

# Transfer matrix eigenvectors of the Baxter-Bazhanov-Stroganov $\tau_2$ -model for $N = 2$

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**Abstract.** We find a representation of the row-to-row transfer matrix of the Baxter-Bazhanov-Stroganov  $\tau_2$ -model for  $N = 2$  in terms of an integral over two commuting sets of Grassmann variables. Using this representation, we explicitly calculate transfer matrix eigenvectors and normalize them. It is also shown how form factors of the model can be expressed in terms of determinants and inverses of certain Toeplitz matrices.

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## 1. Preliminaries

The  $\tau_2$ -model was originally introduced by Baxter in the work [1], where it appeared in relation to the superintegrable case of the chiral Potts model. Later it was used by Bazhanov and Stroganov to establish a connection between six-vertex model and chiral Potts model [7] (see also [19]). This connection has allowed to obtain a system of functional relations for transfer matrices of these models [2] and has led to the derivation of exact formulas for the free energy [3] and order parameter [4] of the chiral Potts model. Following the authors of [11, 16], we will use instead of the name ‘ $\tau_2$ -model’ the name ‘Baxter-Bazhanov-Stroganov model’ (or simply ‘BBS model’).

BBS model is a system of spins, living on a square lattice and taking on  $N$  values  $0, 1, \dots, N - 1$ . The interactions exist only between nearest neighbours. In addition, the difference  $b_2 - b_1$  of neighbouring spins, living on the same vertical line ( $b_2$  is higher than  $b_1$ ) is allowed to take on only the values 0 and 1 (mod  $N$ ). Consider an elementary plaquette of the lattice, drawn in the Figure 1a. Boltzmann weights  $W(b_1, b_2, b_3, b_4)$ , associated to this plaquette, are defined in the following table (our  $b_1, b_2, b_3, b_4$  correspond to  $d, a, b, c$  of [5] and to  $b_4, b_1, b_2, b_3$  of [11]):

$b_2 - b_1$	$b_3 - b_4$	$W(b_1, b_2, b_3, b_4)$
0	0	$1 - \omega^{b_1 - b_3 + 1} t / (yy')$
0	1	$(y - \omega^{b_1 - b_3 + 1} x') \mu' / (yy')$
1	0	$-(y' - \omega^{b_1 - b_3 + 1} x) \omega \mu t / (yy')$
1	1	$-(t - \omega^{b_1 - b_3 + 1} x x') \omega \mu \mu' / (yy')$

Here  $\omega = e^{2\pi i/N}$  and  $t, x, x', y, y', \mu, \mu'$  are parameters. It is easily seen that the model is  $\mathbb{Z}_N$ -symmetric: if one shifts the spins  $b_1, \dots, b_4$  by 1, all plaquette Boltzmann weights remain unchanged.

Figure 1: a) numeration of spins of an elementary plaquette b) graphical representation of the transfer matrix

Consider now the case  $N = 2$ . Let us use instead of  $b_1, \dots, b_4$  new spin variables  $\sigma_j = (-1)^{b_j}$  ( $j = 1, \dots, 4$ ), taking on the values  $\pm 1$ . The most general  $\mathbb{Z}_2$ -symmetric Boltzmann weight is given by the following formula

$$W(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = a_0 \left( 1 + \sum_{1 \leq i < j \leq 4} a_{ij} \sigma_i \sigma_j + a_4 \sigma_1 \sigma_2 \sigma_3 \sigma_4 \right). \quad (1.1)$$

The coefficients  $a_0, \{a_{ij}\}, a_4$ , which correspond to BBS<sub>2</sub> model, can be written as

$$a_0 = (y + \mu t)(y' + \mu') / (4yy'), \quad (1.2)$$

$$a_0 a_4 = (y - \mu t)(y' - \mu') / (4yy'), \quad (1.3)$$

$$a_0 a_{12} = (y - \mu t)(y' + \mu') / (4yy'), \quad (1.4)$$

$$a_0 a_{34} = (y + \mu t)(y' - \mu') / (4yy'), \quad (1.5)$$

$$a_0 a_{13} = (1 + x\mu)(t + x'\mu') / (4yy'), \quad (1.6)$$

$$a_0 a_{24} = (1 - x\mu)(t - x'\mu') / (4yy'), \quad (1.7)$$

$$a_0 a_{14} = (1 + x\mu)(t - x'\mu') / (4yy'), \quad (1.8)$$

$$a_0 a_{23} = (1 - x\mu)(t + x'\mu') / (4yy'). \quad (1.9)$$

It was pointed out in [11] that these coefficients satisfy a ‘free-fermion condition’

$$a_4 = a_{12} a_{34} - a_{13} a_{24} + a_{14} a_{23}. \quad (1.10)$$

Free-fermion models of statistical mechanics have been extensively studied in the literature from different points of view [6, 8, 13, 14]. In particular, the partition function of the model with plaquette weight (1.1), satisfying the condition (1.10), was calculated by Bugrij [10] even in the case of a finite lattice. This result has allowed to obtain

the eigenvalues of the  $\text{BBS}_2$  transfer matrix without solving any functional relations [11]. Technically, the condition (1.10) means that the Boltzmann weight (1.1) can be represented as the following integral over four auxiliary Grassmann variables  $\psi^1, \dot{\psi}^2, \dot{\psi}^3, \psi^4$ :

$$W(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = \int d\psi^1 d\dot{\psi}^2 d\dot{\psi}^3 d\psi^4 \exp\left\{a_{12}\sigma_1\sigma_2\psi^1\dot{\psi}^2 - a_{13}\sigma_1\sigma_3\psi^1\dot{\psi}^3 + \right. \quad (1.11) \\ \left. + a_{14}\sigma_1\sigma_4\psi^1\psi^4 + a_{23}\sigma_2\sigma_3\dot{\psi}^2\dot{\psi}^3 - a_{24}\sigma_2\sigma_4\dot{\psi}^2\psi^4 + a_{34}\sigma_3\sigma_4\dot{\psi}^3\psi^4\right\} e^{\psi^1} e^{\dot{\psi}^2} e^{\dot{\psi}^3} e^{\psi^4}.$$

Throughout this paper, we will use the convention that ‘dotted’ Grassmann variables commute with the usual ones, and that the variables inside each set anticommute:

$$\psi^\alpha\psi^\beta = -\psi^\beta\psi^\alpha, \quad \dot{\psi}^\alpha\dot{\psi}^\beta = -\dot{\psi}^\beta\dot{\psi}^\alpha, \quad \psi^\alpha\dot{\psi}^\beta = \dot{\psi}^\beta\psi^\alpha \quad \forall \alpha, \beta.$$

The method of Grassmann variables was initially designed as a method of simple calculation of the partition function of the 2D Ising model. It was discovered and improved by different authors (see, for example, [9, 15, 20]). The use of two commuting sets of Grassmann variables, which is crucial for our further discussion, was suggested in [9].

The main drawback of the method of Grassmann integration is that it does not give the eigenvectors of the transfer matrix, which are necessary ingredients in the computation of correlation functions and form factors. Even in the case of the Ising model, the only known practical way of calculation of these eigenvectors is the algebraic method of Kaufman [18] (it should be mentioned, however, that recently a considerable progress has been achieved [17] in the calculation of eigenvectors of the BBS transfer matrix using Sklyanin’s method of separation of variables). It consists of two steps. First one should remark that the transfer matrix induces a rotation in a certain Clifford algebra. Then the eigenvectors are given by certain vectors from a Fock space, associated to the basis of this algebra, in which the above rotation is diagonal. Although Kaufman’s method was later extended to some other free-fermion models [21], it does not seem to work neither for the general free-fermion model nor in the case of the  $\text{BBS}_2$  model $\ddagger$ . The main complication, as compared to the Ising model case, is that one should *guess* the explicit form of the appropriate rotation of the Clifford algebra.

Having spent some time trying to guess the answer for the rotation, the author has finally found another method, which links Grassmann integral approach with the transfer matrix formalism and allows to obtain the eigenvectors of the transfer matrix of the general free-fermion model (i. e. the model with plaquette weight (1.1), satisfying the condition (1.10)). The present paper is devoted to the exposition of this method.

$\ddagger$  Kaufman’s method has also some drawbacks. First, it does not give a convenient representation of eigenvectors in terms of initial spin variables. Therefore, one is forced to do all the calculations in purely algebraic setting. Second, in the Ising case the transfer matrix spectrum is highly degenerate, so the eigenvectors are not determined uniquely. However, in the calculation of correlation functions and form factors one is typically interested in a very precise basis of eigenstates; in addition to the transfer matrix, they should also diagonalize the operator of discrete translations. Kaufman’s method does not guarantee this last condition.

This paper is organized as follows. In the next section, we find a convenient representation of the row-to-row transfer matrix of the periodic BBS<sub>2</sub> model (or, rather, general free-fermion model) in terms of a Grassmann integral, involving two commuting sets of variables (formulas (2.13), (2.15)). In Section 3, the eigenvectors of this transfer matrix are calculated (basic ansatz is given by (3.3)). It should be pointed out that the form of the answer depends on whether the number of sites in one row of the lattice is even or odd. In Section 4, we find a dual basis of eigenvectors and normalize them. It is also shown that one can express form factors of the model in terms of determinants and inverses of certain Toeplitz matrices. Finally, in the last section the above results are specialized to two particular cases (BBS<sub>2</sub> model and Ising model) and are rewritten in more common notation.

## 2. Grassmann integral representation for the transfer matrix

Let us introduce the row-to-row transfer matrix of the BBS<sub>2</sub> model. It is given by the product of plaquette Boltzmann weights over one row (see Figure 1b),

$$T[\sigma, \sigma'] = \prod_{j=1}^L W(\sigma_j, \sigma'_j, \sigma'_{j+1}, \sigma_{j+1}), \quad (2.1)$$

where periodic boundary conditions are imposed on spin variables:

$$\sigma_{L+1} = \sigma_1, \quad \sigma'_{L+1} = \sigma'_1.$$

This matrix naturally acts in the  $2^L$ -dimensional vector space  $V$ , composed of functions of  $L$  spin variables  $\sigma_1, \dots, \sigma_L$ . Namely, for any  $f[\sigma] \in V$  we define the left action

$$(Tf)[\sigma] = \sum_{[\sigma']} T[\sigma, \sigma'] f[\sigma'].$$

Partition function of the BBS<sub>2</sub> model on  $L \times M$  lattice, wrapped on the torus, may be expressed in terms of the eigenvalues of  $T$ :

$$Z(L, M) = \text{Tr } T^M = \sum_{[\sigma^{(1)}]} \dots \sum_{[\sigma^{(M)}]} T[\sigma^{(1)}, \sigma^{(2)}] \dots T[\sigma^{(M)}, \sigma^{(1)}].$$

In order to compute various correlation functions, one also needs to know matrix elements of local field operators in the normalized basis of eigenstates of  $T$ . To obtain these eigenstates in an explicit form, let us first find a convenient representation of the transfer matrix.

