Determinant solution for the Totally Asymmetric Exclusion Process with parallel update.

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Abstract

We consider the totally asymmetric exclusion process in discrete time with the parallel update. Constructing an appropriate transformation of the evolution operator, we reduce the problem to that solvable by the Bethe ansatz. The non-stationary solution of the master equation for the infinite 1D lattice is obtained in a determinant form. Using a modified combinatorial treatment of the Bethe ansatz, we give an alternative derivation of the resulting determinant expression.

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I. INTRODUCTION

The first result on the Bethe ansatz solutions of the asymmetric simple exclusion process (ASEP) is to be probably attributed to the short remark by Dhar [1]. The detailed calculations were later given in a seminal paper of Gwa and Spohn [2], where the issue about the scaling of the next to the largest eigenvalues of the evolution operator was addressed. The subject has attracted a significant attention since then and a number of the exact results has become available. Among the results obtained with the Bethe ansatz, there are the scaling function describing the crossover of the next to the largest eigenvalue from the Kardar-Parisi-Zhang to the Edwards-Wilkinson regime [3], the large deviation function for the distance travelled by a particle [4],[5]. These results describe the behaviour of the system in the large time limit. For the finite time evolution, we first of all should mention the Schütz's determinant solution of the master equation for the ASEP on the infinite lattice [6]: the conditional probability of an arbitrary particle configuration, given an initial particle configuration, was obtained. This solution was later generalized to the ring geometry [7], to the ASEP with particle dependent velocities [8] and used to evaluate the distribution of the current of particles across the arbitrary bond [9] at the infinite lattice.

All the above mentioned results concern the continuous time ASEP, where the jump of a particle is described by the Poisson process. The continuous time solution can be straightforwardly extended to the discrete time dynamics with random sequential update [5]. The Bethe ansatz solution for the ASEP with backward ordered update was given in [10]. The history of studies of the ASEP with fully parallel update rises probably from 90-th, when it was widely discussed as a simplest version of the Nagel-Schreckenberg traffic model with maximal velocity of cars: $v_{\max} = 1$ [11]. In that case the approximate mean-field method proposed to find the stationary measure of the traffic flow models [12] resulted in the expression, which becomes exact in the thermodynamic limit. Later it was realized that the mapping to the zero range process (ZRP) allows one to represent the stationary measure of the parallel ASEP at the ring in form of the product measure [13]. In the case of open boundary conditions the stationary measure was obtained in form of the matrix product [14]. At the same time, the description of the time evolution of the ASEP with parallel update remained challenging unsolved problem for a long time. Recently, it has been shown that the parallel ASEP is a particular case of the general stochastic process with parallel

update, which is solvable by the Bethe ansatz [15].

In present article we use the Bethe ansatz to obtain the solution of the master equation for the ASEP with parallel update on an infinite lattice. Given an initial configuration, we obtain the probability of an arbitrary configuration of particles. The solution is expressed in terms of the determinant of the matrix, which depends on the initial and final configurations of particles and time. Though the general idea of the solution is similar to that of [6], the parallel dynamics leads to peculiarities, which make a generalization of previous results nontrivial. Below we give two independent derivations of the final result using two different treatments of the Bethe ansatz: the analytic representation based on the expansion of the solution over the Bethe ansatz eigenbasis and the geometric one based on the direct enumeration of particle trajectories. While the former reveals the peculiarities of the eigenspace of the evolution operator, the latter clarifies combinatorial aspects of the problem. Of course both methods lead to the same final result.

The article is organized as follows. In the Section II, we formulate the model and sketch the results of the article. In the Sections III and IV we derive the solution using the analytic and geometric Bethe ansatz respectively. In the Section V we summarize the results and discuss perspectives.

II. FORMULATION OF THE MODEL AND RESULTS.

Let us consider P particles on 1D infinite lattice. Below we imply that the particles move from left to right, which will be also referred to as a forward direction. At each step of the discrete time each particle may take one step forward with probability v or stay with probability $(1-v)$ provided that the next site is vacant. If the next site is occupied, the particle stays with probability 1. All sites are updated simultaneously at each time step. We define a configuration X of particles by a set of their coordinates $X = \{x_1, x_2, \ldots, x_P\}$. Below we imply them to be ordered:

$$-\infty < x_1 < x_2 < \cdots < x_P < \infty \tag{1}$$

The probability $P_t(X)$ for the system to be in a configuration X at time t obeys the Markov equation

$$P_{t+1}(X) = \sum_{\{X'\}} T(X, X') P_t(X'), \tag{2}$$

where $T(X, X')$ is the probability of the transition from X' to X for one time step. The transition probabilities $T(X, X')$ are defined by the above dynamical rules.

The goal of the article is to show that the conditional probability, $P(X, t|X^0, 0)$, for the system to be in a configuration $X = \{x_1, x_2, \ldots, x_P\}$ at time t, given it was in a configuration $X_0 = \{x_1^0, x_2^0, \ldots, x_P^0\}$ at time 0, is the ratio of two determinants

$$P(X, t|X^0, 0) = \frac{\det \mathbf{F}\left(X, X^0, t\right)}{\det \mathbf{F}\left(X, X, 0\right)}. \tag{3}$$

The matrix elements of the matrix $\mathbf{F}\left(X, Y, t\right)$, which depend on two configurations $X = \{x_1, x_2, \ldots, x_P\}$ and $Y = \{y_1, y_2, \ldots, y_P\}$ and on time t, are

$$\left(\mathbf{F}\left(X, Y, t\right)\right)_{i,j} = f(i - j, x_i - y_j, t), \tag{4}$$

where the function $f\left(a, b, t\right)$ is expressed in terms of the Gauss hypergeometric functions:

$$f(a, b, t) = (1 - v)^t (-1)^a \begin{cases} \left(\frac{v}{v-1}\right)^b \frac{(-t-a)_b}{b!} \, {}_2F_1\left(\begin{matrix} a, -t - a + b \\ b + 1 \end{matrix}; \frac{v}{v-1}\right) & b > 0 \\ \frac{(a)_{-b}}{(-b)!} \, {}_2F_1\left(\begin{matrix} a - b, -t - a \\ -b + 1 \end{matrix}; \frac{v}{v-1}\right) & b \leq 0 \end{cases}. \tag{5}$$

The notation $(a)_n$ means the shifted factorial $(a)_n = a(a + 1) \cdots (a + n - 1)$.

III. ANALYTIC APPROACH.

A. Hilbert space and the vector of state.

Let us introduce the Hilbert space supplied with the complete left and right bases consisting of vectors $\langle X|$ and $|X\rangle$ respectively, where X runs over all particle configurations, with inner product

$$\langle X|X'\rangle = \delta(X, X'). \tag{6}$$

The state of the system at any time step can be associated with the state vector

$$|P_t\rangle = \sum_{\{X'\}} P_t(X) \, |X\rangle \, . \tag{7}$$

In terms of the state vectors, the master equation takes a simple operator form

$$|P_{t+1}\rangle = \mathbf{T}\,|P_t\rangle\,, \tag{8}$$

where the evolution operator $\mathbf{T}$ is defined as follows

$$\mathbf{T} = \sum_{\{X\},\{X'\}} |X\rangle\, T(X,X')\,\langle X'|\,. \tag{9}$$

The conditional probability $P(X,t|X^0,0)$ can be represented as the matrix element

$$P(X,t|X^0,0) = \langle X|\,\mathbf{T}^t\,\left|X^0\right\rangle. \tag{10}$$

To evaluate matrix elements, we construct the set of left eigenvectors $|B_Z\rangle$ of the operator $\mathbf{T}$

$$\mathbf{T}\,|B_Z\rangle = \Lambda(Z)\,|B_Z\rangle \tag{11}$$

and the adjoint set of right eigenvectors $\left\langle\overline{B}_Z\right|$

$$\left\langle\overline{B}_Z\right|\mathbf{T} = \Lambda(Z)\left\langle\overline{B}_Z\right|, \tag{12}$$

which, as will be seen below, are parametrized by the P-fold parameter Z. We will prove that the set forms a complete basis, i.e. provides the expansion of the identity operator

$$\sum_Z \langle X|B_Z\rangle\left\langle\overline{B}_Z|X'\right\rangle = \langle X|X'\rangle\,. \tag{13}$$

As a result, the transition probability (10) we are looking for can be reduced to the evaluation of the sum

$$\langle X|\,\mathbf{T}^t\,\left|X^0\right\rangle = \sum_Z \left\langle X|\mathbf{T}^t|B_Z\right\rangle\left\langle\overline{B}_Z|X^0\right\rangle = \sum_Z \Lambda^t\,(Z)\,\langle X|B_Z\rangle\left\langle\overline{B}_Z|X^0\right\rangle. \tag{14}$$

B. Bethe ansatz.

The solution of the eigenproblem (11) corresponding to the master equation of the general integrable ASEP-ZRP model with parallel update was given in [15]. Here we sketch the solution for the particular case of the parallel ASEP defined above.

For further purposes it is convenient to introduce a notion of cluster of particles. A sequence of n occupied sites situated one after another with two empty sites on the ends

without any other empty sites between them will be referred to as a cluster of the length n. A single isolated particle is the cluster of the unit length. During the evolution an isolated particle can approach and join a cluster from behind resulting in decrease of the number of clusters at the lattice by one. The opposite process is when a cluster splits up by a first particle detaching the cluster ahead creating an isolated particle and increasing the number of clusters by one. If a particle jumps between two clusters, which are separated by only one empty site, the number of clusters does not change.

The explicit form of the transition probability $T(X, X')$ is a product of factors, each corresponding to a particular cluster of particles in the initial configuration X'. The value of these factors is either v or $(1-v)$ depending on whether or not the first particle of a given cluster jumps during the transition from X' to X.

$$T(X, X') = \prod_{i=1}^{\mathcal{N}_c(X')} (1-v)^{1-m_i} \, v^{m_i} \tag{15}$$

Here $m_i = 0, 1$ is the number of particles hopping from i-th cluster of X'.

