history of this result goes back at least to the articles by Redlich [3] and Jackiw [4], which focused on massless non-abelian gauge theory in 2+1 dimensions, but included discussions of QED with a mass term. The Hall conductivity for a homogeneous external electromagnetic field is implied by their calculations. An initial normalization error by a factor of 1/2 disappeared in [5]. The half-integrality of the result required the resolution of a paradox, since general arguments suggest integral values [2, 6]. It was stated that when spin is included, the conductivity is doubled [4], but the corresponding Zeeman term may induce more complicated changes [2]. The result for zero magnetic field has been stated explicitely only in [7]. However, this derivation is purely computational and unnecessarily complicated and does not give any insight into the half integral nature of the result. In particular, a straightforward geometrical interpretation of the fractional value does not exist in the literature. A clear explanation should prevent the common errors by factors of 2 in QHE calculations.
Our starting point is the massive ($m \neq 0$) Dirac equation

$$[-i(\partial_\mu + ieA_\mu)\gamma^\mu + m]\psi = 0, \tag{1}$$

where e is the electron charge and $\boldsymbol{A}$ denotes an abelian external electromagnetic field. We use the convention $\gamma^\mu := \sigma_3\sigma_\mu$, for $\mu = 0, 1, 2$, where σ_μ is the μ-th Pauli matrix, and $\{\gamma^\mu\gamma^\nu\} = 2g^{\mu\nu}$ for $g_{\mu\nu} = diag(1, -1, -1)$. In a homogeneous background field $F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$, the ground state current is [5, 8]

$$\langle j_\mu \rangle = \frac{1}{8\pi}\, sgn(m)\, \varepsilon_{\mu\nu\eta}(eF^{\nu\eta}). \tag{2}$$

When $\mu = 1, 2$, this equation becomes the Ohm-Hall law with Hall conductivity $\sigma_H := \sigma_{21}/e^2 = \frac{1}{2} sgn(m)\frac{1}{h}$. The crucial observation is that by (2), σ_H is independent of the magnetic field strength F^{12}, which reveals a *zero-field Hall effect* [9]. From the point of view of solid state physics this is uncommon, since usually a magnetic field perpendicular to the plane is necessary in order to obtain a nonzero σ_H. If the Hamiltonian is the Schrödinger operator with $U(1)$ gauge field $\vec{A}$, then σ_H vanishes indeed if $\vec{A}$ does [10]. However, a time reversal breaking term may in general suffice to produce a nonzero Hall conductivity [2].
Several attempts have been made to find a solid state analog of (2+1)-dimensional electrodynamics. In the planar honeycomb lattice introduced by Semenoff [11], the generation mechanism of a Hall current fails only due to fermion doubling. The addition of a local magnetic flux density normal to the plane results in $\sigma_H \neq 0$ [2]. Though a magnetic field is again necessary, the sophisticated model yields a zero net magnetic flux through the unit cell. More recently, the nonmagnetic Dirac operator of a nearest neighbour discretization

was shown to project down onto two Schrödinger operators with magnetic fluxes π and $-\pi$, respectively, in a tight-binding model. Here the mass term is the obstruction for a vanishing of σ_H in the double-layer system [12].
These models will be of considerable practical relevance, when a zero-field Hall effect can be produced in modified carbon monolayers, as envisaged in [11, 13]. At present, a strong external magnetic field is indispensable for the experimental realization of quantum Hall phenomena in graphite [14, 15]. Very recent results on graphene can be found in [16].
In our context it is sufficient to consider a relativistic electron gas without interaction between the electrons, since higher loop terms do not contribute [17]. In particular, the ground state is not degenerate. In solid state physics, this is the decisive argument for predicting the integrality of $2\pi\sigma_H$ (when $\hbar \doteq 1$). Its actual deviation from being integral needs to be explained.
Our result does not contradict this general property. A single relativistic fermion cannot approximate the behaviour of a system with a finite number of degrees of freedom per volume. Indeed, the use of such systems in lattice gauge theory always produces an even number of relativistic fermions. The presence of spectator fermions [7, 18] reestablishes an integer QHE. When a band gap closes and reopens, the change in the Hall conductivity remains integral [19, 20].
In the first section of this paper the geometric nature of the half-integral value of $2\pi\sigma_H$ is discussed. In the second part, the Kubo formula is used in the context of three-dimensional QED. In particular, this yields the corrections to σ_H in a finite area with periodic boundary conditions. Moreover, we shall see that for our system the old time-ordered perturbation methods are equivalent to the relativistically covariant ones, but more efficient. This is very useful for systems with boundaries, where Lorentz invariance does anyhow not apply [21].
The Hamiltonian of the Dirac equation (1) is given by

$$H^{\boldsymbol{A}} := -i(\vec{\nabla} + ie\vec{A}) \cdot \vec{\sigma} + \sigma_3 m + A_0, \tag{3}$$