From the formulas (1.11) and (2.1) it follows that one can write  $T$  in the form of a Grassmann integral,

$$T[\sigma, \sigma'] = a_0^L \int \mathcal{D}\psi \mathcal{D}\psi' \exp \left\{ \sum_{j=1}^L \left( a_{12} \sigma_j \sigma'_j \psi_j^1 \psi_j^2 - a_{13} \sigma_j \sigma'_{j+1} \psi_j^1 \psi_{j+1}^3 + a_{14} \sigma_j \sigma_{j+1} \psi_j^1 \psi_{j+1}^4 + \right. \right. \\ \left. \left. + a_{23} \sigma'_j \sigma'_{j+1} \psi_j^2 \psi_{j+1}^3 - a_{24} \sigma'_j \sigma_{j+1} \psi_j^2 \psi_{j+1}^4 + a_{34} \sigma'_{j+1} \sigma_{j+1} \psi_{j+1}^3 \psi_{j+1}^4 \right) \right\} \prod_{j=1}^L e^{\psi_j^1} e^{\psi_j^2} e^{\psi_{j+1}^3} e^{\psi_{j+1}^4},$$

where the measure is given by

$$\mathcal{D}\psi \mathcal{D}\dot{\psi} = \prod_{j=1}^L \left( d\psi_j^1 d\dot{\psi}_j^2 d\dot{\psi}_{j+1}^3 d\psi_{j+1}^4 \right).$$

Now let us make the change of integration variables

$$\psi_j^1 \rightarrow \sigma_j \psi_j^1, \quad \dot{\psi}_j^2 \rightarrow \sigma'_j \dot{\psi}_j^2, \quad \dot{\psi}_j^3 \rightarrow \sigma'_j \dot{\psi}_j^3, \quad \psi_j^4 \rightarrow \sigma_j \psi_j^4$$

for all  $j = 1, \dots, L$ . This change does not affect the measure, since every  $\sigma_j$  and  $\sigma'_j$  appears in it twice. Spin variables disappear from the first (quadratic) exponential under the integral, but emerge in the ‘tail’. Namely, one obtains

$$\begin{aligned} T[\sigma, \sigma'] &= a_0^L \int \mathcal{D}\psi \mathcal{D}\dot{\psi} e^{S_1[\psi, \dot{\psi}]} \left[ e^{\sigma_1 \psi_1^1} \prod_{j=2}^L \left( e^{\sigma_j \psi_j^4} e^{\sigma_j \psi_j^1} \right) e^{\sigma_1 \psi_1^4} \right] \times \\ &\quad \times \left[ e^{\sigma'_1 \dot{\psi}_1^2} \prod_{j=2}^L \left( e^{\sigma'_j \dot{\psi}_j^3} e^{\sigma'_j \dot{\psi}_j^2} \right) e^{\sigma'_1 \dot{\psi}_1^3} \right], \end{aligned} \quad (2.2)$$

where  $S_1[\psi, \dot{\psi}]$  can be schematically represented as

$$S_1[\psi, \dot{\psi}] = \frac{1}{2} \begin{pmatrix} \psi & \dot{\psi} \end{pmatrix} \hat{D}_1 \begin{pmatrix} \psi \\ \dot{\psi} \end{pmatrix}, \quad (2.3)$$

with  $\begin{pmatrix} \psi & \dot{\psi} \end{pmatrix} = \begin{pmatrix} \psi^1 & \psi^2 & \dot{\psi}^3 & \psi^4 \end{pmatrix}$  and

$$\hat{D}_1 = \begin{pmatrix} 0 & a_{12} & -a_{13} \nabla_x & a_{14} \nabla_x \\ a_{12} & 0 & a_{23} \nabla_x & -a_{24} \nabla_x \\ -a_{13} \nabla_{-x} & -a_{23} \nabla_{-x} & 0 & a_{34} \\ -a_{14} \nabla_{-x} & -a_{24} \nabla_{-x} & a_{34} & 0 \end{pmatrix}. \quad (2.4)$$

Here  $\nabla_x$  denotes the operator, shifting the lower indices of Grassmann variables by 1. It obeys periodic boundary condition  $(\nabla_x)^L = 1$ . For example, one has

$$\psi^1 \nabla_x \dot{\psi}^3 = \sum_{j=1}^{L-1} \psi_j^1 \dot{\psi}_{j+1}^3 + \psi_L^1 \dot{\psi}_1^3.$$

Note that in (2.2) we have rearranged the tail, assembling together the exponentials, containing the same spin variables. It seems, however, that the exponentials  $e^{\sigma'_1 \dot{\psi}_1^3}$  and  $e^{\sigma_1 \psi_1^4}$  are not on their ‘right’ places. One may correct this, observing that for any function  $F[\psi]$  and for any Grassmann variable  $\psi_\alpha$  we have the identity

$$F[\psi] e^{\psi_\alpha} = e^{\psi_\alpha} \frac{F[\psi] + F[-\psi]}{2} + e^{-\psi_\alpha} \frac{F[\psi] - F[-\psi]}{2}.$$

Now, introducing the notation

$$F_1[\psi] = e^{\sigma_1 \psi_1^1} \prod_{j=2}^L \left( e^{\sigma_j \psi_j^4} e^{\sigma_j \psi_j^1} \right), \quad F_2[\dot{\psi}] = e^{\sigma'_1 \dot{\psi}_1^2} \prod_{j=2}^L \left( e^{\sigma'_j \dot{\psi}_j^3} e^{\sigma'_j \dot{\psi}_j^2} \right),$$

one may rewrite the tail as

$$F_1[\psi] e^{\sigma_1 \psi_1^4} F_2[\dot{\psi}] e^{\sigma'_1 \dot{\psi}_1^3} = \left\{ e^{\sigma_1 \psi_1^4} \frac{F_1[\psi] + F_1[-\psi]}{2} + e^{-\sigma_1 \psi_1^4} \frac{F_1[\psi] - F_1[-\psi]}{2} \right\} \times \quad (2.5)$$

$$\times \left\{ e^{\sigma'_1 \dot{\psi}_1^3} \frac{F_2[\dot{\psi}] + F_2[-\dot{\psi}]}{2} + e^{-\sigma'_1 \dot{\psi}_1^3} \frac{F_2[\dot{\psi}] - F_2[-\dot{\psi}]}{2} \right\}.$$

Expanding this last expression, one obtains 16 terms. However, some of these terms are equivalent, since the simultaneous change of the signs of all  $\psi$  and  $\dot{\psi}$  does not affect the value of the integral. Then one may easily check that (2.5) may be replaced (after appropriate change of variables) by the following combination, containing only 4 terms:

$$F_1[\psi] e^{\sigma_1 \psi_1^4} F_2[\dot{\psi}] e^{\sigma'_1 \dot{\psi}_1^3} \rightarrow \frac{1}{2} \left\{ e^{-\sigma_1 \psi_1^4} F_1[\psi] e^{-\sigma'_1 \dot{\psi}_1^3} F_2[\dot{\psi}] + e^{\sigma_1 \psi_1^4} F_1[-\psi] e^{-\sigma'_1 \dot{\psi}_1^3} F_2[\dot{\psi}] + \right. \quad (2.6)$$

$$\left. + e^{\sigma_1 \psi_1^4} F_1[\psi] e^{\sigma'_1 \dot{\psi}_1^3} F_2[\dot{\psi}] - e^{-\sigma_1 \psi_1^4} F_1[-\psi] e^{\sigma'_1 \dot{\psi}_1^3} F_2[\dot{\psi}] \right\}.$$

The third term of the last expression has the desired form and there is no need to transform it further. If we make the substitution  $\dot{\psi}_1^3 \rightarrow -\dot{\psi}_1^3$ ,  $\psi_1^4 \rightarrow -\psi_1^4$  in the integral, corresponding to the first term, it will have almost the same structure. The only difference is that the boundary condition for the shift operator  $\nabla_x$  becomes antiperiodic:  $(\nabla_x)^L = -1$ .

Next one should remark that the second and the fourth term in (2.6) can be obtained from the first and the third one, respectively, by changing the signs of the spins  $\sigma_1, \dots, \sigma_L$ . This change can be realized, using the operator of spin reflection  $U$ , whose defining property is that  $(Uf)[\sigma] = f[-\sigma]$  for any vector  $f[\sigma] \in V$ . Matrix elements of  $U$  may be explicitly written as

$$U[\sigma, \sigma'] = \prod_{j=1}^L \frac{1 - \sigma_j \sigma'_j}{2}.$$

Summarizing the above observations, we obtain the following representation for the transfer matrix:

$$T = \frac{1+U}{2} T^{NS} + \frac{1-U}{2} T^R, \quad (2.7)$$

where

$$T^{NS(R)}[\sigma, \sigma'] = a_0^L \int \mathcal{D}\psi \mathcal{D}\dot{\psi} \exp \left\{ S_1^{NS(R)}[\psi, \dot{\psi}] \right\} \prod_{j=1}^L \left( e^{\sigma_j \psi_j^4} e^{\sigma_j \dot{\psi}_j^3} \right) \prod_{j=1}^L \left( e^{\sigma'_j \dot{\psi}_j^3} e^{\sigma'_j \psi_j^2} \right), \quad (2.8)$$

and both actions  $S_1^{NS(R)}[\psi, \dot{\psi}]$  are defined by the formulas (2.3)–(2.4). Upper indices NS and R correspond to antiperiodic (Neveu-Schwartz) and periodic (Ramond) boundary conditions, satisfied by the shift operator  $\nabla_x$ .

The matrices  $P_{\pm} = \frac{1 \pm U}{2}$  have the properties of projectors, i. e.  $P_{\pm}^2 = P_{\pm}$ ; thus their eigenvalues are equal to either 0 or 1. The eigenvectors, corresponding to zero eigenvalues of  $P_+$  ( $P_-$ ), are odd (even) under spin reflection. It means that a vector  $f[\sigma] \in V$  will satisfy  $(P_+ f)[\sigma] = 0$  ( $(P_- f)[\sigma] = 0$ ) iff  $f[\sigma] = -f[-\sigma]$  (respectively,  $f[\sigma] = f[-\sigma]$ ). Analogously, the eigenvectors of  $P_+$  ( $P_-$ ) with eigenvalue 1 are even (odd) under spin reflection.