To proceed with the solution, we introduce a transformation [15] given by the diagonal in the configurational basis operator $\mathcal{D}$

$$\mathcal{D} = \sum_{\{X\}} \frac{|X\rangle \langle X|}{W(X)}.$$

where the wight $W(X)$ is the stationary measure of the process. For the fixed number of particles at the infinite lattice, it can be given in terms of the number of clusters $\mathcal{N}_c(X)$ in configuration X

$$W(X) = (1-v)^{P - \mathcal{N}_c(X)}. \tag{16}$$

Technically, it is more convenient to solve first the right eigenproblem

$$\mathbf{T}_0 \left| B_Z^0 \right\rangle = \Lambda(Z) \left| B_Z^0 \right\rangle, \tag{17}$$

for the operator $\mathbf{T}_0$ defined as follows

$$\mathbf{T}_0 = \mathcal{D} \mathbf{T} \mathcal{D}^{-1}$$

rather than for the operator $\mathbf{T}$, which have the same eigenvalue but different eigenvectors. Then the eigenvectors $|B_Z\rangle$ of $\mathbf{T}$ can be related to eigenvectors $|B_Z^0\rangle$ of $\mathbf{T}_0$

$$|B_Z\rangle = \mathcal{D}^{-1} \left|B_Z^0\right\rangle . \tag{18}$$

Explicitly the matrix elements

$$T_0(X, X') = \frac{W(X')}{W(X)} T(X, X'). \tag{19}$$

can be constructed in terms of the cluster structure of final configuration X according to the following rule:

If a cluster of particles in X receives a particle from behind, it contributes the factor v to $T_0(X, X')$ and $(1-v)$ otherwise.

$$T_0(X, X') = \prod_{i=1}^{\mathcal{N}_c(X)} (1-v)^{1-k_i} \, v^{k_i} \tag{20}$$

where $k_i = 0, 1$ is the number of particles approaching i-th cluster of X from behind. Indeed, the event when an isolated particle joins a cluster, the first particle of which stays, contributes $v(1-v)$ to $T(X, X')$. At the same time, as the two clusters merge into one, it follows from (16,19) that the additional factor of $(1-v)^{-1}$ should be incorporated into $T_0(X, X')$, so that the final contribution will be v (see Fig 1). If the first particle of a cluster of the initial configuration jumps forward, while no particle is received from behind, the probability of the jump v enters into $T(X, X')$. However, as one cluster in the initial configuration has split up into two clusters in the final one, the factor $(1-v)$ should be added in $T_0(X, X')$. The processes, where the both first and the last particles of the cluster stay, proceed without change of the number of clusters $\mathcal{N}_c(X)$, and hence their contributions to $T(X, X')$ and to $T_0(X, X')$ are equal as well as when the cluster of the initial configuration receives a particle from behind while its first particle detaches it at the front. The expression (20) can also be obtained in more formal way using the correspondence between the ASEP to the ZRP as it was done in [15]. As a result, Eq. (17) can be written in the form

$$\Lambda(Z) \langle \ldots, \underbrace{x, x+1, \ldots, x+n_i}_{i-th\ cluster}, \ldots | B_Z^0 \rangle = \sum_{\{k_i\}} \prod_{i=1}^{\mathcal{N}_c(X)} v^{k_i} (1-v)^{1-k_i} \langle \ldots, x-k_i, x+1, \ldots, x+n_i, \ldots | B_Z^0 \rangle, \tag{21}$$

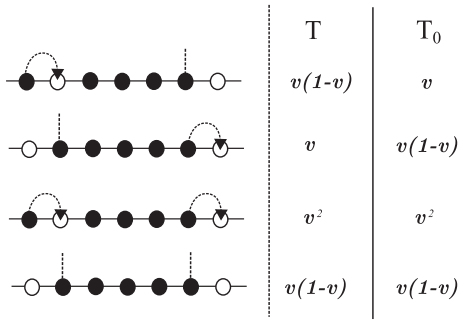

FIG. 1: The transformation of the transition probabilities.

where the index i in the product runs over all the clusters of particles of the configuration X. The number k_i takes on the value of 1 or 0 depending on whether or not the i-th cluster receives the particle from behind. The summation is performed over all possible sets $\{k_i\}$. The i-th cluster consisting of n_i particles is written explicitly in the argument in formula (21).

As usual, the strategy of the Bethe ansatz solution is the following. We note that when all the particles are at least one empty site apart from each other, they behave like noninteracting, i.e. each particle decides whether it is going to jump independently of the location of the others. In other words, if the coordinates of particles $\{x_1, \ldots, x_P\}$ of the configuration X satisfy the inequality $(x_i + 1) < x_{i+1}$ for every i, $|B_Z^0\rangle$ obeys the following equation

$$\Lambda(Z)\left\langle x_1, \ldots, x_P | B_Z^0 \right\rangle = \sum_{\{k_i\}} \prod_{i=1}^{P} v^{k_i} (1-v)^{1-k_i} \times \left\langle x_1 - k_1, \ldots, x_P - k_P | B_Z^0 \right\rangle . \tag{22}$$

Here the product is taken over all P particles, and the summation is over all k_i, which take on the values of 1 and 0 depending on whether or not the i-th particle decided to jump. The necessary trick is to find a way for this equation to be satisfied in the whole set of particle configurations including the domains where the equality $(x_i + 1) = x_{i+1}$ takes place. If we consider formally the equation (22) for such a configuration, the terms like $\langle \ldots, x, x, \ldots | B_Z^0 \rangle$ will appear. They are beyond the allowed domain (1) where the sets of particle coordinates are defined. Therefore they should be redefined and cancelled so that the difference with Eq. (21) disappears. Particularly, one needs to make sure that the result of the one cluster

summation in Eq.(21) coincides with the corresponding term of the free equation,

$$v\left\langle \ldots, x-1, x+1, \ldots, x+n, \ldots \right| B_Z^0 \rangle + (1-v)\left\langle \ldots, x, x+1, \ldots, x+n, \ldots \right| B_Z^0 \rangle$$
$$= \sum_{\{k_i\}} \prod_{i=0}^{n} v^{k_i} (1-v)^{1-k_i} \left\langle \ldots, x-k_0, x+1-k_2, \ldots, x+n-k_n, \ldots \right| B_Z^0 \rangle . \quad (23)$$

This relation can be satisfied with the only constraint on $B_Z^0(x_1, \ldots, x_P)$

$$v^2 \left[\left\langle \ldots, x-1, x, \ldots \right| B_Z^0 \rangle - \left\langle \ldots, x-1, x+1 \ldots \right| B_Z^0 \rangle \right]$$
$$+ v(1-v) \left[\left\langle \ldots, x, x, \ldots \right| B_Z^0 \rangle - \left\langle \ldots, x, x+1, \ldots \right| B_Z^0 \rangle \right] = 0, \quad (24)$$

which can be checked by a direct calculation for the size of the cluster $n = 2$, and then proven by induction for larger n (see [15]).

After that one can look for $B_Z^0(x_1, \ldots, x_P)$ in the form of the Bethe ansatz,

$$\langle x_1, \ldots, x_P | B_Z^0 \rangle = \sum_{\{\sigma\}} A_{\sigma_1 \ldots \sigma_P} z_{\sigma_1}^{-x_1} \ldots z_{\sigma_P}^{-x_P}, \quad (25)$$

parametrized by the parameter Z, which is a set of P complex numbers $Z = \{z_1, \ldots, z_P\}$. Here the summation is performed over all permutations σ of the numbers $1, \ldots, P$. The substitution of (25) to the free equation (22) yields the expression for the eigenvalues in terms of parameters $\{z_1, \ldots, z_P\}$

$$\Lambda(Z) = (1-v)^P \prod_{i=1}^{P} (1 + \lambda z_i), \quad (26)$$

where

$$\lambda = \frac{v}{1-v}. \quad (27)$$

The substitution of the Bethe ansatz into the constraint (24) results in the relation between the amplitudes $A_{\sigma_1 \ldots \sigma_P}$, which differ from each other only by two neighboring indices permuted

$$\frac{A_{\ldots ij \ldots}}{A_{\ldots ji \ldots}} = -S(z_i, z_j), \quad (28)$$

where

$$S(z_i, z_j) \equiv \frac{1 - 1/z_i}{1 - 1/z_j} \frac{1 + \lambda z_j}{1 + \lambda z_i}. \quad (29)$$

To obtain an explicit form of the amplitudes A_σ, consider a particular permutation $(\sigma_1, \ldots, \sigma_P)$. Let us associate the factor $\xi_i \xi_j^{-1}$ to an elementary transposition $(\ldots i, j \ldots) \to$

$(\ldots j, i \ldots)$. Apparently the power of ξ_i will then coincide with the position of i in the permutation. Thus, we have the following correspondence

$$\begin{pmatrix} 1, \ldots, P \\ \sigma_1, \ldots, \sigma_P \end{pmatrix} . \rightarrow \xi_{\sigma_1}^{1-\sigma_1} \xi_{\sigma_2}^{2-\sigma_2} \ldots \xi_{\sigma_P}^{2-\sigma_P} . \tag{30}$$

From Eqs.(28),we have

$$\xi_i = -\frac{1+\lambda z_i}{1-1/z_i}. \tag{31}$$

and

$$A_{\sigma_1 \ldots \sigma_P} = (-1)^{\mathcal{P}(\{\sigma\})} \prod_{i=1}^{P} \left(-\frac{1+\lambda z_{\sigma_i}}{1-1/z_{\sigma_i}} \right)^{i-\sigma_i}, \tag{32}$$

where $\mathcal{P}(\{\sigma\})$ is the parity of a permutation $\{\sigma\}$. The formulas (18,25,32) together with the definition of the weight (16) is all what we need to write the right eigenvector $|B_Z\rangle$ as a set of its coordinates $\langle X|B_Z\rangle$ in the configurational basis $\{|X\rangle\}$.