acting on the smooth functions in $\mathcal{H} := L^2(\mathbb{R}^2, \Delta_2)$, where $\Delta_2 = \mathbb{C}^2$ is the spinor space. Time inversion is implemented by the anti-unitary operator $U_T : \mathcal{H} \to \mathcal{H}$, defined by $U_T(\psi) := \sigma_2\bar{\psi}$. Here $\bar{\psi}$ is the complex conjugate of ψ. In absence of an external magnetic field, (3) reduces to

$$H := -i\vec{\nabla} \cdot \vec{\sigma} + m\sigma_3. \tag{4}$$

The key observation is that the mass term of (4) breaks time reversal symmetry: $U_T \circ H \circ U_T^{-1} = -i\vec{\nabla} \cdot \vec{\sigma} - \sigma_3 m \neq H$.

II. GEOMETRY OF THE ZERO-FIELD HALL EFFECT

To calculate the Hall conductivity for the the nonmagnetic Dirac operator (4), we restrict to wave functions on a finite torus $T = \mathbb{R}^2/\Lambda$, for some lattice $\Lambda \cong \mathbb{Z}^2$. This leads to the direct integral decomposition of $\mathcal{H}$ over the dual torus $T^* := \mathbb{R}^2/\Lambda^*$ (here $\Lambda^* := \{\vec{K} \in \mathbb{R}^2 \mid \vec{K} \cdot \vec{R} \in 2\pi\mathbb{Z}, \forall \vec{R} \in \Lambda\}$ is the dual lattice), where T^* parametrizes the quasi-periodic boundary conditions. Thus wave functions for $\vec{k} \in T^*$ decompose as $\psi(\vec{x}) = e^{i\vec{k}\cdot\vec{x}}u(\vec{x})$ with $u \in \mathcal{H}' := L^2(T, \mathbb{C}^2)$ acted upon by $H(\vec{k})$, the conjugate of H by $e^{-i\vec{k}\cdot\vec{x}}$. Fourier transformation maps $\mathcal{H}'$ onto $\ell^2(\vec{k}+\Lambda^*)\otimes\mathbb{C}^2$, transforming $H(\vec{k})$ into $\oplus_{\vec{K}\in\Lambda^*}((\vec{k}+\vec{K})\cdot\vec{\sigma}+m\sigma_3)$.

The contribution of the positive resp. negative energies of $H(\vec{k})$ to the conductivity $\sigma_{\mu\nu}(\vec{k})$ is given by

$$\sigma_{21}^{(\pm)}(\vec{k}) = ie^2 \sum_{\vec{K}\in\Lambda^*} \mathrm{Tr}_{\mathbb{C}^2}\left(\widehat{P}^{(\pm)}_{\vec{k}+\vec{K}}\left[\partial_{k_1}\widehat{P}^{(\pm)}_{\vec{k}+\vec{K}}, \partial_{k_2}\widehat{P}^{(\pm)}_{\vec{k}+\vec{K}}\right]\right) \tag{5}$$

$$= \mp\frac{e^2}{2}\sum_{\vec{K}\in\Lambda^*}\frac{m}{[(\vec{k}+\vec{K})^2+m^2]^{3/2}}. \tag{6}$$

Here $\widehat{P}^{(\pm)}_{\vec{k}+\vec{K}}$ denotes the spectral projector to the energy $E^{(\pm)}_{\vec{K}}(\vec{k}) := \pm[(\vec{k}+\vec{K})^2+m^2]^{1/2}$ (the hat referring to Fourier space). Using Poisson summation, we obtain

$$\sigma_{21}^{(\pm)}(\vec{k}) = \mp\,\frac{e^2}{4\pi}\,\mathrm{sgn}(m)\sum_{\vec{R}\in\Lambda} e^{-|m||\vec{R}|-i\vec{k}\cdot\vec{R}}.$$

The Hall conductivity on the infinite plane can be obtained either by taking $\Lambda\to\infty$ (i.e., all lattice periods large) or by averaging over the boundary conditions, which yields

$$2\pi\sigma_H^{(\pm)} = \frac{1}{2\pi}\int_{T^*}\sigma_H^{(\pm)}(\vec{k})\,d^2k = \mp\,\frac{1}{2}\,\mathrm{sgn}(m). \tag{7}$$