The operator  $U$  commutes with the transfer matrix  $T$ . Therefore, these two matrices can be diagonalized simultaneously and one may choose the eigenvectors of  $T$  so that they are either even or odd under the action of  $U$ . Let us take an even eigenvector  $f_e$  of  $T$ , and denote by  $\lambda_{f_e}$  the corresponding eigenvalue. Acting on  $f_e$  by both sides of the relation (2.7), and using the fact that  $U$  commutes with  $T^{NS}$  and  $T^R$  as well, one obtains

$$\lambda_{f_e} f_e = T f_e = (T^{NS} P_+ + T^R P_-) f_e = T^{NS} f_e,$$

that is, any even eigenvector of  $T$  is an eigenvector of  $T^{NS}$  with the same eigenvalue. Similarly, any odd eigenvector of  $T$  is an eigenvector of  $T^R$ . Conversely, any even eigenvector of  $T^{NS}$  and any odd eigenvector of  $T^R$  are eigenvectors of  $T$ . Therefore, the set of all transfer matrix eigenstates splits into two parts: NS-sector (even eigenvectors of  $T^{NS}$ ) and R-sector (odd eigenvectors of  $T^R$ ). The problem of diagonalization of  $T$  is then reduced to the calculation of eigenvectors and eigenvalues of matrices  $T^{NS}$  and  $T^R$ , given by the formula (2.8).

**Remark.** Having diagonalized  $T^{NS}$  and  $T^R$ , one also gets for free the solution of the BBS<sub>2</sub> model with antiperiodic boundary conditions for spin variables (in one direction). It is easy to understand that the transfer matrix of such model is given by

$$T^a = \frac{1-U}{2} T^{NS} + \frac{1+U}{2} T^R,$$

and, therefore, the set of its eigenstates is composed of odd eigenvectors of  $T^{NS}$  and even eigenvectors of  $T^R$ .

The representation (2.8) can be simplified even further by integrating over fermionic degrees of freedom, which are not coupled to spin variables. We mean the following: elementary factors from the products of (2.8) can be written as

$$e^{\sigma_j \psi_j^4} e^{\sigma_j \psi_j^1} = e^{-\psi_j^1 \psi_j^4} e^{\sigma_j (\psi_j^1 + \psi_j^4)}, \quad e^{\sigma_j' \psi_j^3} e^{\sigma_j' \psi_j^2} = e^{-\psi_j^2 \psi_j^3} e^{\sigma_j' (\psi_j^2 + \psi_j^3)}.$$

Let us now introduce instead of  $\psi$  and  $\dot{\psi}$  new Grassmann variables

$$\varphi_j = \psi_j^1 + \psi_j^4, \quad \dot{\varphi}_j = \psi_j^2 + \psi_j^3, \quad \eta_j = \psi_j^4, \quad \dot{\eta}_j = \psi_j^3, \quad j = 1, \dots, L. \quad (2.9)$$

Since the jacobian of the transformation (2.9) is equal to 1, the integration measure transforms as

$$\mathcal{D}\psi \mathcal{D}\dot{\psi} \rightarrow \mathcal{D}\varphi \mathcal{D}\dot{\varphi} \mathcal{D}\eta \mathcal{D}\dot{\eta} = \prod_{j=1}^L (d\varphi_j d\dot{\varphi}_j d\eta_j d\dot{\eta}_j).$$

Then the integral (2.8) may be rewritten as

$$T^{NS(R)}[\sigma, \sigma'] = a_0^L \int \mathcal{D}\varphi \mathcal{D}\dot{\varphi} \mathcal{D}\eta \mathcal{D}\dot{\eta} \exp \left\{ S_2^{NS(R)}[\varphi, \dot{\varphi}, \eta, \dot{\eta}] \right\} \prod_{j=1}^L e^{\sigma_j \varphi_j} \prod_{j=1}^L e^{\sigma_j' \dot{\varphi}_j}, \quad (2.10)$$

where the action  $S_2^{NS(R)}[\varphi, \dot{\varphi}, \eta, \dot{\eta}]$  is given by

$$S_2^{NS(R)}[\varphi, \dot{\varphi}, \eta, \dot{\eta}] = \frac{1}{2} \begin{pmatrix} \varphi & \dot{\varphi} & \eta & \dot{\eta} \end{pmatrix} \hat{D}_2 \begin{pmatrix} \varphi & \dot{\varphi} & \eta & \dot{\eta} \end{pmatrix}^T,$$

$$\hat{D}_2 = \begin{pmatrix} 0 & a_{12} & -a_{12} - a_{13} \nabla_x & -1 + a_{14} \nabla_x \\ a_{12} & 0 & -1 + a_{23} \nabla_x & -a_{12} - a_{24} \nabla_x \\ -a_{12} - a_{13} \nabla_{-x} & 1 - a_{23} \nabla_{-x} & -a_{23}(\nabla_x - \nabla_{-x}) & a_{12} + a_{34} + a_{13} \nabla_{-x} + a_{24} \nabla_x \\ 1 - a_{14} \nabla_{-x} & -a_{12} - a_{24} \nabla_{-x} & a_{12} + a_{34} + a_{13} \nabla_x + a_{24} \nabla_{-x} & -a_{14}(\nabla_x - \nabla_{-x}) \end{pmatrix}.$$

Let us now integrate over  $\eta$  and  $\dot{\eta}$  in the representation (2.10). This integration can be done relatively easily, since  $S_2^{NS(R)}[\varphi, \dot{\varphi}, \eta, \dot{\eta}]$  is diagonalized by Fourier transformation. Namely, if one denotes

$$\begin{pmatrix} \varphi_p & \dot{\varphi}_p & \dot{\eta}_p & \eta_p \end{pmatrix} = \frac{1}{\sqrt{L}} \sum_{j=1}^L e^{-ipj} \begin{pmatrix} \varphi_j & \dot{\varphi}_j & \dot{\eta}_j & \eta_j \end{pmatrix}, \quad (2.11)$$

then

$$S_2^{NS(R)}[\varphi, \dot{\varphi}, \eta, \dot{\eta}] = \frac{1}{2} \sum_p^{NS(R)} \begin{pmatrix} \varphi_{-p} & \dot{\varphi}_{-p} & \dot{\eta}_{-p} & \eta_{-p} \end{pmatrix} \hat{D}_2(p) \begin{pmatrix} \varphi_p & \dot{\varphi}_p & \dot{\eta}_p & \eta_p \end{pmatrix}^T, \quad (2.12)$$

where the one-mode matrix  $\hat{D}_2(p)$  is given by

$$\hat{D}_2(p) = \begin{pmatrix} 0 & a_{12} & -a_{12} - a_{13} e^{ip} & -1 + a_{14} e^{ip} \\ a_{12} & 0 & -1 + a_{23} e^{ip} & -a_{12} - a_{24} e^{ip} \\ -a_{12} - a_{13} e^{-ip} & 1 - a_{23} e^{-ip} & -2i a_{23} \sin p & a_{12} + a_{34} + a_{13} e^{-ip} + a_{24} e^{ip} \\ 1 - a_{14} e^{-ip} & -a_{12} - a_{24} e^{-ip} & a_{12} + a_{34} + a_{13} e^{ip} + a_{24} e^{-ip} & -2i a_{14} \sin p \end{pmatrix}$$

and the indices NS and R in the sum (2.12) mean that the corresponding quasimomenta run over Neveu-Schwartz values ( $p = \frac{2\pi}{L} (j + \frac{1}{2})$ ,  $j = 0, 1, \dots, L-1$ ) or, correspondingly, Ramond values ( $p = \frac{2\pi}{L} j$ ,  $j = 0, 1, \dots, L-1$ ). Note also that the integration measure can be written as

$$\mathcal{D}\varphi \mathcal{D}\dot{\varphi} \mathcal{D}\eta \mathcal{D}\dot{\eta} = \prod_p^{NS(R)} \left( d\varphi_p d\dot{\varphi}_p d\eta_p d\dot{\eta}_p \right)$$

Thus the  $2L$ -fold integral over  $\eta$  and  $\dot{\eta}$  in the representation (2.10) factorizes into a product of 4-fold (over  $\eta_{\pm p}$ ,  $\dot{\eta}_{\pm p}$ ) and 2-fold integrals. Double integrals correspond to the mode  $p = 0$  (always present in the Ramond sector) and  $p = \pi$  (present in the Ramond sector for even  $L$  and in the Neveu-Schwartz sector for odd  $L$ ). After a little bit cumbersome but nevertheless straightforward calculation one then obtains

$$T^{NS(R)}[\sigma, \sigma'] = \zeta^{NS(R)} \int \mathcal{D}^{NS(R)} \varphi \mathcal{D}^{NS(R)} \dot{\varphi} \exp \left\{ S^{NS(R)}[\varphi, \dot{\varphi}] \right\} \prod_{j=1}^L e^{\sigma_j \varphi_j} \prod_{j=1}^L e^{\sigma'_j \dot{\varphi}_j}, \quad (2.13)$$

where

$$\begin{aligned} \mathcal{D}^{NS(R)} \varphi &= \prod_p^{NS(R)} d\varphi_p, & \mathcal{D}^{NS(R)} \dot{\varphi} &= \prod_p^{NS(R)} d\dot{\varphi}_p, \\ \zeta^{NS(R)} &= a_0^L \prod_p^{NS(R)} \chi_p^{1/2}, \end{aligned} \quad (2.14)$$

$$\chi_p = [a_{12} + a_{34} + (a_{13} + a_{24}) \cos p]^2 + [(a_{13} - a_{24})^2 + 4a_{14}a_{23}] \sin^2 p,$$



and the action  $S^{NS(R)}[\varphi, \dot{\varphi}]$  is given by

$$S^{NS(R)}[\varphi, \dot{\varphi}] = \frac{1}{2} \sum_p^{NS(R)} \begin{pmatrix} \varphi_{-p} & \dot{\varphi}_{-p} \end{pmatrix} \begin{pmatrix} G_{11}(p) & G_{12}(p) \\ G_{21}(p) & G_{22}(p) \end{pmatrix} \begin{pmatrix} \varphi_p \\ \dot{\varphi}_p \end{pmatrix}, \quad (2.15)$$

with

$$\chi_p G_{11}(p) = 2i \sin p \left[ a_{23} + a_{12}a_{24} + a_{13}a_{34} + a_{14}a_4 - 2(a_{14}a_{23} - a_{13}a_{24}) \cos p \right],$$

$$\chi_p G_{22}(p) = 2i \sin p \left[ a_{14} + a_{12}a_{13} + a_{34}a_{24} + a_{23}a_4 - 2(a_{14}a_{23} - a_{13}a_{24}) \cos p \right],$$

$$\begin{aligned} \chi_p G_{12}(p) = \chi_p G_{21}(-p) &= \left[ (a_{12} + a_{34})(a_4 + 1) - (a_{14} + a_{23})(a_{13} + a_{24}) \right] + \\ &+ \left[ (a_{13} + a_{24})(a_4 + 1) - (a_{14} + a_{23})(a_{12} + a_{34}) \right] \cos p + \\ &+ \left[ (a_{24} - a_{13})(a_4 - 1) + (a_{14} - a_{23})(a_{12} - a_{34}) \right] i \sin p. \end{aligned}$$

As we will see in the next section, this final representation for  $T^{NS(R)}$  (given by the formulas (2.13), (2.15)) allows to obtain all transfer matrix eigenvectors almost immediately. Concrete form of the functions  $G_{ij}$  ( $i, j = 1, 2$ ) does not play any essential role.