To construct the adjoint set of the left eigenvectors $\langle \overline{B}_Z|$ one has to solve Eq.(12) or equivalently the right eigenproblem for the transposed matrix $\mathbf{T}^*$

$$\mathbf{T}^* \left| \overline{B}_Z \right\rangle = \Lambda(Z) \left| \overline{B}_Z \right\rangle . \tag{33}$$

To this end, we note that in the original process corresponding to the matrix $\mathbf{T}$, the first particle of a cluster contributes the factor v to the transition probability $T(X, X')$ if it jumps forward and $(1-v)$ if it stays. In the adjoint process, which corresponds to the transposed matrix $\mathbf{T}^*$ the particles move in the opposite direction. The last particle of a cluster contributes the factor v or $(1-v)$ to a matrix element if it jumps approaching the cluster or stays in it respectively. One can see that the action of the operator $\mathbf{T}^*$ is similar, up to the inversion of the coordinates, to the one of the operator $\mathbf{T}_0$, Eq.(19). In other words, the matrices the matrices $\mathbf{T}_0$ and $\mathbf{T}^*$ are related by the identity

$$\mathbf{T}^* = \mathbf{I}\mathbf{T}_0\mathbf{I} \tag{34}$$

where $\mathbf{I}$ is the operator of the inversion of coordinates, $\mathbf{I}\,|x_1, \ldots, x_p\rangle = |-x_p, \ldots, -x_1\rangle$. It is then straightforward to show that the solution of the equation (33), corresponding to the eigenvalue (26) is given by the Bethe ansatz

$$\langle \overline{B}_Z | X \rangle = \frac{1}{P!} \sum_{\{\sigma\}} \overline{A}_{\sigma_1 \ldots \sigma_P} z_{\sigma_1}^{x_1} \ldots z_{\sigma_P}^{x_P} \tag{35}$$

with the amplitudes $\overline{A}_\sigma$ obeying the relation

$$\frac{\overline{A}_{\ldots ij\ldots}}{\overline{A}_{\ldots ji\ldots}} = -S\left(z_j, z_i\right). \tag{36}$$

where the S-matrix is the same as in the relation for A_σ, (29), with the permuted arguments. As a result we obtain

$$A_{\sigma_1 \ldots \sigma_P} = 1/\overline{A}_{\sigma_1 \ldots \sigma_P}$$

The constant factor $(P!)^{-1}$ is introduced for further convenience. Finally $\left\langle \overline{B}_Z | X \right\rangle$ and $\langle X | B_Z \rangle$ can be represented in the determinant form

$$\langle X | B_Z \rangle \;=\; W(X) \det \mathbf{B}, \tag{37}$$

$$\left\langle \overline{B}_Z | X \right\rangle \;=\; \frac{1}{P!} \det \overline{\mathbf{B}} \tag{38}$$

where the matrix elements B_{ij} and $\overline{B}_{ij}$ of the matrices $\mathbf{B}$ and $\overline{\mathbf{B}}$ respectively are given by

$$B_{ij} = 1/\overline{B}_{ij} = \left(-\frac{1+\lambda z_j}{1-1/z_j}\right)^{i-j} z_j^{-x_i}. \tag{39}$$

C. Completeness of the eigenbasis and the transition probabilities.

To proceed, we should define the range of the summation over Z in Eq. (13). For the infinite lattice, the spectrum of Z can be taken continuous, and the summation over Z in Eqs.(13,14) should be replaced by the $P-$ fold contour integral

$$\sum_Z \to \prod_{i=1}^{P} \oint \frac{dz_i}{2\pi i z_i}. \tag{40}$$

The contour of the integration over each z_i is closed around the points $z = 0$ and $z = 1$, so that the point $z = -1/\lambda$ stays outside, Fig.(2). This is a generalization of the solution for the continuous time ASEP. In the limit $\lambda \to 0$ the point $z = -1/\lambda$ goes to infinity, and the situation becomes similar to the one studied in [6]. Below, we prove the completeness of the eigenbasis, which follows from Eq.(13). As soon as it is established, one can directly substitute (25,26,32) into the expression for the probability (14) and obtain

$$\left\langle X \right| \mathbf{T}^t \left| X^0 \right\rangle = W(X) \sum_{\{\sigma\}} (-1)^\sigma \prod_{i=1}^{P} f(i - \sigma_i, x_i - x^0_{\sigma_i}, t) \tag{41}$$

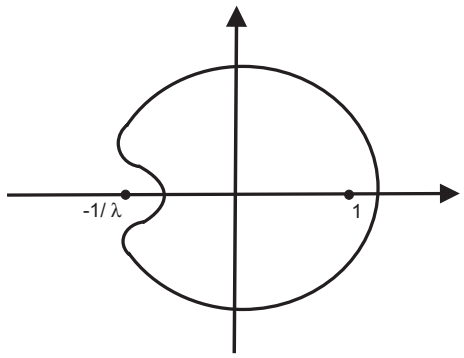

FIG. 2: Contour of integration over the parameters $z_1, \ldots, z_M$.

where $(-1)^\sigma$ is the sign of the permutation σ and the function $f(a,b,t)$ is defined as follows

$$f(a,b,t) = (1-v)^t \oint \frac{dz}{2\pi i z} (1+\lambda z)^t \left(-\frac{1+\lambda z}{1-1/z}\right)^a z^{-b}. \tag{42}$$

We should note that initially there should be a double sum over permutations σ and σ' as the terms $\langle X|B_Z\rangle$ and $\langle \overline{B}_Z|X\rangle$ contains one sum each. However, there is no any dependence on σ' in the product of the factors $f(\sigma'_i - \sigma_{\sigma'_i}, x_{\sigma'_i} - x^0_{\sigma_{\sigma'_i}}, t)$ over all i-s. As a result we have $P!$ equal terms under the sum over σ', which just eliminate the prefactor $1/(P!)$ in (35) and the only one sum (41) over permutations remains.

Consider two configurations

$$X = \{x_1, < x_2, < \ldots, < x_P\}, Y = \{y_1, < y_2, < \ldots, < y_P\}. \tag{43}$$

The explicit form of (13) is given by the particular case of formulas (41,42) taken at $t = 0$, $X^0 = Y$. Consider a term of the sum corresponding to a particular permutation σ, which is the product of P integrals of the form.

$$f(i - \sigma_i, x_i - y_{\sigma_i}, 0) = \oint \frac{dz}{2\pi i z} \left(-\frac{1+\lambda z}{1-1/z}\right)^{i-\sigma_i} z^{-x_i + y_{\sigma_i}}. \tag{44}$$

Apparently, the summand to be nonzero, the integrals for all i-s should be nonzero simultaneously. The following statement can be proved.

Proposition 1 *The expression (44) yields a nonzero result for all $i = 1, \ldots, P$ only when $X = Y$ and only for those permutations, which permute the particles inside their clusters.*

It means that the relations

$$x_i = y_i, \tag{45}$$

$$x_i - x_{\sigma_i} = i - \sigma_i. \tag{46}$$

hold for any $i = 1, \ldots, P$.

We should remark that in the case of the continuous time ASEP, which corresponds to the particular limit of our model with $\lambda \to 0$, [6], the contribution to the continuous time version of (13) comes from the only summand, which corresponds to the identical permutation $\sigma_i = i$. The proof of this fact though is not quite trivial was not given in [6], where the formula (13) was stated. In the Appendix, we prove the above proposition for the general case under consideration, while the continuous time case follows from the proof.

Given the configuration $X = Y$ with any pair of coordinates x_i, x_{σ_i} satisfying (45,46), we can calculate the integral (44), which is equal to $(-\lambda)^{i-\sigma_i}$ for $i > \sigma_i$ and 1 for $i \leq \sigma_i$. The summation over all permutations of particles within the clusters leads to the determinant of $P \times P$ matrix

$$\int dZ \, \langle X|B_Z\rangle \, \langle \overline{B}_Z|X\rangle = W(X) \det \mathbf{F}_0, \tag{47}$$

which has a block diagonal form

$$\mathbf{F}_0 = \begin{vmatrix} \mathbf{S}(n_1) & & & 0 \\ & \mathbf{S}(n_2) & & \\ & & \mathbf{S}(n_3) & \\ 0 & & & \ddots \end{vmatrix}. \tag{48}$$

Here, $\mathbf{S}(n)$ is the $n \times n$ matrix

$$\mathbf{S}(n) = \begin{vmatrix} 1 & -\lambda & (-\lambda)^2 & (-\lambda)^3 & \ldots \\ 1 & 1 & -\lambda & (-\lambda)^2 & \ldots \\ 1 & 1 & 1 & -\lambda & \ldots \\ 1 & 1 & 1 & 1 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}, \tag{49}$$

which corresponds to a cluster consisting of n particles. It can be transformed to the triangular form by subtracting neighboring columns, which allows the calculation of its determinant:

$$\det \mathbf{S}(n) = (1+\lambda)^{n-1} = (1-v)^{-n+1} \tag{50}$$

The product of such terms over all clusters yields just the value inverse to the weight $W(X)$ defined above:

$$\det \mathbf{F}_0 = (1-v)^{\mathcal{N}_c(X)-P} = 1/W(X) \tag{51}$$

The substitution to the formula (47) completes the proof of the relation (13). Whe should note that the matrix (49) has a lower triangular form in the limit $\lambda \to 0$, with all the units on the main diagonal, as should be in the continuous time limit.