We want to understand the geometric origin of this result. Averaging over the boundary conditions in (5) results in replacing the sum in (6) by an integral over $\mathbb{R}^2$, with an additional factor of $(2\pi)^{-2}$. We extend to $\mathbb{R}^3$ by writing $\widehat{H}(\boldsymbol{k}) = \boldsymbol{k}\cdot\boldsymbol{\sigma}$ for $\boldsymbol{\sigma} := (\sigma_j)_{j=1}^3$ and $\boldsymbol{k}\in F_m := \{\boldsymbol{k}\in\mathbb{R}^3 | k_3 = m\}\cong\mathbb{R}^2$. By (6), the two-form $(2\pi)^{-1}\sigma_H^{(\pm)}(\vec{k})\,dk_1\wedge dk_2$ generalizes naturally to

$$\eta^{(\pm)} := \mp\frac{1}{8\pi}\,\varepsilon^{\alpha\beta\gamma}\,\frac{k_\alpha dk_\beta\wedge dk_\gamma}{|\boldsymbol{k}|^3},$$

which is rotationally invariant. By homogeneity of $\eta^{(\pm)}$ and the fact that $\int_{S^2}\eta^{(+)} = -1$, we obtain

$$\int_{F_m\subset\mathbb{R}^3}\eta^{(+)} = \int_{\tilde{F}_m\subset S^2}\eta^{(+)} = -\frac{1}{2}\mathrm{sgn}(m).$$

Here $\tilde{F}_m\subset S^2$ denotes for $m>0$ and $m<0$ the open upper and lower half-sphere, respectively, onto which the hypersurface F_m projects homeomorphically. This proves (7) geometrically, describing $\sigma_H^{(\pm)}$ as a solid angle. The half-spheres are orbits of the Lorentz group, so that the quantization of $2\pi\sigma_H^{(\pm)}$ follows from Lorentz invariance.

We assume, as for the rest of this paper, that the Fermi energy lies in a spectral gap and that the temperature is zero. In this situation, the Hall conductivity is classically an integer. Namely, the one-particle states ideally form a line bundle of eigenspaces of the Hamiltonian over T^*, and the corresponding contribution to the Hall conductivity is the Chern number of this bundle [6, 22, 23], which is an integer. In our system, the one-particle states under consideration cannot be separated into such line bundles, since degeneracies occur over T^*. Therefore, the argument doesn't apply.
Another way to understand the deviation from integrality is from the point of view of multi-particle states. Here the argument is more subtle. If we take $P\equiv\widehat{P}^{(-)}_{\vec{k}}$ to be the

projector onto the multi-fermion ground state, which we assume again to be non-degenerate, then

$$\sigma_H^{(-)}(\vec{k}) = i\mathrm{Tr}(P[\partial_{k_1}P, \partial_{k_2}P])$$

is the curvature of the adiabatic connection $P \circ \nabla_{\vec{k}}$ as long as the particle number is finite, and its integral yields a topological number [2, 11]. In QED$_3$, non-degeneracy is assured by the Pauli exclusion principle. But our example shows that the argument breaks down for an infinite number of particles. Indeed, one can construct easily examples with arbitrary σ_H by changing the solid angle discussed above. Of course, this procedure breaks Lorentz invariance.

In solid state physics, when a one-particle description is possible, the set of momenta forms a closed surface in momentum space, which can be described as a union of half-spheres of the type considered above. Thus our calculation is consistent with the integer result in [2], due to fermion doubling.

The Dirac see ground state cannot quite be described as a line bundle over T^*. If, for $\vec{k} \in T^*$, $|\vec{k}+\vec{K}\rangle \in \mathcal{H}'$ denotes the eigenfunction of $(\vec{k}+\vec{K})\cdot\vec{\sigma} + m\sigma_3$ to energy $E_{\vec{K}}(\vec{k}) < 0$, then the multi-particle ground state is $|0(\vec{k})\rangle \equiv \wedge_{\vec{K}\in\Lambda^*}|\vec{k}+\vec{K}\rangle$. Here the order of the wedge factors cannot be fixed in a consistent way: The energy crossings over T^* give rise to an infinite number of pairwise interchanges of wedge factors when $\vec{k}$ runs around a period. This yields a sign ambiguity, so that only by taking the square, we obtain a line bundle. The latter is trivial, since obviously it possesses a nowhere vanishing global section. Note that the sign ambiguity does not affect P and the formula for σ_H.

III. THE KUBO FORMULA IN QED$_3$

In solid-state physics, the Hall conductivity $\sigma_H(\boldsymbol{x})$ (at a fixed time) is derived by "old-fashioned"perturbation theory in the Schrödinger picture. We have seen that this approach is very convenient for our system, too. For more general problems in QED$_3$, e.g. non-homogeneous external fields, it is instructive to relate the resulting Kubo formula to the standard relativistically invariant treatment. In first order perturbation theory one has

$$\delta\langle 0|j_\mu(\boldsymbol{x})|0\rangle = \langle 0|j_\mu(\boldsymbol{x})(E_0 - H)^{-1}(P_{|0\rangle})^{\perp}\delta H|0\rangle + c.c., \tag{8}$$