For further convenience and making parallels with the work [11], let us also introduce the notation  $v_p = 4\chi_p G_{12}(p)$  and

$$\begin{aligned} u_p &= 2\chi_p \left( 1 - G_{11}(p)G_{22}(p) + G_{12}(p)G_{21}(p) \right) = \\ &= 2 \left[ (1 + a_4)^2 + (a_{12} + a_{34})^2 + (a_{13} + a_{24})^2 + (a_{14} + a_{23})^2 \right] + \\ &+ 4 \left[ (a_{12} + a_{34})(a_{13} + a_{24}) - (a_{14} + a_{23})(1 + a_4) \right] \cos p. \end{aligned}$$

**Remark.** It should be pointed out that we did not care about the correct overall sign of  $T^{NS(R)}$  in the representation (2.13). However, using the fact that for  $a_{12} = a_{13} = a_{14} = a_{23} = a_{24} = a_{34} = 0$  all the eigenvalues of  $T$  should be equal to  $2^L a_0^L$ , one can restore this sign at any stage.

### 3. Transfer matrix eigenvectors

It appears that the form of the eigenvectors of  $T^{NS}$  and  $T^R$  depends on whether  $L$  is even or odd. Moreover, quasiparticle interpretation of the eigenvectors and eigenvalues is different in different regions of parameters of the  $BBS_2$  model. Below we will consider various cases in order of increasing difficulty.

#### 3.1. $NS$ -sector, even $L$

If  $L$  is even, then the Neveu-Schwartz spectrum of quasimomenta does not contain the values 0 and  $\pi$  (the only values with the property  $p = -p \pmod{2\pi}$ ).

The simplest ansatz for an eigenvector  $f[\sigma] \in V$  of the matrix  $T^{NS}$  is given by an integral over  $L$  auxiliary Grassmann variables  $\xi_1, \dots, \xi_L$ :

$$f[\sigma] = \int \mathcal{D}^{NS} \xi \exp \left\{ \sum_p \frac{NS}{2} \xi_{-p} A(p) \xi_p \right\} \prod_{j=1}^L e^{\sigma_j \xi_j}. \quad (3.1)$$

Here  $A(p) = -A(-p)$  is an unknown odd function to be determined, and  $\{\xi_p\}$  denote Fourier components of  $\xi$ . The indices  $\frac{NS}{2}$  and  $\frac{R}{2}$  in sums and products will be used to indicate that the corresponding operations involve only those Neveu-Schwartz and Ramond quasimomenta, which lie in the *open* interval  $(0, \pi)$  (for the NS-sector and even  $L$ , this is exactly one half of the Brillouin zone). Note that for even  $L$  the function (3.1) is even with respect to the action of  $U$ : the reversal of all spins is equivalent to the change of variables  $\xi \rightarrow -\xi$ .

Let us now act on  $f[\sigma]$  by the matrix  $T^{NS}$ . Since the fields  $\dot{\varphi}$  and  $\xi$  commute, the sum  $\sum_{[\sigma']} T^{NS}[\sigma, \sigma'] f[\sigma']$  can be easily evaluated and one obtains

$$\begin{aligned} (T^{NS} f)[\sigma] &= 2^L \zeta^{NS} \int \mathcal{D}^{NS} \varphi \mathcal{D}^{NS} \dot{\varphi} e^{S^{NS}[\varphi, \dot{\varphi}]} \prod_{j=1}^L e^{\sigma_j \varphi_j} \times \\ &\quad \times \int \mathcal{D}^{NS} \xi \exp \left\{ \sum_p \frac{NS}{2} \xi_{-p} A(p) \xi_p + \sum_{j=1}^L \dot{\varphi}_j \xi_j \right\}. \end{aligned}$$

After integration over  $\xi$  one finds the exponential of a quadratic form in  $\dot{\varphi}$ ,

$$\begin{aligned} &\int \mathcal{D}^{NS} \xi \exp \left\{ \sum_p \frac{NS}{2} \xi_{-p} A(p) \xi_p + \sum_{j=1}^L \dot{\varphi}_j \xi_j \right\} = \\ &= \int \mathcal{D}^{NS} \xi \exp \left\{ \sum_p \frac{NS}{2} \left( \xi_{-p} A(p) \xi_p + \dot{\varphi}_{-p} \xi_p + \dot{\varphi}_p \xi_{-p} \right) \right\} = \\ &= \left( \prod_p \frac{NS}{2} A(p) \right) \exp \left\{ - \sum_p \frac{NS}{2} \dot{\varphi}_{-p} A^{-1}(p) \dot{\varphi}_p \right\}, \end{aligned}$$

which can then be pulled through the ‘linear’ exponentials. Then one may integrate over  $\dot{\varphi}$  and obtain

$$(T^{NS} f)[\sigma] = 2^L \zeta^{NS} \prod_p \frac{NS}{2} \left( 1 - A(p) G_{22}(p) \right) \int \mathcal{D}^{NS} \varphi \exp \left\{ \sum_p \frac{NS}{2} \varphi_{-p} A'(p) \varphi_p \right\} \prod_{j=1}^L e^{\sigma_j \varphi_j},$$

with

$$A'(p) = G_{11}(p) + \frac{A(p) G_{12}(p) G_{21}(p)}{1 - A(p) G_{22}(p)}.$$

Therefore, the function (3.1) will be an eigenvector of  $T^{NS}$  iff for all NS-values of  $p$  from the interval  $(0, \pi)$  one has  $A(p) = A'(p)$ . This equation is quadratic in  $A(p)$ , and its roots are given by

$$A^\pm(p) = \frac{1 + G(p) \mp \sqrt{(1 - G(p))^2 - 4G_{12}(p)G_{21}(p)}}{2G_{22}(p)}, \quad (3.2)$$

where we have introduced the notation

$$G(p) = G_{11}(p)G_{22}(p) - G_{12}(p)G_{21}(p).$$

Thus the formula (3.1) gives  $2^{L/2}$  eigenvectors of  $T^{NS}$ , corresponding to different choices of the set of one-mode roots.

One can take, for instance,  $A(p) = A^+(p)$  for all  $p \in (0, \pi)$ . The vector, corresponding to this particular choice, will be denoted by  $|vac\rangle_{NS}$ , since under some conditions, satisfied by the parameters of the BBS<sub>2</sub> model, it corresponds to the eigenvalue with maximum modulus. Similarly, if we choose  $A(p) = A^-(p)$  for some values  $p_1, \dots, p_k \in (0, \pi)$ , and  $A(p) = A^+(p)$  for all the other NS-quasimomenta from the interval  $(0, \pi)$ , then the corresponding eigenvector will be denoted by  $|p_1, -p_1; \dots, p_k, -p_k\rangle_{NS}$ . The origin of this notation will become clear soon.

In order to find all the eigenvectors of  $T^{NS}$ , only a slight generalization of the ansatz (3.1) is needed. Namely, let us define

$$f_{\{i_p\}}^{NS}[\sigma] = \int \mathcal{D}^{NS} \xi \prod_p^{NS} F_{i_p}(\xi_{-p}, \xi_p) \prod_{j=1}^L e^{\sigma_j \xi_j}. \quad (3.3)$$

Here each of the indices  $\{i_p\}$  can take any of the four values, which we will conventionally denote by 1, 2, 3, and 4. Corresponding functions  $F_i(\xi_{-p}, \xi_p)$  are defined as follows:

$$F_1(\xi_{-p}, \xi_p) = \exp(\xi_{-p} A^+(p) \xi_p), \quad (3.4)$$

$$F_2(\xi_{-p}, \xi_p) = \xi_{-p}, \quad (3.5)$$

$$F_3(\xi_{-p}, \xi_p) = \xi_p, \quad (3.6)$$

$$F_4(\xi_{-p}, \xi_p) = \exp(\xi_{-p} A^-(p) \xi_p). \quad (3.7)$$

Similarly to the above, one should act on  $f_{\{i_p\}}^{NS}[\sigma]$  by  $T^{NS}$ , then to sum over the intermediate spin variables, to integrate the result over  $\xi$  and, finally, over  $\dot{\varphi}$ . Then it is straightforward to verify that the formulas (3.3)–(3.7) indeed define an eigenvector of  $T^{NS}$  with the eigenvalue

$$\Lambda_{\{i_p\}}^{NS} = 2^L \zeta^{NS} \prod_p^{NS} \lambda_{i_p}(p), \quad (3.8)$$

where ‘one-mode’ eigenvalues are given by

$$\lambda_1(p) = 1 - A^+(p)G_{22}(p), \quad (3.9)$$

$$\lambda_2(p) = G_{12}(p), \quad (3.10)$$

$$\lambda_3(p) = G_{21}(p), \quad (3.11)$$

$$\lambda_4(p) = 1 - A^-(p)G_{22}(p). \quad (3.12)$$

One can also rewrite them in the following way (see the end of the previous section for the notations):

$$\lambda_1(p) = \frac{u_p + \sqrt{u_p^2 - v_p v_{-p}}}{4\chi_p}, \quad (3.13)$$

$$\lambda_2(p) = \frac{v_p}{4\chi_p}, \quad \lambda_3(p) = \frac{v_{-p}}{4\chi_p}, \quad (3.14)$$

$$\lambda_4(p) = \frac{u_p - \sqrt{u_p^2 - v_p v_{-p}}}{4\chi_p}. \quad (3.15)$$

The total number of found eigenstates is equal to  $4^{L/2} = 2^L$ , and thus the diagonalization of the matrix  $T^{NS}$  is completed.