The integral (42) can be evaluated in terms of the hypergeometric functions. The resulting determinant expression was given in the Section 2, Eqs.(3-5).

IV. GEOMETRIC APPROACH.

In this section, we give an alternative solution of the ASEP with the parallel update based on a combinatorial analysis of particle trajectories. We introduce a geometrical treatment considering sequentially three models of interacting particles with increasing complexity of interaction. The simplest one is the free-fermion model of non-intersecting trajectories, known in physical literature as the problem of "vicious walkers" [16] and in mathematical literature as the Gessel-Viennot theorem [17]. Then, we consider the ASEP with the backward sequential update to demonstrate how the Bethe ansatz appears with a complication of interaction. Finally, we apply the developed approach to our main problem of the ASEP with the parallel update.

A. Combinatorial ansatz for vicious walkers

Consider P particles labelled $\{1, 2, \ldots, P\}$ hopping in one direction on an infinite one-dimensional lattice. The discrete space-time dynamics of their motion can be described as a set of continuous broken trajectories on a triangle lattice Λ, which is obtained from the square lattice by adding a diagonal bond between the upper left corner and the lower right corner of each elementary square. Let (x, t) be integer space-time coordinates of a particle on Λ, where the vertical time axis is directed down and the horizontal space axis is directed right. A trajectory of each particle is a sequence of connected vertical and diagonal bonds of Λ. The diagonal bonds correspond to jumps of particles to their right for a unit time-step with probability v. The vertical bonds correspond to stays of particles at fixed

sites during the unit time-step with probability $1-v$. The initial positions of particles are $x_1^0 < x_2^0 < \ldots < x_P^0$. Let our particles be found at sites $x_1 < x_2 < \ldots < x_P$ by the moment of time t. The problem of vicious walkers is to find the conditional probability for the particles to reach the positions $X = (x_1, x_2, \ldots, x_P); x_1 < x_2 < \ldots < x_P$ from the initial positions $X^0 = (x_1^0, x_2^0, \ldots, x_P^0)$ for time t so that no pairs of trajectories are intersected during time t. In other words, one assumes that every site can be occupied by at most one particle at every moment of discrete time and if this rule is violated at a moment $t' < t$, the process stops.

We start with consideration of the one-particle motion on the infinite lattice. Let $\mathcal{T}_t(x|x^0)$ be a set of one-particle trajectories, which are starting at $(x^0, 0)$ and finishing at (x, t). Each trajectory $q \in \mathcal{T}_t(x|x^0)$ is realized with probability $v^{x-x^0}(1-v)^{t-x+x^0}$. Therefore, the total probability for the particle to reach x from x^0 for time t is

$$P(x, t|x^0, 0) = v^{x-x^0}(1-v)^{t-x+x^0}\|\mathcal{T}_t(x|x^0)\| = F_0(x - x^0|t). \tag{52}$$

where

$$F_0(x|t) = \binom{t}{x} v^x (1-v)^{t-x} \tag{53}$$

is the weight of one-particle trajectories in the set $\mathcal{T}_t(x|x^0)$. For the case of P particles, the set $\mathbb{S}_P$ of all possible free trajectories is a direct product of one-particle sets of trajectories reaching (X, t) from $(X^0, 0)$:

$$\mathbb{S}_P = \mathcal{T}_t(x_1|x_1^0) \otimes \mathcal{T}_t(x_2|x_2^0) \otimes \ldots \otimes \mathcal{T}_t(x_P|x_P^0) \tag{54}$$

The set $\mathbb{S}_P$ contains non-intersecting and intersecting trajectories. The latter cases are forbidden by the single-occupation rule and should be subtracted in evaluations of probability $P(X, t|X^0, 0)$.

Consider first the case $P = 2$. We denote the set of forbidden elements by $\mathbb{U}_{12}$, $\mathbb{U}_{12} \subset \mathbb{S}_2$ where indices $1, 2$ emphasize that the order of final points of intersecting trajectories coincides with the initial one. To cancel the contribution of forbidden elements, we introduce an auxiliary set of pairs of trajectories

$$\mathbb{A}_{21} = \mathcal{T}_t(x_2|x_1^0) \otimes \mathcal{T}_t(x_1|x_2^0), \tag{55}$$

where the final coordinates are permuted with respect to those in $\mathbb{S}_2$. It is easy to see that all elements of the set $\mathbb{A}_{21}$ are pairs of crossing trajectories. Each intersecting pair $q \in \mathbb{S}_2$

has a first collision point (x_c, t_c), i.e. the space-time point, where the trajectories meet for a first time. Consider the set $\mathbb{S}_2(x_c, t_c)$ of intersecting trajectories with the fixed first collision point (x_c, t_c). The sets $\mathbb{S}_2(x_c, t_c)$, $(x_c, t_c) \in \Lambda$ break the set $\mathbb{U}_{12}$ into subsets parameterized by coordinates (x_c, t_c). For all $(x_c', t_c') \neq (x_c'', t_c'')$ we have

$$\mathbb{S}_2(x_c', t_c') \cap \mathbb{S}_2(x_c'', t_c'') = \emptyset \tag{56}$$

and

$$\bigcup_{(x_c,t_c)\in\Lambda} \mathbb{S}_2(x_c, t_c) = \mathbb{U}_{12} \tag{57}$$

For every set $\mathbb{S}_2(x_c, t_c)$, there exists a uniquely defined set $\mathbb{A}_{21}(x_c, t_c) \subset \mathbb{A}_{21}$ obtained from $\mathbb{S}_2(x_c, t_c)$ by permutation of tails of all trajectories beginning at the first collision point. The sets $\mathbb{S}_2(x_c, t_c)$ and $\mathbb{A}_{21}(x_c, t_c)$ are geometrically equivalent,as there is one-to-one correspondence between their elements. Therefore $\|\mathbb{S}_2(x_c, t_c)\| = \|\mathbb{A}_2(x_c, t_c)\|$. Like the sets $\mathbb{S}_2(x_c, t_c)$, the sets $\mathbb{A}_{21}(x_c, t_c)$, $(x_c, t_c) \in \Lambda$ break the set $\mathbb{A}_{21}$ into subsets: for all $(x_c', t_c') \neq (x_c'', t_c'')$ we have

$$\mathbb{A}_{21}(x_c', t_c') \cap \mathbb{A}_{21}(x_c'', t_c'') = \emptyset \tag{58}$$

and

$$\bigcup_{(x_c,t_c)\in\Lambda} \mathbb{A}_{21}(x_c, t_c) = \mathbb{A}_{21}. \tag{59}$$

It follows from Eqs.(57,58,59) that the whole sets $\mathbb{U}_{12}$ and $\mathbb{A}_{21}$ are geometrically equivalent and $\|\mathbb{U}_{12}\| = \|\mathbb{A}_{21}\|$. Given (X, t) and $(X^0, 0)$, all elements of $\mathbb{S}_2$ have the same weight

$$Q = \prod_{i=1}^{2} v^{x_i - x_i^0} (1 - v)^{t - x_i + x_i^0}.$$

Then, we have for the probability

$$\begin{aligned} P(X, t|X^0, 0) &= Q \times (\|\mathbb{S}_2\| - \|\mathbb{A}_{21}\|) = \\ &= \big(F_0(x_1 - x_1^0|t)F_0(x_2 - x_2^0|t) - F_0(x_2 - x_1^0|t)F_0(x_1 - x_2^0|t)\big) = \\ &= det\, \mathbf{M}, \end{aligned} \tag{60}$$

where the elements of the 2×2 matrix $\mathbf{M}$ are

$$M_{i,j} = F_0(x_j - x_i^0|t), \quad i, j = 1, 2. \tag{61}$$

This result may be easily generalized for the P-particle system of vicious walkers [16]. The set $\mathbb{S}_P$ contains the subset of intersecting trajectories $\mathbb{U}_{12...P}$. We introduce $P! - 1$ auxiliary sets

$$\mathbb{A}_\sigma = \mathcal{T}_t(x_{\sigma_1}|x_1^0) \otimes \mathcal{T}_t(x_{\sigma_2}|x_2^0) \otimes \ldots \otimes \mathcal{T}_t(x_{\sigma_P}|x_P^0) \tag{62}$$

where $\sigma = \{\sigma_1, \ldots, \sigma_P\}$ is any permutation of numbers $1, 2, ..., P$ beside the identical one. Each element of the set

$$\mathbb{U}_{12...P} \bigcup \{ \bigcup_{\sigma \neq 1} \mathbb{A}_\sigma \} \tag{63}$$

containing trajectories intersecting at given point (x_c, t_c), has a unique geometrically identical counterpart with a pair of permuted indices, e.g. $12...i...j...P$ and $12...j...i...P$. Introducing the sign of permutation $(-1)^\sigma$, we can ascribe opposite signs to two counterparts. Then, we obtain the known result [16]:

$$\begin{aligned} P(X, t|X^0, 0) &= Q \times \left(\|\mathbb{S}_P\| + \sum_{\sigma \neq 1} (-1)^\sigma \|\mathbb{A}_\sigma\| \right) = \\ &= \det \mathbf{M}, \end{aligned} \tag{64}$$

where

$$M_{i,j} = F_0(x_j - x_i^0|t), \quad i, j = 1, 2, ...P. \tag{65}$$

B. Combinatorial treatment of Bethe ansatz

The vicious walkers are locally interacting particles. In this section, we consider the system of particles with the interaction of a non-zero range, which takes place in the case of the totally asymmetric exclusion process with the backward sequential update [18]. Consider again P particles hopping in one direction on an infinite one-dimensional lattice. The interaction between particles can be defined by the following rules:

1. Trajectories of particles do not intersect (every site can be occupied by at most one particle).

2. A particle stays at its own site with probability 1 if the target site is occupied by another particle during the step of discrete time.