where $P_{|0\rangle}$ denotes the projector onto the ground state. In a finite volume with quasiperiodic boundary conditions the vacuum state can be written as a wedge product over $\vec{K} \in \Lambda^*$ of one-particle ground states $|\vec{k}+\vec{K}\rangle$ in $\mathcal{H}' = L^2(T, \mathbb{C}^2)$, as discussed above in Fourier space. For $\delta H(\vec{x})\,|_{\mathcal{H}'} := -ex_1E^1$, variation of (8) w.r.t. the electric field gives, after division by e^2, the Kubo-formula for the Hall-conductivity at zero temperature

$$\sigma_H(\boldsymbol{x},\vec{k}) = -2\int_{\mathbb{R}^2}\sum_{E_{\vec{K}}<0}\sum_{E_{\vec{N}}>0}\Im\left(\frac{\langle\vec{k}+\vec{K}|v_2(\boldsymbol{x},\vec{k})|\vec{k}+\vec{N}\rangle\langle\vec{k}+\vec{N}|v_1(\vec{x}\,',\vec{k})|\vec{k}+\vec{K}\rangle}{[E_{\vec{K}}(\vec{k}) - E_{\vec{N}}(\vec{k})]^2}\right)d^2x',$$

where $v_\mu = e^{-1}j_\mu$ is the velocity operator. Note that, by the Pauli exclusion principle, the interior sum runs over positive energies only, and symmetry of the spectrum is considered in our notation.

For a better understanding of the Kubo-formula in QED, and a result for $\sigma_H(\boldsymbol{x})$, we have to

rewrite the r.h.s. of equat. (8) as an integral over the entire space-time. The Dirac operator (4) does not depend on time, and changing from Schrödinger into Heisenberg picture,

$$
\begin{aligned}
\delta\langle 0|j_\mu(\boldsymbol{x})|0\rangle &= -i\int_{-\infty}^{\infty}\langle 0|T\left[j_\mu(\boldsymbol{x})(P_{|0\rangle})^{\perp}\delta H(t)\right]|0\rangle\, dt \\
&= -i\delta e\left(\int_{\mathbb{R}^3}\langle 0|T[j_\mu(\boldsymbol{x})\Phi(\boldsymbol{x}')]|0\rangle\, d^3x'\right)_{\text{regul.}} \\
&= \delta e\left(\int_{\mathbb{R}^3}\langle 0|j_\mu(\boldsymbol{x})\Phi(\boldsymbol{x}')|0\rangle_{\text{Euclid.}}\, d^3x'\right)_{\text{regul.}}, \qquad (9)
\end{aligned}
$$

using Wick rotation in (9). Here $\Phi(\boldsymbol{x}) = A^\nu(\boldsymbol{x})j_\nu(\boldsymbol{x})$ is the multi-particle operator with $A^0(\boldsymbol{x}) = -ex_1E^1$ and $A^j \equiv const.$ for $j = 1, 2$. (9) is the relativistically invariant version of the quantum mechanical Kubo formula in three-dimensional QED. To obtain the Hall conductivity, one needs [8, form. (2.5)]

$$[\langle 0|j^\mu(\boldsymbol{x})j^\nu(\boldsymbol{x}')|0\rangle_{\text{Euclid.}}]_{\text{regul.}} \sim \operatorname{sgn}(m)\,\varepsilon^{\mu\eta\nu}\frac{\partial}{\partial x^\eta}\,\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}').$$

Note that the $\operatorname{sgn}(m)$ factor on the r.h.s. is necessary, since a change in space-time orientation can be compensated by a sign change of m. Using

$$\langle j^\mu(\boldsymbol{x})\rangle := \frac{\delta S_{\text{eff}}[\boldsymbol{A}]}{e\delta A_\mu(\boldsymbol{x})},$$

the regularized effective action turns out to be, to order $O(e^3)$, the Chern-Simons action [5]

$$S_{CS}[\boldsymbol{A}] = \operatorname{sgn}(m)\frac{e^2}{8\pi}\varepsilon^{\mu\nu\eta}\int_{\mathbb{R}^3}(\partial_\mu A_\nu)A_\eta\, d^3x.$$

This proves (2) and in particular the result in (7).

IV. CONCLUSION

The Hall effect for relativistic massive fermions is important both for quantum field theory and condensed matter physics. It can be treated by the same formalism in both contexts, which allows to visualize both the close analogies and the profound differences of the two physical systems. The corresponding half integral value of of the Hall conductivity has an elegant geometric interpretation. Integrality breaks down since the quantum field theory does not allow for the existence of a global Hilbert space which is independent of the boundary conditions. This is analogous to the situation in Haag's theorem (for Lorentz invariant theories): There is no natural way to identify the Hilbert spaces of quantum field theories with different physical parameters. Currently, relevant experimental results are restricted to graphene, where reflection invariance requires the use of strong external magnetic fields. In suitable non-symmetric materials, however, the effect may not need such a field.

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