Let us now turn to quasiparticle interpretation of eigenvalues and eigenvectors. It follows from (2.14), (3.8), (3.13)–(3.15) that the eigenvalues can be written in the form

$$\Lambda_{\{i_p\}}^{NS} = \Lambda_{max}^{NS} \prod_{p|i_p=2}^{\frac{NS}{2}} \frac{v_p}{\rho_p} \prod_{p|i_p=3}^{\frac{NS}{2}} \frac{v_{-p}}{\rho_p} \prod_{p|i_p=4}^{\frac{NS}{2}} \frac{v_p v_{-p}}{\rho_p^2}, \quad (3.16)$$

where

$$\Lambda_{max}^{NS} = a_0^L \prod_p^{NS} \rho_p^{1/2}, \quad \rho_p = u_p + \sqrt{u_p^2 - v_p v_{-p}}. \quad (3.17)$$

**Remark.** The expression  $u_p^2 - v_p v_{-p}$  is a quadratic polynomial in  $\cos p$ . For the sake of simplicity, it will be assumed that the parameters  $\{a_{ij}\}_{1 \leq i < j \leq 4}$  are all real and chosen so that this polynomial has no roots inside the interval  $(-1, 1)$  (for example, this condition is satisfied, if one takes  $a_{12} = a_{34}$  and  $a_{13} = a_{24}$ ). It means, in particular, that  $u_p^2 - v_p v_{-p}$  is non-negative and has local extrema only at the points  $p = 0$  and  $p = \pi$ .

Consider also the operator of translations in discrete space  $R$ . Its action on an arbitrary vector  $f[\sigma] \in V$  is defined as

$$(Rf)(\sigma_1, \sigma_2, \dots, \sigma_L) = f(\sigma_2, \sigma_3, \dots, \sigma_1). \quad (3.18)$$

This operator commutes with the transfer matrix  $T$ , with the matrices  $T^{NS}$  and  $T^R$ , and also with the operator  $U$  of spin reflection. Since we have already diagonalized  $T^{NS}$  and obtained nondegenerate spectrum, the eigenvectors (3.3) should diagonalize  $R$  as well. Actually, it is not difficult to verify that

$$(Rf_{\{i_p\}}^{NS})[\sigma] = \prod_{p|i_p=2}^{\frac{NS}{2}} e^{ip} \prod_{p|i_p=3}^{\frac{NS}{2}} e^{-ip} f_{\{i_p\}}^{NS}[\sigma].$$

Now it is clear that the eigenvectors of  $T^{NS}$  can be labelled by the collections of distinct NS-quasimomenta and interpreted as multiparticle states. One-particle energy is given by

$$\varepsilon(p) = -\ln \frac{v_p}{\rho_p}.$$

It may have a non-zero imaginary part, which is a general consequence of the fact that the transfer matrix  $T$  of the BBS<sub>2</sub> model is not symmetric. The eigenstate, which contains particles with the momenta  $p_1, \dots, p_k$ , will be denoted by  $|p_1, \dots, p_k\rangle_{NS}$ . In order to determine, which one of the functions (3.3) gives the explicit form of this vector, one should decompose the set of momenta of particles from the state  $|p_1, \dots, p_k\rangle_{NS}$  into three parts: pairs of the form  $\pm p_j$ , ‘unpaired’ momenta from the interval  $(0, \pi)$ , and ‘unpaired’ momenta from the interval  $(\pi, 2\pi)$ . Then in the ansatz (3.3) one should set

- $i_p = 4$ , if  $\pm p$  appears in the first part,
- $i_p = 2$ , if  $p$  appears in the second part,
- $i_p = 3$ , if  $-p$  appears in the third part,
- $i_p = 1$  for all the other values of  $p$ .

This procedure establishes the correspondence between the formulas (3.3) and usual quasiparticle notation.

**Remark.** Recall that only even eigenvectors of  $T^{NS}$  diagonalize the full transfer matrix  $T$  as well. It means that the total number of appearances of  $i_p = 2$  and  $i_p = 3$  in the functions (3.3) should be even. In other words, NS-eigenstates of  $T$  should contain even number of particles.

### 3.2. NS-sector, odd $L$

When  $L$  is odd, the Neveu-Schwartz spectrum of quasimomenta contains the value  $p = \pi$ . To take into account this special mode, it is sufficient to slightly modify the ansatz (3.3). Let us consider

$$f_{\{i_p\}}^{NS}[\sigma] = \int \mathcal{D}^{NS} \xi \tilde{F}_{i_\pi}(\xi_\pi) \prod_p^{NS} F_{i_p}(\xi_{-p}, \xi_p) \prod_{j=1}^L e^{\sigma_j \xi_j}, \quad (3.19)$$

where all the indices  $\{i_p\}$ , except  $i_\pi$ , take on four values as above, and the functions  $F_1 \dots F_4$  are given by (3.4)–(3.7). The index  $i_\pi$  can have only two values, 1 and 2, and the corresponding functions  $\tilde{F}_1$  and  $\tilde{F}_2$  are simply

$$\tilde{F}_1(\xi) = 1, \quad \tilde{F}_2(\xi) = \xi.$$

One may verify that the function (3.19) gives an eigenvector of  $T^{NS}$  with the eigenvalue

$$\Lambda_{\{i_p\}}^{NS} = 2^L \zeta^{NS} \tilde{\lambda}_{i_\pi}(\pi) \prod_p^{NS} \lambda_{i_p}(p),$$

where all  $\lambda_i(p)$  are defined as above and

$$\tilde{\lambda}_1(\pi) = 1, \quad \tilde{\lambda}_2(\pi) = G_{12}(\pi) = \frac{v_\pi}{4\chi_\pi}.$$

Since the total number of eigenvectors (3.19) is equal to  $2 \times 4^{(L-1)/2} = 2^L$ , the diagonalization of  $T^{NS}$  is completed.

The first thing that may seem unusual is that if we set  $i_p = 1$  for all  $p$ , including  $p = \pi$ , the corresponding eigenvector of  $T^{NS}$  will not always represent the physical vacuum. Moreover, this vector is *odd* under spin reflection and, therefore, it is not an eigenvector of the full transfer matrix  $T$ . Note also that

$$\rho_\pi^{1/2} = 2\chi_\pi^{1/2} \max \left\{ \frac{|v_\pi|}{4\chi_\pi}, 1 \right\}.$$

Therefore, if one tries to write the eigenvalues in the form, analogous to (3.16), then the result will be different in different regions of parameters. Namely, for  $|v_\pi|/4\chi_\pi \geq 1$  one obtains

$$\Lambda_{\{i_p\}}^{NS} = \Lambda_{max}^{NS} \left( \frac{v_\pi}{\rho_\pi} \right)^{2-i_\pi} \prod_{p|i_p=2}^{\frac{NS}{2}} \frac{v_p}{\rho_p} \prod_{p|i_p=3}^{\frac{NS}{2}} \frac{v_{-p}}{\rho_p} \prod_{p|i_p=4}^{\frac{NS}{2}} \frac{v_p v_{-p}}{\rho_p^2},$$

and for  $|v_\pi|/4\chi_\pi \leq 1$  we have

$$\Lambda_{\{i_p\}}^{NS} = \Lambda_{max}^{NS} \left( \frac{v_\pi}{\rho_\pi} \right)^{i_\pi-1} \prod_{p|i_p=2}^{\frac{NS}{2}} \frac{v_p}{\rho_p} \prod_{p|i_p=3}^{\frac{NS}{2}} \frac{v_{-p}}{\rho_p} \prod_{p|i_p=4}^{\frac{NS}{2}} \frac{v_p v_{-p}}{\rho_p^2},$$

where  $\Lambda_{max}^{NS}$  is defined by the formula (3.17). Thus one can again interpret the eigenvectors of  $T^{NS}$  as multiparticle states  $|p_1, \dots, p_k\rangle_{NS}$ . The main differences with the previous case are the following:

- If  $|v_\pi|/4\chi_\pi \geq 1$ , then the states, containing a particle with the momentum  $p = \pi$ , are given by the ansatz (3.19) with  $i_\pi = 1$ ; for  $|v_\pi|/4\chi_\pi \leq 1$  they correspond to the choice  $i_\pi = 2$ .
- For  $|v_\pi|/4\chi_\pi \geq 1$  the eigenstates, which are even (odd) under spin reflection, contain even (odd) number of particles, while for  $|v_\pi|/4\chi_\pi \leq 1$  this number should be odd (even).

### 3.3. $R$ -sector, odd $L$

The treatment of this case is completely analogous to the previous one, since for odd  $L$  Ramond spectrum contains only one ‘special’ mode  $p = 0$ . All the eigenvectors and eigenvalues of the matrix  $T^R$  are given by

$$f_{\{i_p\}}^R[\sigma] = \int \mathcal{D}^R \xi \tilde{F}_{i_0}(\xi_0) \prod_p^{\frac{R}{2}} F_{i_p}(\xi_{-p}, \xi_p) \prod_{j=1}^L e^{\sigma_j \xi_j}, \quad (3.20)$$

$$\Lambda_{\{i_p\}}^R = 2^L \zeta^R \tilde{\lambda}_{i_0}(0) \prod_p^{\frac{R}{2}} \lambda_{i_p}(p).$$

Here the indices  $\{i_p\}_{p \neq 0}$  take on four values,  $i_0 = 1, 2$ , the functions  $\{\tilde{F}_j\}$ ,  $\{F_j\}$ ,  $\{\lambda_j\}$  are defined as above and

$$\tilde{\lambda}_1(0) = 1, \quad \tilde{\lambda}_2(0) = G_{12}(0) = \frac{v_0}{4\chi_0}.$$

Again, since we have

$$\rho_0^{1/2} = 2\chi_0^{1/2} \max \left\{ \frac{|v_0|}{4\chi_0}, 1 \right\},$$

the quasiparticle interpretation of eigenvalues and eigenvectors is different in the regions  $|v_0|/4\chi_0 \geq 1$  and  $|v_0|/4\chi_0 \leq 1$ . Namely, one has

$$\Lambda_{\{i_p\}}^R = \Lambda_{max}^R \left( \frac{v_0}{\rho_0} \right)^{2-i_0} \prod_{p|i_p=2}^{\frac{R}{2}} \frac{v_p}{\rho_p} \prod_{p|i_p=3}^{\frac{R}{2}} \frac{v_{-p}}{\rho_p} \prod_{p|i_p=4}^{\frac{R}{2}} \frac{v_p v_{-p}}{\rho_p^2} \quad \text{for } |v_0|/4\chi_0 \geq 1,$$

$$\Lambda_{\{i_p\}}^R = \Lambda_{max}^R \left( \frac{v_0}{\rho_0} \right)^{i_0-1} \prod_{p|i_p=2}^{\frac{R}{2}} \frac{v_p}{\rho_p} \prod_{p|i_p=3}^{\frac{R}{2}} \frac{v_{-p}}{\rho_p} \prod_{p|i_p=4}^{\frac{R}{2}} \frac{v_p v_{-p}}{\rho_p^2} \quad \text{for } |v_0|/4\chi_0 \leq 1,$$

where the eigenvalue with the maximum modulus,  $\Lambda_{max}^R$ , is given by

$$\Lambda_{max}^R = \prod_p^R \rho_p^{1/2}.$$

Similarly to the above, let us denote by  $|p_1, \dots, p_k\rangle_R$  the eigenstate of  $T^R$ , containing  $k$  particles with distinct R-momenta  $p_1, \dots, p_k$ .