Like the previous case, the trajectory of each particle is a sequence of vertical and diagonal bonds on the lattice Λ. Each diagonal bond has weight v and the vertical bond weight $1 - v$. In view of more complicated interaction, it is convenient to decompose the set of all

free trajectories into more detailed subsets. For each vertical bond $[(x,t'),(x,t'+1)]$, the trajectory of i-th particle passing this bond can be decomposed into two parts $[(x_i^0,0)\to(x,t')|1|(x,t'+1)\to(x_i,t)]$ and $[(x_i^0,0)\to(x,t')|-v|(x,t'+1)\to(x_i,t)]$ where the value between vertical bars means the weight of the bond $[(x,t'),(x,t'+1)]$. These new trajectories are geometrically equivalent, but the first trajectory passes the selected vertical bond with weight 1 and second one with weight $-v$. We make this decomposition for each vertical bond of each trajectory and denote the whole set of decomposed trajectories of i-th particle by $\mathcal{T}_t(x_i|x_i^0)$ using the same notation as for the set of one-particle trajectories in the previous section.

The weight of the trajectory $q\in\mathcal{T}_t(x_i|x_i^0)$ is a product of the weights of its bonds.

$$\mu(q)=\prod_{k=1}^{t}\mu(k\text{−th}\quad\text{bond}\quad\text{of}\quad q) \tag{66}$$

The weight of the set of trajectories is a sum of weights of its elements. In accordance with the above definition (52), the weight of the whole set $\mathcal{T}_t(x_i|x_i^0)$ is

$$\mu(\mathcal{T}_t(x_i|x_i^0))=F_0(x_i-x_i^0|t) \tag{67}$$

The set of free P-particle trajectories $\mathbb{S}_P$,

$$\mathbb{S}_P=\mathcal{T}_t(x_1|x_1^0)\otimes\mathcal{T}_t(x_2|x_2^0)\otimes\ldots\otimes\mathcal{T}_t(x_P|x_P^0) \tag{68}$$

has the weight

$$\mu(\mathbb{S}_P)=\sum_{q\in\mathbb{S}_P}\mu(q)=\prod_{i=1}^{P}v^{x_i-x_i^0}(1-v)^{t-x_i+x_i^0}\binom{t}{x_i-x_i^0}. \tag{69}$$

As in the case of vicious walkers, the set $\mathbb{S}_P$ contains a subset of unallowed elements,$\mathbb{U}_{12\ldots P}$, which should be excluded. By the rules 1 and 2, an element of $\mathbb{S}_P$ is unallowed, if there is at least one pair of intersecting trajectories, or if there are two neighboring vertical bonds from which the left one has weight $-v$. To cancel unallowed elements we start as above with the case $P=2$ and construct an auxiliary set of trajectories $\mathbb{A}_{21}$. We will see that, due to nonlocality, the auxiliary set has more complicated structure

$$\mathbb{A}_{21}=\bigcup_{k_1=0}^{\infty}\bigcup_{k_2=0}^{1}\mathcal{T}_t(x_2|x_1^0-k_1)\otimes\mathcal{T}_t(x_1|x_2^0-k_2). \tag{70}$$

Beginning the construction, we notice that every unallowed pair has a first collision point $(x_c, t_c) \in \Lambda$, where the particles come for the first time to neighboring sites at a moment $t_c < t$. The first trajectory reaches the site (x_c, t_c) from $(x_1^0, 0)$ and second one reaches the site $(x_c + 1, t_c)$ from $(x_2^0, 0)$ and then it makes the vertical step to the site $(x_c + 1, t_c + 1)$.

The first trajectory has, just after the collision, a diagonal bond $[(x_c, t_c), (x_c + 1, t_c + 1)]$ with weight v (refered to as a collision of the first type) or vertical bond $[(x_c, t_c), (x_c, t_c + 1)]$ with weight $-v$ (refered to as the collision of the second type). Notice, that trajectories, which have the vertical bond $[(x_c, t_c), (x_c, t_c + 1)]$ with weight 1 are allowed. Thus, we have two types of unallowed elements of $\mathbb{U}_{12}$ for fixed (x_c, t_c). Denote these subsets by $\mathbb{V}(x_c, t_c)$ for the first type and $\mathbb{W}(x_c, t_c)$ for the second one. For all $(x_c', t_c') \neq (x_c'', t_c'')$ we have

$$\mathbb{V}(x_c', t_c') \cap \mathbb{V}(x_c'', t_c'') = \emptyset,$$
$$\mathbb{W}(x_c', t_c') \cap \mathbb{W}(x_c'', t_c'') = \emptyset,$$
$$\mathbb{V}(x_c', t_c') \cap \mathbb{W}(x_c'', t_c'') = \emptyset,$$

and

$$\bigcup_{(x_c,t_c)\in\Lambda} (\mathbb{V}(x_c, t_c) \cup \mathbb{W}(x_c, t_c)) = \mathbb{U}_{12} \tag{71}$$

For each space-time point $(x_c, t_c) \in \Lambda$, we construct the sequence of pairs of trajectories $\mathbb{A}_v(k_1, k_2)$ and $\mathbb{A}_w(k_1, k_2)$, ($k_1 = 0, 1, 2, \ldots$ and $k_2 = 0, 1$,arguments x_c, t_c are omitted), obtained from $\mathbb{V}$ and $\mathbb{W}$ by permutation of tails in each pair of trajectories and shifting the initial parts in negative direction.

Explicitly, the transformation $\mathbb{V} \Rightarrow \mathbb{A}_v(k_1, k_2)$ is

$[(x_1^0, 0) \to (x_c, t_c) \to (x_c + 1, t_c + 1) \to (x_1, t)] \otimes [(x_2^0, 0) \to (x_c + 1, t_c) \to (x_c + 1, t_c + 1) \to (x_2, t)] \Rightarrow [(x_1^0 - k_1, 0) \to (x_c - k_1, t_c) \to (x_c + 1 - k_1, t_c + 1) \to (x_1, t)] \otimes [(x_2^0 - k_2, 0) \to (x_c + 1 - k_2, t_c) \to (x_c + 1 - k_2, t_c + 1) \to (x_2, t)]$,and

the transformation $\mathbb{W} \Rightarrow \mathbb{A}_w(k_1, k_2)$ is

$[(x_1^0, 0) \to (x_c, t_c) \to (x_c, t_c + 1) \to (x_1, t)] \otimes [(x_2^0, 0) \to (x_c + 1, t_c) \to (x_c + 1, t_c + 1) \to (x_2, t)] \Rightarrow [(x_1^0 - k_1, 0) \to (x_c - k_1, t_c) \to (x_c - k_1, t_c + 1) \to (x_1, t)] \otimes [(x_2^0 - k_2, 0) \to (x_c + 1 - k_2, t_c) \to (x_c + 1 - k_2, t_c + 1) \to (x_2, t)]$.

It follows from the definitions that

$$\bigcup_{(x_c,t_c)\in\Lambda} \bigcup_{k_1=0}^{\infty} \bigcup_{k_2=0}^{1} (\mathbb{A}_v(k_1, k_2) \cup \mathbb{A}_w(k_1, k_2)) = \mathbb{A}_{21}. \tag{72}$$

The weights of introduced sets obey the following relations

$$\mu\left(\ \mathbb{A}_v(0,0)\right) \ = \ \mu\left(\ \mathbb{V}(x_c,t_c)\right), \tag{73}$$

$$\mu\left(\ \mathbb{A}_v(0,1)\right) \ = \ -\mu\left(\ \mathbb{W}(x_c,t_c)\right), \tag{74}$$

$$\mu\left(\ \mathbb{A}_w(k,0)\right) \ = \ -\mu\left(\ \mathbb{A}_v(k+1,0)\right), \ k=0,1,2\dots\ , \tag{75}$$

$$\mu\left(\ \mathbb{A}_w(k,1)\right) \ = \ -\mu\left(\ \mathbb{A}_v(k+1,1)\right), \ k=0,1,2\dots\ . \tag{76}$$

Using these relations, we can express the probability

$$P(X,t|X^0,0) = \mu(\ \mathbb{S}_2 \setminus\ \mathbb{U}_{12}) \tag{77}$$

via the weight of all unallowed configurations

$$\mu(\ \mathbb{U}_{12}) = \sum_{(x_c,t_c)}\sum_{k=0}^{\infty}\left(\mu(\ \mathbb{A}_v(k,0)) + \mu(\ \mathbb{A}_w(k,0)) - \mu(\ \mathbb{A}_v(k,1)) - \mu(\ \mathbb{A}_w(k,1))\right). \tag{78}$$

Our next aim is to bring this expression to a determinant form similar to (65). Consider operator $\hat{a}_i$, which acts on the set of the free trajectories of i-th particle and gives the set of free trajectories with the shifted origin by one lattice space to the left:

$$\hat{a}_i\mathcal{T}_t(x_j|x_i^0) = \mathcal{T}_t(x_j|x_i^0-1) \tag{79}$$

The operator representation allows us to write (78) in compact form:

$$\mu(\ \mathbb{U}_{12}) = \mu\left(\frac{1-\hat{a}_2}{1-\hat{a}_1}\mathcal{T}_t(x_2|x_1^0)\otimes\mathcal{T}_t(x_1|x_2^0)\right), \tag{80}$$

where the denominator is defined by its expansion

$$\frac{1}{1-\hat{a}_1} = \sum_{k=0}^{\infty}\hat{a}_1^k. \tag{81}$$

and the summation of operators means the joining of the sets.