Note that even (odd) eigenstates of  $T^R$  should contain even (odd) number of particles for  $|v_0|/4\chi_0 \geq 1$ , and odd (even) number of particles for  $|v_0|/4\chi_0 \leq 1$ . This change can be easily understood if we take, say,  $|v_0|/4\chi_0 \leq 1$ , and consider two eigenstates,  $|p_1, \dots, p_k\rangle_R$  and  $|0, p_1, \dots, p_k\rangle_R$  ( $p_j \neq 0, j = 1, \dots, k$ ). Then let us gradually increase  $|v_0|/4\chi_0$ . When this parameter approaches the critical value 1, two eigenstates correspond to the same eigenvalue, and when it exceeds 1, the roles of two vectors swap around: the particle with zero momentum disappears from the second vector (thus decreasing the number of particles by 1) and appears in the first (the number of particles increases by 1).

#### 3.4. R-sector, even $L$

Since for even  $L$  the Ramond spectrum of quasimomenta contains both  $p = 0$  and  $p = \pi$ , the eigenvectors and eigenvalues of  $T^R$  in this case can be written in the following way:

$$f_{\{i_p\}}^R[\sigma] = \int \mathcal{D}^R \xi \tilde{F}_{i_0}(\xi_0) \tilde{F}_{i_\pi}(\xi_\pi) \prod_p^{\frac{R}{2}} F_{i_p}(\xi_{-p}, \xi_p) \prod_{j=1}^L e^{\sigma_j \xi_j}, \quad (3.21)$$

$$\Lambda_{\{i_p\}}^R = 2^L \zeta^R \tilde{\lambda}_{i_0}(0) \tilde{\lambda}_{i_\pi}(\pi) \prod_p^{\frac{R}{2}} \lambda_{i_p}(p).$$

From the physical point of view, here one should distinguish four different regions in the space of parameters. They have the following properties:

- $|v_0| \geq 4\chi_0, |v_\pi| \geq 4\chi_\pi$ . The eigenstates of  $T^R$ , containing a particle with the momentum  $p = 0$  ( $p = \pi$ ), are given by the formula (3.21) with  $i_0 = 1$  ( $i_\pi = 1$ ). The eigenvectors, which are even (odd) under spin reflection, should contain even (odd) number of particles.
- $|v_0| \geq 4\chi_0, |v_\pi| \leq 4\chi_\pi$ . The eigenstates, containing a particle with the momentum  $p = 0$  ( $p = \pi$ ), correspond to  $i_0 = 1$  ( $i_\pi = 2$ ). Even (odd) eigenvectors contain odd (even) number of particles.
- $|v_0| \leq 4\chi_0, |v_\pi| \geq 4\chi_\pi$ . Particle with the momentum  $p = 0$  ( $p = \pi$ ) corresponds to  $i_0 = 2$  ( $i_\pi = 1$ ). Even (odd) eigenvectors contain odd (even) number of particles.
- $|v_0| \leq 4\chi_0, |v_\pi| \leq 4\chi_\pi$ . Particle with the momentum  $p = 0$  ( $p = \pi$ ) corresponds to  $i_0 = 2$  ( $i_\pi = 2$ ). Even (odd) eigenvectors contain even (odd) number of particles.

#### 4. Norms and form factors

In the present section, the problem of computation of correlation functions of the  $\text{BBS}_2$  model is addressed. Local fields will be represented by spin variables  $\sigma_{i,j}$  ( $i = 1, \dots, L$ ;  $j = 1, \dots, M$ ). In the transfer matrix formalism,  $2k$ -point correlation functions  $\langle \sigma_{i_1, j_1} \sigma_{i_2, j_2} \dots \sigma_{i_{2k}, j_{2k}} \rangle$  can be written in the following way<sup>§</sup>:

$$\begin{aligned} \langle \sigma_{i_1, j_1} \dots \sigma_{i_{2k}, j_{2k}} \rangle &= Z^{-1}(L, M) \times \\ &\times \sum_{[\sigma^{(1)}]} \dots \sum_{[\sigma^{(2k)}]} \sigma_{i_1}^{(1)} T^{j_2 - j_1}[\sigma^{(1)}, \sigma^{(2)}] \sigma_{i_2}^{(2)} T^{j_2 - j_1}[\sigma^{(2)}, \sigma^{(3)}] \dots \sigma_{i_{2k}}^{(2k)} T^{M - (j_{2k} - j_1)}[\sigma^{(2k)}, \sigma^{(1)}], \end{aligned} \quad (4.1)$$

where it was assumed that  $j_1 \leq j_2 \leq \dots \leq j_{2k}$ . Let us introduce spin operator

$$S_{1,1}[\sigma, \sigma'] = \sigma_1 \delta_{[\sigma], [\sigma']} = \sigma_1 \prod_{j=1}^L \frac{1 + \sigma_j \sigma'_j}{2},$$

acting on functions  $f[\sigma] \in V$  from the left in the usual way. If we make use of the translation operator  $R$  (see formula (3.18)) to define

$$S_{i,j} = T^{j-1} R^{i-1} S_{1,1} R^{1-i} T^{1-j}, \quad i = 1, \dots, L, \quad j = 1, \dots, M,$$

then one may rewrite (4.1) as

$$\langle \sigma_{i_1, j_1} \dots \sigma_{i_{2k}, j_{2k}} \rangle = \frac{\text{Tr} (S_{i_1, j_1} S_{i_2, j_2} \dots S_{i_{2k}, j_{2k}} T^M)}{\text{Tr} T^M}. \quad (4.2)$$

Since all the eigenvalues of the transfer matrix  $T$  are known, the problem reduces to the calculation of the trace in the numerator. One would want to compute this trace in the basis of eigenstates of  $T$ . However, such computation is not quite straightforward, since the transfer matrix of the  $\text{BBS}_2$  model is not symmetric and thus its eigenvectors are not necessarily orthogonal. Therefore, in order to construct the dual basis, one should separately find the eigenvectors for the *right* action of  $T$ , (since  $T$  is not symmetric, they can not be obtained from the eigenvectors, found in the previous section, by simple transposition).

Assume for a moment that we have found all ‘left’ and ‘right’ eigenvectors of  $T$ . Let us denote them by  $|n\rangle$  and  $\langle n|$ , where  $n$  is any convenient set of quantum numbers, identifying the eigenstate (for example, the number of particles and their quasimomenta). The resolution of the identity matrix in this basis of eigenstates has the form

$$\mathbf{1} = \sum_n b_n |n\rangle \langle n|, \quad b_n = 1/\langle n|n\rangle. \quad (4.3)$$

The relation (4.3) means, in particular, that the trace of any matrix  $X$  can be written as

$$\text{Tr} X = \sum_n b_n \langle n|X|n\rangle.$$

<sup>§</sup> All  $(2k + 1)$ -point correlation functions vanish due to  $\mathbb{Z}_2$ -symmetry of the model.



Recall also that the eigenvectors of  $T$  diagonalize as well the translation operator  $R$ . Therefore, inserting the resolution of the identity matrix into the representation (4.2)  $k$  times, one can rewrite  $2k$ -point correlation function in the form of the so-called form factor expansion. For example, for the 2-point correlation function one has

$$\langle \sigma_{i_1, j_1} \sigma_{i_2, j_2} \rangle = \frac{\sum_{m, n} b_m b_n \langle n | S_{1,1} | m \rangle \langle m | S_{1,1} | n \rangle e^{-E_m(j_2 - j_1) - E_n(M - j_2 + j_1) + i(P_m - P_n)(i_2 - i_1)}}{\sum_n e^{-ME_n}}. \quad (4.4)$$

Matrix elements  $\langle n | S_{1,1} | m \rangle$ , entering this formula, hereinafter will be referred to as form factors. Parameters  $E_n$  and  $P_n$  have the meaning of energy and total momentum of the state  $|n\rangle$  (and  $\langle n|$ ). They are related to the eigenvalues of  $T$  and  $R$  in the following way:

$$T|n\rangle = \Lambda_{max} e^{-E_n} |n\rangle, \quad R|n\rangle = e^{iP_n} |n\rangle,$$

where  $\Lambda_{max}$  denotes the eigenvalue of  $T$  with the maximum modulus. In the BBS<sub>2</sub> model, the values of  $E_n$  and  $P_n$ , corresponding to the multiparticle state  $|n\rangle = |p_1, \dots, p_k\rangle$ , are given by the sums of one-particle energies and momenta.

The generalization of the form factor expansion (4.4) to the multipoint case is straightforward. Thus in order to find all correlation functions, only three further steps should be made. First one should find the eigenvectors for the right action of the transfer matrix  $T$ , i. e. the functions  $f[\sigma] \in V$  such that  $\sum_{[\sigma]} f[\sigma] T[\sigma, \sigma'] = \lambda_f f[\sigma']$ . Then one needs to compute scalar products  $\langle n | n \rangle$ . Finally, the most difficult task is the calculation of form factors  $\langle n | S_{1,1} | m \rangle$ . All these problems are treated (the third one with only a partial success) in the following subsections.

#### 4.1. Eigenvectors for the right action of $T$

The variables  $[\sigma]$  and  $[\sigma']$  enter into the representation (2.13) for the matrices  $T^{NS}$  and  $T^R$  in a similar way. Therefore, one may construct the eigenvectors for the right action of  $T$  along the lines of Section 3. However, there exists even more straightforward way to obtain them. Note that the right action of  $T$  on  $f[\sigma] \in V$  coincides with the left action of the transfer matrix  $\dot{T} = T^T$  of another BBS<sub>2</sub> model (see Figure 1b), characterized by the parameters

$$\dot{a}_{12} = a_{12}, \quad \dot{a}_{13} = a_{24}, \quad \dot{a}_{14} = a_{23}, \quad (4.5)$$

$$\dot{a}_{23} = a_{14}, \quad \dot{a}_{24} = a_{23}, \quad \dot{a}_{34} = a_{34}. \quad (4.6)$$

Thus the eigenvectors we are looking for may be obtained from already found ones by the substitution (4.5)–(4.6) and matching the eigenvalues.

It is easy to verify that under the above substitution various quantities, used in the construction of eigenvectors and eigenvalues, change as follows:

$$\begin{pmatrix} G_{11}(p) & G_{12}(p) \\ G_{21}(p) & G_{22}(p) \end{pmatrix} \rightarrow \begin{pmatrix} \dot{G}_{11}(p) & \dot{G}_{12}(p) \\ \dot{G}_{21}(p) & \dot{G}_{22}(p) \end{pmatrix} = \begin{pmatrix} G_{22}(p) & G_{21}(p) \\ G_{12}(p) & G_{11}(p) \end{pmatrix},$$

$$A^\pm(p) \rightarrow \dot{A}^\pm(p) = \frac{1 + G(p) \mp \sqrt{(1 - G(p))^2 - 4G_{12}(p)G_{21}(p)}}{2G_{11}(p)}, \quad (4.7)$$

$$\chi_p \rightarrow \dot{\chi}_p = \chi_p, \quad u_p \rightarrow \dot{u}_p = u_p, \quad v_p \rightarrow \dot{v}_p = v_{-p}.$$

Let us now consider, for instance, NS-sector and assume that  $L$  is even. Let  $f_{\{i_p\}}^{NS}[\sigma]$  denote the ‘left’ eigenvector (3.3), corresponding to the eigenvalue  $\Lambda_{\{i_p\}}^{NS}$ . Under the substitution (4.5)–(4.6) ‘partial’ eigenvalues  $\lambda_2(p)$  and  $\lambda_3(p)$  (formulas (3.10), (3.11)) exchange their roles, while  $\lambda_1(p)$  and  $\lambda_4(p)$  remain unchanged. Then it becomes clear that the ‘right’ eigenvector  $\dot{f}_{\{i_p\}}^{NS}[\sigma]$  of  $T^{NS}$ , corresponding to the same eigenvalue as  $f_{\{i_p\}}^{NS}[\sigma]$ , is given by

$$\dot{f}_{\{i_p\}}^{NS}[\sigma] = \int \mathcal{D}^{NS} \dot{\xi} \prod_p^{\frac{NS}{2}} \dot{F}_{i_p}(\dot{\xi}_{-p}, \dot{\xi}_p) \prod_{j=1}^L e^{\sigma_j \dot{\xi}_j}, \quad (4.8)$$

with

$$\dot{F}_1(\dot{\xi}_{-p}, \dot{\xi}_p) = \exp\left(\dot{\xi}_{-p} \dot{A}^+(p) \dot{\xi}_p\right), \quad (4.9)$$

$$\dot{F}_2(\dot{\xi}_{-p}, \dot{\xi}_p) = \dot{\xi}_p, \quad (4.10)$$

$$\dot{F}_3(\dot{\xi}_{-p}, \dot{\xi}_p) = \dot{\xi}_{-p}, \quad (4.11)$$

$$\dot{F}_4(\dot{\xi}_{-p}, \dot{\xi}_p) = \exp\left(\dot{\xi}_{-p} \dot{A}^-(p) \dot{\xi}_p\right), \quad (4.12)$$

the functions  $\dot{A}^\pm(p)$  being defined by the formula (4.7). Dotted Grassmann variables  $\dot{\xi}$  are used in the representation (4.8) for further convenience in the computation of scalar products and form factors.

In order to obtain a similar answer for the other cases (Neveu-Schwartz sector for odd  $L$  and Ramond sector), it is sufficient to substitute in (3.19), (3.20) and (3.21) instead of  $F_1 \dots F_4$  new functions  $\dot{F}_1 \dots \dot{F}_4$ . The functions  $\tilde{F}_1$  and  $\tilde{F}_2$ , which are responsible for the special modes  $p = 0, \pi$ , remain unchanged.

#### 4.2. Normalization

It is instructive to consider not only the norms  $\langle n|n \rangle$ , but also general scalar products  $\langle m|n \rangle$ , and to verify by hand that  $\langle m|n \rangle = 0$  for  $m \neq n$ . First one should remark that the eigenvectors of  $T$ , which belong to different sectors, are orthogonal, since they correspond to different eigenvalues of  $U$ . Thus one may look at each sector separately. Let us now consider, for instance, the Neveu-Schwartz sector for even  $L$ . Let us take a ‘right’ eigenvector  ${}_{NS}\langle \{i_p\} | \stackrel{def}{=} \dot{f}_{\{i_p\}}^{NS}[\sigma]$  (given by the formula (4.8)) and a ‘left’ eigenvector  $| \{j_p\} \rangle_{NS} \stackrel{def}{=} f_{\{j_p\}}^{NS}[\sigma]$  (given by the formula (3.3)), and then compute their scalar product

$${}_{NS}\langle \{i_p\} | \{j_p\} \rangle_{NS} = \sum_{[\sigma]} \dot{f}_{\{i_p\}}^{NS}[\sigma] f_{\{j_p\}}^{NS}[\sigma].$$

Since the fields  $\xi$  and  $\dot{\xi}$  in the representations (3.3) and (4.8) commute, the summation over intermediate spins can be easily done and one obtains

$${}_{NS}\langle\{i_p\}|\{j_p\}\rangle_{NS} = 2^L \int \mathcal{D}^{NS}\xi \mathcal{D}^{NS}\dot{\xi} \prod_p^{\frac{NS}{2}} \left( \dot{F}_{i_p}(\dot{\xi}_{-p}, \dot{\xi}_p) F_{j_p}(\xi_{-p}, \xi_p) e^{\dot{\xi}_{-p}\xi_p + \xi_{-p}\dot{\xi}_p} \right).$$

Calculation of this factorized integral gives

$${}_{NS}\langle\{i_p\}|\{j_p\}\rangle_{NS} = 2^L \prod_p^{\frac{NS}{2}} \alpha_{i_p j_p}(p),$$

where the functions  $\alpha_{ij}(p)$  can be assembled into a  $4 \times 4$  matrix

$$\|\alpha_{ij}(p)\|_{i,j=1,\dots,4} = \begin{pmatrix} \dot{A}^+(p)A^+(p) - 1 & 0 & 0 & \dot{A}^+(p)A^-(p) - 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \dot{A}^-(p)A^+(p) - 1 & 0 & 0 & \dot{A}^-(p)A^-(p) - 1 \end{pmatrix}.$$

Using the explicit formulas for  $\dot{A}^\pm(p)$  and  $A^\pm(p)$ , one may check that  $\dot{A}^\pm(p)A^\mp(p) = 1$ . Therefore, ‘right’ and ‘left’ eigenvectors, corresponding to different eigenvalues, are orthogonal (as it should be). The norm  ${}_{NS}\langle\{i_p\}|\{i_p\}\rangle_{NS}$  is given by

$${}_{NS}\langle\{i_p\}|\{i_p\}\rangle_{NS} = 2^L \prod_p^{\frac{NS}{2}} \alpha_{i_p}(p), \quad (4.13)$$

where we have introduced the notation

$$\begin{aligned} \alpha_1(p) &= 1 - \dot{A}^+(p)A^+(p), \\ \alpha_2(p) &= \alpha_3(p) = 1, \\ \alpha_4(p) &= 1 - \dot{A}^-(p)A^-(p), \end{aligned}$$

and corrected the overall sign. One may check that the answer for the Neveu-Schwartz sector and odd  $L$  is given by the same formula (4.13). The only things that change in the Ramond sector are the values of quasimomenta.

### 4.3. Form factors

Since the eigenstates of  $T$  from the same sector are all simultaneously even or odd under spin reflection, all form factors of type NS–NS and R–R are equal to zero. Now assume for definiteness that  $L$  is even and consider a ‘right’ eigenvector  ${}_{NS}\langle\{i_p\}| = \dot{f}_{\{i_p\}}^{NS}[\sigma]$  from the Neveu-Schwartz sector and a ‘left’ eigenvector  $|\{j_p\}\rangle_R = f_{\{j_p\}}^R[\sigma]$  from the Ramond sector. Let us calculate the form factor

$${}_{NS}\langle\{i_p\}|S_{1,1}|\{j_p\}\rangle_R = \sum_{[\sigma]} \sigma_1 \dot{f}_{\{i_p\}}^{NS}[\sigma] f_{\{j_p\}}^R[\sigma].$$

After summation over intermediate spins one obtains

$$\begin{aligned} {}_{NS}\langle\{i_p\}|S_{1,1}|\{j_p\}\rangle_R &= 2^L \int \mathcal{D}^R\xi \mathcal{D}^{NS}\dot{\xi} \tilde{F}_{j_0}(\xi_0) \tilde{F}_{j_\pi}(\xi_\pi) \prod_p^{\frac{R}{2}} F_{j_p}(\xi_{-p}, \xi_p) \times \\ &\times \prod_q^{\frac{NS}{2}} \dot{F}_{i_q}(\dot{\xi}_{-q}, \dot{\xi}_q)(\xi_1 + \dot{\xi}_1) \exp \left\{ \sum_{k=1}^L \xi_k \dot{\xi}_k \right\}. \end{aligned} \quad (4.14)$$

Unfortunately, we have not managed to find a compact expression for this gaussian integral, although we strongly suspect it is possible. In order to illustrate emerging difficulties, let us assume that  $|v_0| < 4\chi_0$ ,  $|v_\pi| > 4\chi_\pi$  (this region mimics ferromagnetic phase), and consider the simplest possible form factor  ${}_{NS}\langle vac|S_{1,1}|vac\rangle_R$ , which corresponds to the following choice:  $j_0 = 1$ ,  $j_\pi = 2$ ,  $j_p = 1$  for all  $p \in (0, \pi)$ ,  $i_q = 1$  for all  $q \in (0, \pi)$ . One then obtains

$$\begin{aligned} {}_{NS}\langle vac|S_{1,1}|vac\rangle_R &= 2^L \int \mathcal{D}^R \xi \mathcal{D}^{NS} \dot{\xi} \xi_\pi (\xi_1 + \dot{\xi}_1) \times \\ &\times \exp \left\{ \sum_p^{\frac{R}{2}} \xi_{-p} A^+(p) \xi_p + \sum_q^{\frac{NS}{2}} \dot{\xi}_{-q} \dot{A}^+(q) \dot{\xi}_q + \sum_{k=1}^L \xi_k \dot{\xi}_k \right\}. \end{aligned}$$