The crucial property of the operator expression in Eq.(80) is its factorization with respect to indices 1 and 2. Introducing the functions

$$F_1(x_2-x_1^0|t) = \mu\left((1-\hat{a}_1)^{-1}\mathcal{T}_t(x_2|x_1^0)\right) = \sum_{k=0}^{\infty}F_0(x_2-x_1^0+k|t), \tag{82}$$

and

$$F_{-1}(x_1-x_2^0|t) = \mu\left((1-\hat{a}_2)\mathcal{T}_t(x_1|x_2^0)\right) = \sum_{k=0}^{1}F_0(x_1-x_2^0+k|t) \tag{83}$$

we obtain the two-particle probability $P(X,t|X^0,0) = \mu(\ \mathbb{S}_2) - \mu(\ \mathbb{U}_{12})$ in the form

$$P(X,t|X^0,0) = F_0(x_1 - x_1^0|t)F_0(x_2 - x_2^0|t) - F_1(x_2 - x_1^0|t)F_{-1}(x_1 - x_2^0|t) \qquad (84)$$

or

$$P(X,t|X^0,0) = det\ \mathbf{M}, \qquad (85)$$

where

$$M_{i,j} = F_{i-j}(x_i - x_j^0|t) \qquad i,j = 1,2 \qquad (86)$$

To generalize the determinant formula (85) to the general P-particle case, we should take into consideration several additional properties of intersecting trajectories. First,we may organize the procedure of exclusion of unallowed elements of $\mathbb{U}_{12...P}$ in an ordered way. Starting with the top line of the lattice Λ, we examine all free trajectories of the set $\mathbb{S}_P$, row by row, until we meet the first collision point where unallowed trajectories are cancelled with elements of the auxiliary set $\mathbb{A}_\sigma$.

If the number of particles $P > 2$, the elementary squares associated with the collision point $(x_c,t_c),(x_c,t_c)(x_c+1,t_c)(x_c,t_c+1)(x_c+1,t_c+1)$, may occur several times in one horizontal strip of Λ. If squares filled by interacting trajectories are separated one from another by a gap of empty sites, the above arguments can be applied to each pair of interacting trajectories independently. The crucial case for the Bethe ansatz is a situation when the elementary squares are nearest neighbors. The specific property of the totally ASEP is that, in each pair of interacting trajectories, the right trajectory remains free and interacts with the next trajectory independently on its left neighbors. Therefore, we can analyse the interaction between particles considering successively elementary squares in each row from left to right starting from an arbitrary empty square until all unwanted trajectories will be removed.

After t' steps from the top to bottom, one obtains the set of path configurations $q \in \mathbb{S}_P$ which are allowed in the first t' rows, and the set of auxiliary configurations yet not involved into cancellation procedures. Remembering that all elements of $\mathbb{A}_\sigma$ for all $\sigma \neq 1$ have end points permuted with respect to the original order $12...P$, we conclude that each element of $\mathbb{A}_\sigma$ contains at least one collision point. Therefore, all elements of the auxiliary set will be cancelled after $t' = t$ steps with unallowed elements of $\mathbb{U}_{12...P}$.

The described way of exclusion of unallowed configurations implies a successive construction of the auxiliary set $\mathbb{A}_\sigma$. We notice that each two intersecting trajectories in $\mathbb{A}_{21}$ are

non-equivalent: one of them belongs to the particle which overtakes another and can be called "active". On the contrary, the second particle is "passive". In the case $P > 2$, one trajectory can overtake m others, and we call it "m-active". Similarly, the "m-passive" trajectories appear.

Assume, that the trajectory of a given particle has m active intersections. It means that it participates m times in the cancellation procedure and its starting point is shifted m times to arbitrary distances in the negative direction. As a result, the auxiliary set associated with the free trajectory between points x_i^0 and x_j becomes

$$\frac{1}{(1-\hat{a}_i)^m}\mathcal{T}_t(x_j|x_i^0) \tag{87}$$

Similarly, for trajectories having m passive intersections we get

$$(1-\hat{a}_i)^m\mathcal{T}_t(x_j|x_i^0) \tag{88}$$

The weights of sets of m-active and m-passive trajectories are given by functions introduced in [6]:

$$F_m(x_j - x_i^0|t) = \mu\left(\frac{1}{(1-\hat{a}_i)^m}\mathcal{T}_t(x_j|x_i^0)\right) = \sum_{k=0}^{\infty}\left(\begin{array}{c} k+m-1 \\ m-1 \end{array}\right) F_0(x_j - x_i^0 + k|t), \tag{89}$$

and

$$F_{-m}(x_j - x_i^0|t) = \mu\left((1-\hat{a}_i)^m\mathcal{T}_t(x_j|x_i^0)\right) = \sum_{k=0}^{m}\left(\begin{array}{c} m \\ k \end{array}\right) F_0(x_j - x_i^0 + k|t), \tag{90}$$

Activity m of each trajectory is defined uniquely by the permutation of numbers σ, so we have for the weight of auxiliary set $\mathbb{A}_\sigma$

$$\mu(\ \mathbb{A}_\sigma) = \prod_{i=1}^{P} F_{\sigma_i - i}(x_{\sigma_i} - x_i^0|t) \tag{91}$$

This product is a term of expansion of the determinant $det\ \mathbf{M}$ with matrix elements for $\sigma \neq 1$

$$M_{i,j} = F_{i-j}(x_i - x_j^0|t) \qquad i,j = 1,2,...,P \tag{92}$$

It follows from Eq.(68) that the term corresponding to the identical permutation $\sigma = 1$ is

$$\mu(\ \mathbb{S}_P) = \prod_{i=1}^{P} F_0(x_i - x_i^0|t) \tag{93}$$

Collecting the contributions from the set $\mathbb{S}_P$ and all auxiliary sets $\mathbb{A}_\sigma$, we obtain the the determinant formula [6]

$$P(X,t|X^0,0) = det\,\mathbf{M}, \tag{94}$$

which is valid for all $P \geq 1$.

C. Combinatorial solution for parallel update

The TASEP with parallel update is characterized by the rules

1. Trajectories of particles do not intersect.

2. A particle stays at its own site with probability 1 if the target site is occupied by another particle just before the step of discrete time.

The first rule coincides with that for the the backward sequential update, however the second rules are different. The combinatorial solution for parallel update accumulates all main steps of two preceding sections. We start with the construction of different types of unallowed elements of $\mathbb{U}_{12}$ at the fixed collision point (x_c, t_c). Two of them, $\mathbb{V}(x_c, t_c)$ and $\mathbb{W}(x_c, t_c)$ are the same as for the sequential update. Two new sets $\mathbb{X}(x_c, t_c)$ and $\mathbb{Y}(x_c, t_c)$ are

$\mathbb{X}(x_c, t_c)$: the first trajectory has, just after the collision, the diagonal bond $[(x_c, t_c), (x_c + 1, t_c + 1)]$ with weight v and the second one has also the diagonal bond $[(x_c + 1, t_c), (x_c + 2, t_c + 1)]$ with weight v;

$\mathbb{Y}(x_c, t_c)$: the first trajectory has, after the collision, the vertical bond $[(x_c, t_c), (x_c, t_c+1)]$ with weight $-v$ and the second one has the diagonal bond $[(x_c + 1, t_c), (x_c + 2, t_c + 1)]$ with weight v.

Like Eq.(71) we have

$$\bigcup_{(x_c,t_c)\in\Lambda} (\ \mathbb{V}(x_c, t_c) \cup\ \mathbb{W}(x_c, t_c) \cup\ \mathbb{X}(x_c, t_c) \cup\ \mathbb{Y}(x_c, t_c)) = \mathbb{U}_{12} \tag{95}$$

Similarly, we introduce along with the auxiliary sets $\mathbb{A}_v(k_1, k_2)$ and $\mathbb{A}_w(k_1, k_2)$ new sets $\mathbb{A}_x(k_1, k_2)$ and $\mathbb{A}_y(k_1, k_2)$ obtained from $\mathbb{X}$ and $\mathbb{Y}$ by permutation of tails in each pair of trajectories and shifting the initial parts by integers k_1 and k_2. Explicitly, the transformation $\mathbb{X} \Rightarrow\ \mathbb{A}_x(k_1, k_2)$ is

$[(x_1^0,0)\to(x_c,t_c)\to(x_c+1,t_c+1)\to(x_1,t)]\otimes[(x_2^0,0)\to(x_c+1,t_c)\to(x_c+2,t_c+1)\to(x_2,t)]\Rightarrow[(x_1^0-k_1,0)\to(x_c-k_1,t_c)\to(x_c+1-k_1,t_c+1)\to(x_1,t)]\otimes[(x_2^0-k_2,0)\to(x_c+1-k_2,t_c)\to(x_c+2-k_2,t_c+1)\to(x_2,t)]$,

and the transformation $\mathbb{Y}\Rightarrow\ \mathbb{A}_y(k_1,k_2)$ is

$[(x_1^0,0)\to(x_c,t_c)\to(x_c,t_c+1)\to(x_1,t)]\otimes[(x_2^0,0)\to(x_c+1,t_c)\to(x_c+2,t_c+1)\to(x_2,t)]\ \Rightarrow\ [(x_1^0-k_1,0)\to(x_c-k_1,t_c)\to(x_c-k_1,t_c+1)\to(x_1,t)]\otimes[(x_2^0-k_2,0)\to(x_c+1-k_2,t_c)\to(x_c+2-k_2,t_c+1)\to(x_2,t)]$.

The whole auxiliary set $\mathbb{A}_{21}$ has, instead of Eq.(72), a more complicated structure. To reveal it, we show schematically in Fig.(3) the positions of sets $\mathbb{V}$, $\mathbb{W}$, $\mathbb{X}$, $\mathbb{Y}$ and $\mathbb{A}_v(0,0)$, $\mathbb{A}_w(0,0)$, $\mathbb{A}_x(0,0)$, $\mathbb{A}_w(0,0)$. It is easy to see that all these sets are characterized by reference points from where trajectories reach the end points x_1 and x_2 after the collision. For the set $\mathbb{V}$, the point x_1 is reached from x_c+1, the point x_2 from x_c+1, for the set $\mathbb{W}$, x_1 is reached from x_c, x_2 from x_c+1; for the set $\mathbb{X}$, x_1 is reached from x_c+1, x_2 from x_c+2; and for the set $\mathbb{Y}$, x_1 is reached from x_c, x_2 from x_c+2. The coordinates of the reference points of $\mathbb{V}$, $\mathbb{W}$, $\mathbb{X}$, $\mathbb{Y}$ are shown in Fig.(3) as full circles.