Quadratic form in the exponential consists of three pieces, which can not be diagonalized simultaneously: the first and the second piece are diagonal in the Fourier basis with Ramond and Neveu-Schwartz values of discrete quasimomenta, and the third one is diagonal in the coordinate representation. Actually, one can now remove the dots, using the following rule: all quadratic terms in the exponential, containing a dotted variable on the left, should change their signs. Performing this operation and passing to the coordinate representation in all terms, one obtains

$$\begin{aligned} &{}_{NS}\langle vac|S_{1,1}|vac\rangle_R = \tag{4.15} \\ &= \frac{2^L}{\sqrt{L}} \int \mathcal{D}\xi \mathcal{D}\eta \sum_{k=1}^L (-1)^k \xi_k (\xi_1 + \eta_1) \exp \left\{ \frac{1}{2} \begin{pmatrix} \xi & \eta \end{pmatrix} \begin{pmatrix} A^+ & \mathbf{1} \\ -\mathbf{1} & -\dot{A}^+ \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right\}, \end{aligned}$$

where antisymmetric  $L \times L$  matrices  $A^+$ ,  $\dot{A}^+$  are given by

$$A_{xx'}^+ = \frac{1}{L} \sum_{p \neq 0, \pi}^R A^+(p) e^{ip(x-x')}, \quad \dot{A}_{xx'}^+ = \frac{1}{L} \sum_q^{NS} \dot{A}^+(q) e^{iq(x-x')}, \quad x, x' = 1, \dots, L.$$

Evaluation of the gaussian integral (4.15) gives

$${}_{NS}\langle vac|S_{1,1}|vac\rangle_R = \frac{2^L}{\sqrt{L}} \text{Pf}(\dot{A}^+) \text{Pf}(H) \sum_{k=1}^L (-1)^k \left[ H^{-1} - (\dot{A}^+)^{-1} H^{-1} \right]_{1k}, \tag{4.16}$$

where  $L \times L$  matrix  $H$  is also antisymmetric and has Toeplitz form:

$$\begin{aligned} H_{xx'} &= (\dot{A}^+)_{xx'}^{-1} - A_{xx'}^+ = \\ &= \frac{1}{L} \sum_q^{NS} A^-(q) e^{iq(x-x')} - \frac{1}{L} \sum_{p \neq 0, \pi}^R A^+(p) e^{ip(x-x')}. \end{aligned}$$

Thus the problem of computation of the form factor  ${}_{NS}\langle vac|S_{1,1}|vac\rangle_R$  is reduced to the calculation of the determinant of  $H$  and inverse matrix  $H^{-1}$ . Actually, this is also the case for more complicated form factors. In spite of the remarkably simple form of the matrix  $H$ , we have not succeeded in the calculation of  $\text{Pf}(H)$  and  $H^{-1}$ . However, we believe that the representations of type (4.16) are still useful, since they effectively reduce initial  $2^L$ -dimensional problem to an  $L$ -dimensional one.

It should also be pointed out that form factor  ${}_{NS}\langle vac|S_{1,1}|vac\rangle_R$  enters into the definition of the order parameter of the BBS<sub>2</sub> model. More precisely, one has

$$\langle\sigma\rangle^2 \stackrel{def}{=} \lim_{i,j\rightarrow\infty} \left( \lim_{L,M\rightarrow\infty} \langle\sigma_{1,1}\sigma_{i,j}\rangle \right) = \lim_{L\rightarrow\infty} \frac{{}_{NS}\langle vac|S_{1,1}|vac\rangle_R {}_R\langle vac|S_{1,1}|vac\rangle_{NS}}{{}_{NS}\langle vac|vac\rangle_{NS} {}_R\langle vac|vac\rangle_R}.$$

Although we have not managed to obtain a closed expression for this form factor, the order parameter can presumably be calculated by another method. We hope to return to this problem elsewhere.

## 5. Special cases

### 5.1. BBS<sub>2</sub> model

Parameters of the general free-fermion model, which correspond to BBS<sub>2</sub> model (via the formulas (1.2)–(1.9)), are not independent. In particular, in addition to free-fermion condition (1.10), they also satisfy the relation  $a_{13}a_{24} = a_{14}a_{23}$ . Therefore, one could expect some simplifications of the above formulas for transfer matrix eigenvectors to occur in this case. Furthermore, it is known that the eigenvalues of the BBS<sub>2</sub> transfer matrix should have polynomial dependence on spectral variable  $t$ , and that the eigenvectors should not depend on it. In order to verify these properties, let us rewrite our formulas in the BBS notation.

The variables  $\chi_p$  and  $G_{ij}(p)$  ( $i, j = 1, 2$ ), which were used in the Grassmann integral representation of the transfer matrix, are expressed in terms of  $t, x, x', y, y', \mu, \mu'$  as

$$\chi_p = \frac{4(t + \mu\mu'xx')^2 + 4(yy' - t\mu\mu')^2 + 8(t + \mu\mu'xx')(yy' - t\mu\mu') \cos p}{(y + \mu t)(y' + \mu')},$$

$$\chi_p G_{11}(p) = \frac{8it(y + \mu\mu'x')(y' - \mu\mu'x) \sin p}{(y + \mu t)^2(y' + \mu')^2}$$

$$\chi_p G_{22}(p) = \frac{8it(y - \mu\mu'x')(y' + \mu\mu'x) \sin p}{(y + \mu t)^2(y' + \mu')^2},$$

$$\chi_p G_{12}(p) = 16(y + \mu t)^{-2}(y' + \mu')^{-2} \times$$

$$\times [t^2 - y^2y'^2 + \mu^2\mu'^2(t^2 - x^2x'^2) - 2\mu\mu'(t^2 + xx'yy') \cos p - 2it\mu\mu'(xy + x'y') \sin p].$$

In order to write down the eigenvalues of  $T^{NS}$  and  $T^R$ , it is sufficient to express in terms of BBS parameters the quantities  $\Lambda_{max}^{NS(R)}$  and  $v_p/\rho_p$ . They are given by (see also [11])

$$\Lambda_{max}^{NS} = \frac{(1 + \mu^L\mu'^L)}{y^L y'^L} \prod_p^{NS} (t + t_p),$$

$$\Lambda_{max}^R = \frac{(1 - \mu^L\mu'^L)}{y^L y'^L} \prod_p^R (t + t_p),$$

$$v_p/\rho_p = \frac{t - t_p}{t + t_p},$$

where  $t_p$  is defined as

$$t_p = \frac{\sqrt{a_p c_p - b_p^2} - i b_p}{a_p},$$

with

$$a_p = 1 - 2\mu\mu' \cos p + \mu^2 \mu'^2,$$

$$b_p = \mu\mu'(xy + x'y') \sin p.$$

$$c_p = y^2 y'^2 + 2\mu\mu' x x' y y' \cos p + \mu^2 \mu'^2 x^2 x'^2.$$

Finally, ‘right’ and ‘left’ eigenvectors of  $T^{NS}$  and  $T^R$  are fully characterized by the functions  $A^\pm(p)$  and  $\dot{A}^\pm(p)$ , which in the case of the BBS<sub>2</sub> model can be written in the following form:

$$A^\pm(p) = \frac{d_p \mp \sqrt{a_p c_p - b_p^2}}{(y - \mu\mu' x')(y' + \mu\mu' x) i \sin p},$$

$$\dot{A}^\pm(p) = \frac{d_p \mp \sqrt{a_p c_p - b_p^2}}{(y + \mu\mu' x')(y' - \mu\mu' x) i \sin p},$$

where

$$d_p = \mu\mu'(x x' - y y') + (y y' - \mu^2 \mu'^2 x x') \cos p.$$

One should note that the functions  $A^+(p)$ ,  $\dot{A}^+(p)$  do not depend on  $t$ , as expected.

## 5.2. Ising model

In the Ising case, another parametrization is typically used. For simplicity, let us consider the isotropic model, characterized by the plaquette weight

$$W(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = \exp\left\{\frac{1}{2} K(\sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_4 + \sigma_4 \sigma_1)\right\}.$$

Parameters of the general free-fermion model, corresponding to this Boltzmann weight, are given by

$$a_0 = \frac{\cosh^2 K + 1}{2}, \quad a_4 = a_{13} = a_{24} = \frac{\sinh^2 K}{\cosh^2 K + 1},$$

$$a_{12} = a_{23} = a_{34} = a_{14} = \frac{\sinh K \cosh K}{\cosh^2 K + 1}.$$

Let us also introduce the function  $\gamma_q$ , given by the positive root of the equation

$$\cosh \gamma_q = \sinh 2K + \sinh^{-1} 2K - \cos q,$$

One can now rewrite the variables  $\chi_p$  and  $G_{ij}(p)$  ( $i, j = 1, 2$ ) from the Grassmann integral representation of  $T^{NS}$  and  $T^R$  in the following way:

$$\chi_p = \frac{\sinh^2 2K [(1 + \tanh K \cos p)^2 + \sin^2 p]}{(\cosh^2 K + 1)^2},$$

$$\chi_p G_{11}(p) = \chi_p G_{22}(p) = \frac{2i \sin p \sinh 2K (\cosh 2K - \tanh K \cos p)}{(\cosh^2 K + 1)^2},$$

$$\chi_p G_{12}(p) = \frac{2 \sinh 2K}{(\cosh^2 K + 1)^2}.$$

The eigenvalues and eigenvectors of  $T^{NS}$  and  $T^R$  may be found from

$$\Lambda_{max}^{NS(R)} = (2 \sinh 2K)^{L/2} \exp \left\{ \frac{1}{2} \sum_p^{NS(R)} \gamma_p \right\}, \quad v_p / \rho_p = e^{-\gamma_p},$$

$$A^\pm(p) = \dot{A}^\pm(p) = \frac{\sinh K \cosh K [(1 + \tanh K \cos p)^2 + \sin^2 p] - e^{\pm\gamma_p}}{i \sin p (\cosh 2K - \tanh K \cos p)}.$$

It should be emphasized once again that the transfer matrix eigenvectors, constructed above, automatically diagonalize the translation operator  $R$  as well. Therefore, we believe that these eigenvectors may turn out to be useful for the construction of a rigorous proof of the recently obtained formula [12] for Ising spin form factors on a finite lattice.

## 6. Summary

We have obtained the transfer matrix eigenvectors of the  $BBS_2$  model on a finite lattice, using the method of Grassmann integration. Our results are exact and explicit, i. e. the eigenvectors are expressed in terms of initial lattice variables. Grassmann integral representation for the eigenvectors immediately gives their norms and allows to considerably advance in the computation of form factors of the  $BBS_2$  model.

The only two things that were actually necessary for our computation are the translational invariance of the model and the representation (1.11) for the plaquette Boltzmann weight. In this respect, the method, developed in the present paper, is quite general and it could be extended to free-fermion models with a more complicated configuration space of order parameter, once these are found.

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