For the auxiliary sets $\mathbb{A}_v(0,0)$, $\mathbb{A}_w(0,0)$, $\mathbb{A}_x(0,0)$, $\mathbb{A}_w(0,0)$, the coordinates of reference points can be obtained by permutation of x_1 and x_2. The positions of auxiliary sets are shown in Fig.(3) as full squares.

The weights of sets in the "scattering zone" between t_c and t_c+1 are: $v(1-v)$ for $\mathbb{V}$,$-v(1-v)$ for $\mathbb{W}$, v^2 for $\mathbb{X}$, $-v^2$ for $\mathbb{Y}$,$v(1-v)$ for $\mathbb{A}_v(0,0)$, $-v(1-v)$ for $\mathbb{A}_w(0,0)$, v^2) for $\mathbb{A}_x(0,0)$ and $-v^2$) for $\mathbb{A}_y(0,0)$.

Our aim is to combine the sets $\mathbb{A}_v(k_1,k_2)$, $\mathbb{A}_w(k_1,k_2)$, $\mathbb{A}_x(k_1,k_2)$, $\mathbb{A}_y(k_1,k_2)$ in such a way that all sets of unallowed trajectories are cancelled with their auxiliary counterparts. Using the shift operators $\hat{a}_i$, we consider sequentially four operator forms:

$$\hat{A}_1=\frac{1}{1+\lambda\hat{a}_2^{-1}}, \tag{96}$$

$$\hat{A}_2=\frac{1}{1-\hat{a}_1}, \tag{97}$$

$$\hat{A}_3=1+\lambda\hat{a}_1^{-1} \tag{98}$$

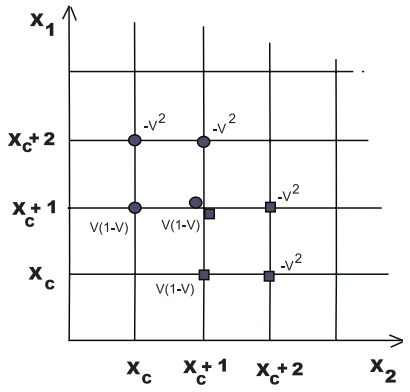

FIG. 3: The reference points and the weights of the sets $\mathbb{V}, \mathbb{W}, \mathbb{X}, \mathbb{Y}$ (full circles) and $\mathbb{A}_v(0,0), \mathbb{A}_w(0,0), \mathbb{A}_x(0,0), \mathbb{A}_w(0,0)$ (full squares)

$$\hat{A}_4 = 1 - \hat{a}_2 \tag{99}$$

where $\lambda = v/(1-v)$.

Each shifted set contains trajectories where an initial part from $t = 0$ to $t = t_c + 1$ is geometrically equivalent to that of non-shifted trajectories. It means that the reference points are also shifted under action of $\hat{a}_i$. As a result, the action of operator $\hat{A}_1$ leads to the diagram Fig.(4a), the action of operator $\hat{A}_2$ leads to the diagram Fig.(4b),and the action of operator $\hat{A}_3\hat{A}_4$ gives the final diagram Fig.(4c) where positions of auxiliary sets coincide with those of unallowed sets $\mathbb{V}$, $\mathbb{W}$, $\mathbb{X}$, $\mathbb{Y}$ together with their weights.

Thus, we may conclude that the whole set of auxiliary trajectories can be obtained by action of operator $\hat{A}_1\hat{A}_2\hat{A}_3\hat{A}_4$, so we have

$$\mu(\mathbb{U}_{12}) = \mu\left(\frac{(1-\hat{a}_2)}{(1-\hat{a}_1)}\frac{(1+\lambda\hat{a}_1^{-1})}{(1+\lambda\hat{a}_2^{-1})}\mathcal{T}_t(x_2|x_1^0)\otimes\mathcal{T}_t(x_1|x_2^0)\right), \tag{100}$$

which should be compared with the analogous expression Eq.(80) for the sequential update.

The operator expression (100) is also factorizable with respect to indices 1 and 2. It allows us to introduce functions which are generalization of F_m and F_{-m} given by Eqs.(89,90):

$$\tilde{F}_m(x_j - x_i^0|t) = \mu\left(\frac{(1+\lambda\hat{a}_i^{-1})^m}{(1-\hat{a}_i)^m}\mathcal{T}_t(x_j|x_i^0)\right) \tag{101}$$

and

$$\tilde{F}_{-m}(x_j - x_i^0|t) = \mu\left(\frac{(1-\hat{a}_i)^m}{(1+\lambda\hat{a}_i^{-1})^m}\mathcal{T}_t(x_j|x_i^0)\right) \tag{102}$$

or, in the explicit form:

$$\tilde{F}_m(N|t) = \sum_{n=0}^{m}\sum_{k=-n}^{\infty}\frac{m(m+k+n-1)!}{(k+n)!n!(m-n)!}\lambda^n F_0(N+k|t), \tag{103}$$

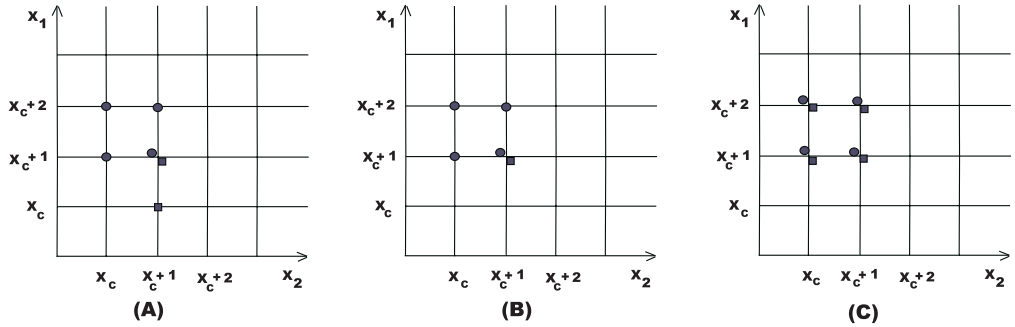

FIG. 4: The transformation of the reference points due to action of operators $\hat{A}_1$ (a),$\hat{A}_2$ (b),and $\hat{A}_3\hat{A}_4$ (c)

and

$$\tilde{F}_{-m}(N|t) = \sum_{n=0}^{m}\sum_{k=-n}^{\infty}(-1)^n\frac{m(m+k+n-1)!}{(k+n)!n!(m-n)!}(-\lambda)^{k+n}F_0(N-k|t), \tag{104}$$

The remainder of derivation coincides with that for the sequential update and we can write for $P(X,t|X^0,0)$ the determinant formula with the matrix elements

$$M_{i,j} = \tilde{F}_{i-j}(x_i - x_j^0|t) \qquad i,j = 1,2,...,P \tag{105}$$

The determinant formula for $P(X,t|X^0,0)$ is valid for all initial states X^0 and for all final states X conditioned by inequalities $x_{i+1} - x_i \geq 1$ for all $i = 1,2,...,P-1$. If $x_{i+1} - x_i = 1$ for some $1 \leq i \leq P-1$, the determinant formula should be corrected. To find the correction, consider the final positions of particles $x_1 = x, x_2 = x+1$ at time t in the two-particle case $P = 2$. For the last time step, the final positions are simultaneously the reference points of unallowed and auxiliary sets of trajectories. However, these sets cannot be superposed as in the case of bulk scattering because the range of auxiliary set exceeds the range of unallowed set (the points $(x, x+2)$ and $(x+1, x+2)$ in Fig.(4) are absent). Explicit calculations for the transitions from $(x-1,t-1),(x+1,t-1)$ to $(x,t),(x+1,t)$ and from $(x,t-1),(x+1,t-1)$ to $(x,t),(x+1,t)$ show that the correct probability differs from the determinant result by the factor $(1-v)$. Using the fact that the scattering of each pair of trajectories can be considered independently, we write the final result as

$$P(X,t|X^0,0) = (1-v)^n det\,\mathbf{M}, \tag{106}$$

where n is the number of pairs of neighboring particles in the final position. The equivalence of the results (3-5) and (106) is verified directly by expanding the hypergeometric functions in (5).

To make above arguments more transparent, consider a transformation $X \Leftrightarrow X^0$ of a realization of the process interchanging the initial and final configurations of particles. As a result, we obtain the same realization overturned (Fig.5a,b).

The initial positions in Fig.(5a) and final positions in Fig.(5b) obey the conditions $x_{i+1} - x_i \geq 1$ for all $i = 1, 2, ..., P-1$. The final positions in Fig.(5a) and initial positions in Fig.(5b) contain pairs of neighboring coordinates. A remarkable property of the path configurations is that the difference in numbers of bonds having weight 1 (yellow bonds in Fig.(5a) and red bonds in Fig.(5b)) equals to the difference in numbers of pairs in final and initial positions. Therefore, we can apply the pure determinant formula to the case when the final position is free of neighboring pairs and then calculate the opposite case multiplying the determinant by $(1-v)^n$ as in Eq.(106).

V. DISCUSSION AND CONCLUSION.

To summarize, we considered the discrete time asymmetric exclusion process with parallel update at the infinite 1-D lattice. We have presented two alternative derivations for the probability of a particle configuration at arbitrary time, given an initial configuration. In the first approach we constructed the transformation of the evolution operator, which allows one to find its left and right eigenvectors in the Bethe ansatz form. We proved the completeness of the eigenbasis, which allows calculation of the matrix elements of an arbitrary power of evolution operator. The second approach is based on the consideration of the ensembles of trajectories of noninteracting particles. Taken with alternating signs their statistical weights are chosen in such a way that they add up to reproduce correctly the dynamics of interacting particles.

The result for the system of P particles is expressed in the form of the determinant of $P \times P$ matrix, whose elements are given in terms of the Gauss hypergeometric functions ${}_2F_1(a, b, c; x)$. To relate this result to the previous ones obtained for the ASEP, we note that in the continuous time limit $t \to \infty$, $\lambda \to 0$, $t\lambda = \tau$ the hypergeometric function ${}_2F_1(a, b, c; x)$ degenerates into the confluent hypergeometric function yielding

$$f(a, b, \tau) = (-1)^a \, e^{-\tau} \begin{cases} \frac{\tau^b}{b!} \, {}_1F_1\,(a, b+1, \tau) & b > 0 \\ \frac{(a)_{-b}}{(-b)!} \, {}_1F_1\,(a-b, -b+1, \tau) & b \leq 0 \end{cases}, \tag{107}$$

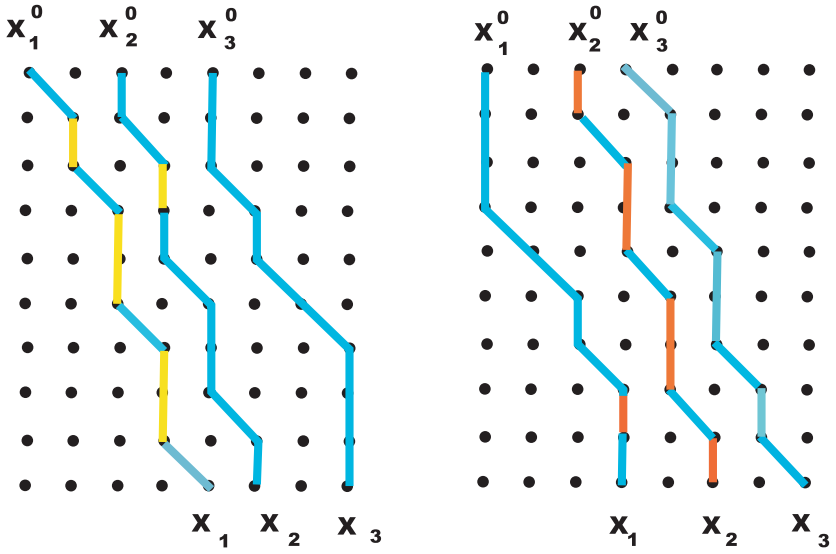

FIG. 5: (a) The initial state contains no pairs of neighboring sites; the final state contains one pair. (b) The reversed dynamics. The yellow and red bonds have weight 1. The difference between their numbers equals to the number of pairs in the final state in (a)

which is equivalent to the results obtained by Schütz, [6]. The relation of Schütz's functions to the confluent hypergeometric functions was already noticed before [19].

We should also remark that there is a direct relation of our results to the determinant solution for the TASEP with backward ordered update [10] referred also to as a fragmentation model [9]. The determinant formula for that process can be obtained from our one by using $f(a,b,t-a)$ instead of $f(a,b,t)$ in the formula for the matrix elements (4). This connection between conditional probabilities seems to be similar to the correspondence noted in [9] on the level of the current distributions. Given such a close relation between the solutions of the master equations, an establishing of the direct mapping between the two original processes would be interesting.

The obtained results open perspectives for derivation of more complicated correlation functions for the density and current of particles. Particularly, the reproducing of the Johansson formula for the current distribution for the parallel ASEP starting from the half filled lattice can be mentioned as a closest goal. Other initial conditions can be also considered [19]. The generalizations of present results can be also useful in connection with the random matrix theory, which relation to the ASEP has been widely discussed [20].

Another direction of research is the generalization of the results for the ring geometry.

In the present article, the proof of the completeness of the Bethe ansatz basis for evolution operator of the parallel ASEP has been given for the case of the infinite chain. Similar proof of the completeness and orthonormality for the case of the ring would be an important issue not only for the stochastic processes community but in the general context of the Bethe ansatz solvable models [21].

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APPENDIX A: PROOF OF THE PROPOSITION 1.

The integral (44) can be evaluated using the Cauchy theorem. To this end, one can expand the expression in the parentheses into the Laurent series. Practically one can first expand $(1+\lambda z)^{i-\sigma_i}$ and $(1-1/z)^{\sigma_i-i}$ separately and then evaluate their product. Then the coefficient of the term of power $(x_i - y_{\sigma_i})$ will be the result. The choice of the integration contour means, that the expansion of $(1+\lambda z)^{i-\sigma_i}$ will contain only nonnegative powers of λz and the expansion of $(1-1/z)^{\sigma_i-i}$ will contain only nonnegative powers of $1/z$. In addition we should take into account that the number of terms of the expansion will be finite when the exponent is positive. Thus, when

$$i = \sigma_i, \tag{A1}$$

the integral to be nonzero we should require

$$x_i = y_i. \tag{A2}$$

In the other cases, the following inequalities should hold.

$$i - \sigma_i \ \geq \ x_i - y_{\sigma_i}, \qquad \text{for} \quad i > \sigma_i \tag{A3}$$

$$\sigma_i - i \ \geq \ y_{\sigma_i} - x_i, \qquad \text{for} \quad i < \sigma_i. \tag{A4}$$

Moreover the domain (43), where the coordinates of particles are defined, implies

$$i - \sigma_i \ \leq \ x_i - x_{\sigma_i}, \quad i - \sigma_i \leq y_i - y_{\sigma_i}, \qquad \text{for} \quad i > \sigma_i \tag{A5}$$

$$\sigma_i - i \ \leq \ x_{\sigma_i} - x_i, \quad \sigma_i - i \leq y_{\sigma_i} - y_i, \qquad \text{for} \quad i < \sigma_i. \tag{A6}$$

Comparing the expressions (A3,A4) with (A5,A6) respectively, we obtain

$$y_{\sigma_i} \geq x_{\sigma_i}, \quad y_i \geq x_i \qquad \text{for} \quad i > \sigma_i \tag{A7}$$

$$y_{\sigma_i} \leq x_{\sigma_i}, \quad y_i \leq x_i \qquad \text{for} \quad i < \sigma_i. \tag{A8}$$

Any permutation can be decomposed into the set of disjoint orbits [22], i.e. the sequences of elements which trade places with one another under a given permutation. Consider an orbit $i_1 \to i_2 \to \cdots \to i_k \to i_1$, such that

$$\sigma_{i_1} = i_2, \sigma_{i_2} = i_3, \cdots, \sigma_{i_k} = i_1 \tag{A9}$$

There exists r such that i_r is the largest in a given orbit. Then from $i_r > i_{r+1} \equiv \sigma_{i_r}$ and the second inequality of Eq. (A7) we have

$$y_{i_r} \geq x_{i_r}, \tag{A10}$$

while from $i_{r-1} < i_r \equiv \sigma_{i_{r-1}}$ and the first inequality of (A8) it follows that

$$y_{i_r} \leq x_{i_r}. \tag{A11}$$

These two are consistent only if

$$y_{i_r} = x_{i_r}. \tag{A12}$$

The latter equality together with the inequality (A3) and the second inequality of (A5) yield

$$y_{i_r} - y_{i_{r+1}} \geq i_r - i_{r+1} \geq y_{i_r} - y_{i_{r+1}}, \tag{A13}$$

which means

$$i_r - i_{r+1} = y_{i_r} - y_{i_{r+1}}. \tag{A14}$$

In the same way using the inequalities (A3,A5) we obtain

$$i_r - i_{r-1} = x_{i_r} - x_{i_{r+1}}. \tag{A15}$$

We should note that the equality like

$$x_i - x_j = i - j \tag{A16}$$

means that there are as many sites between the sites x_i and x_j as many particles. In other words i-th and j-th particles belong to the same cluster. Apparently, the coordinates of any

pair of particles between them also satisfy similar relation. One can see that the particles at sites $y_{i_r}, y_{i_{r+1}}$ belong to the same cluster of the configuration Y, while the particles at sites $x_{i_r}, x_{i_{r-1}}$ belong to the same cluster of the configuration X. Let i_{r+1} be the largest of the numbers i_{r+1}, i_{r-1}. The cluster of particles spreads from x_{i_r} backward to at least $x_{i_{r+1}}$ in both configurations X and Y. Therefore we conclude that

$$x_{i_{r+1}} = y_{i_{r+1}}. \tag{A17}$$

Then we look for i_{r+2}. If i_{r+2} is larger than i_{r+1}, then it is between i_r and i_{r+1} and hence belongs to the same cluster in both X and Y and we should look for i_{r+3}. If i_{r+2} is less than i_{r+1}, using the same arguments as for i_r, i_{r+1} we have

$$y_{i_{r+1}} - y_{i_{r+2}} = i_{r+2} - i_{r+1}, \tag{A18}$$

i.e. we extend the cluster in Y backward up to i_{r+2}-th particle. Then we again compare i_{r+2} with i_{r-1}, which currently limits the cluster in X from behind. If i_{r-1} is between i_{r+1} and i_{r+2} it necessarily belongs to the same cluster also in Y. Then the cluster can be extended up to i_{r-2}, etc. Thus, step by step we extend the cluster backward in Y going forward along the orbit and backward in X going backward along the orbit until we come to the same element in both directions, which means that the orbit is exhausted. Finally we conclude that all the particles in a given orbit should belong to the same cluster and their coordinates in the configurations X and Y coincide.

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