DUALITY IN SUPERSYMMETRIC YANG-MILLS AND THE QUANTUM HALL EFFECT

BRIAN P. DOLAN*
Department of Mathematical Physics, National University of Ireland, Maynooth, Ireland
and
School of Theoretical Physics, Dublin Institute for Advanced Studies, 10, Burlington Rd., Dublin, Ireland
**E-mail: bdolan@thphys.nuim.ie*

It is shown that the long-wavelength, zero-frequency limits of $\mathcal{N}=2$ supersymmetric Yang-Mills in $3+1$-dimensions and the quantum Hall effect in $2+1$ dimensions have many features in common. The phases of both these systems have a hierarchical structure which can be organised and understood in terms of a specific sub-group of the modular group acting on a complex parameter. The complex coupling has positive imaginary part which is the Yang-Mills coupling in the former case and the Ohmic conductivity in the latter. In both case the real part of the complex parameter is associated with a topological term in the effective action for the respective theory. The theoretical scaling flow of SUSY Yang-Mills that follows from modular symmetry is given by a modular form and shows a remarkable similarity to the experimental scaling flow of the quantum Hall effect which, though not necessarily associated with modular forms, is nevertheless strongly constrained by modular symmetry.

Keywords: Duality, Supersymmetry, Quantum Hall Effect.

1. Duality in SUSY Yang-Mills

Pure $SU(2)$ Yang-Mills in 4-dimensional Minkowski space with global $\mathcal{N}=2$ supersymmetry has only 2 parameters in the action: the gauge coupling g and the topological susceptibility θ, all other couplings, including the Higgs ϕ^4 coupling and Yukawa couplings, are determined by g and supersymmetry. These parameters can be combined into a single complex parameter

$$\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2}. \qquad (1)$$

Seiberg and Witten showed[1] that the low energy physics is symmetric under the action of a sub-group $\Gamma(2)$ of the modular group $\Gamma(1) \cong Sl(2,(\mathbf{Z})/\mathbf{Z_2}$ on τ. If $\gamma \in \Gamma(2)$ then

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{sends} \quad \tau \to \frac{a\tau + b}{c\tau + d}, \qquad (2)$$

where a, b, c and d are integers, $\det\gamma = 1$ and b and c are even. The group $\Gamma(2)$ is generated by

$$\mathcal{T}^2 : \tau \to \tau + 2 \text{ and } \mathcal{F}^2 : \tau \to \frac{\tau}{1-2\tau}, \qquad (3)$$

where $\mathcal{F}^2 = \mathcal{S}^{-1}\mathcal{T}^2\mathcal{S}$.

Classically the Higgs potential is minimised by any constant ϕ in the Lie algebra of $SU(2)$ such that $[\phi^\dagger, \phi] = 0$, and since ϕ can always be rotated by a globally well-defined gauge transformation it can always be taken to be $\phi = \frac{1}{2}\boldsymbol{a}\sigma_3$ with σ_3 the usual Pauli matrix and $\boldsymbol{a}$ a complex constant with dimensions of mass. The classical vacuum is thus highly degenerate and can be parameterised by $\boldsymbol{a}$, a non-zero $\boldsymbol{a}$ breaks $SU(2)$ gauge symmetry down to $U(1)$ and $W^\pm$ bosons (and their superpartners) acquire a mass proportional to $\boldsymbol{a}$, leaving one $U(1)$ gauge boson (and its superpartners) massless. Classically, at the special point $\boldsymbol{a}=0$ the gauge symmetry is restored to the full $SU(2)$ symmetry of the original theory. Seiberg and Witten then argue that supersymmetry protects this degeneracy so that it is not lifted by quantum corrections and is still there in the full quantum theory. At low energies, much less than the mass $\boldsymbol{a}$, the only relevant degrees of freedom in the theory are the massless $U(1)$ gauge boson and its superpartners (except for some special values of $\boldsymbol{a}$).

A better, gauge invariant, parameterisation of the quantum vacua is given by $u = tr < \phi^2 >$. For weak coupling (large $\boldsymbol{a}$) $u \approx \frac{1}{2}\boldsymbol{a}^2$, but $< \phi^2 > \neq < \phi >< \phi >$ for strong coupling (small $\boldsymbol{a}$). Seiberg and Witten argued[1] that in the quantum theory the strong coupling regime $g^2 \approx 0$ is associated not with $\boldsymbol{a} = 0$ but instead with two points in the complex u-plane, $u = \pm\Lambda^2$ where Λ is, by definition, the QCD mass scale at which the gauge coupling diverges. Furthermore they found an explicit expression for the full low energy effective action and argued that new massless modes appear at the singular points $u = \pm\Lambda^2$, in addition to the photon and its superpartners. These new massless modes are dyons with the magnetic charge associated with non-perturbative aspects of the classical theory (solitons). Since $g \to \infty$ when $u = \pm\Lambda^2$, τ is real at these points.

Seiberg and Witten's $\Gamma(2)$ symmetry commutes with the scaling flow as u is varied. Taking the logarithmic derivative of $\tau(u)$ with respect to u, and imposing $ad - bc = 1$, we see that

$$u\frac{d\gamma(\tau)}{du} = \frac{1}{(c\tau + d)^2} u\frac{d\tau}{du}. \qquad (4)$$

Meromorphic functions $\tau(u)$ satisfying (4) are well studied in the mathematical literature and are called modular forms of weight -2. For Seiberg and Witten's expression for $\tau(u)$ β-functions were analysed in[2] and it is shown in [3] that the unique possibility, up to a constant, that is finite at weak coupling and at both singular points is

$$-\frac{\Lambda^2}{u}\left(1 - \frac{u^2}{\Lambda^4}\right)\frac{d\tau}{du} = \frac{2}{\pi i}\frac{1}{\left(\vartheta_3^4(\tau) + \vartheta_4^4(\tau)\right)}.$$

where ϑ_3 and ϑ_4 are Jacobi ϑ-functions

$$\vartheta_3(\tau) = \sum_{n=-\infty}^{\infty} e^{i\pi n^2\tau}, \vartheta_4(\tau) = \sum_{n=-\infty}^{\infty} (-1)^n e^{i\pi n^2\tau}.$$

At weak coupling (large u) u is proportional to the square of the gauge boson mass and this flow can be interpreted as giving the Callan-Symanzik β-function in the asymptotic regime. The flow generated by (5) is shown in figure 1, there are fixed points on the real axis at $\tau = q/p$ where the massless dyons have electric quantum number q and magnetic quantum number $-p$. Odd p corresponds to attractive fixed points in the IR direction and even p to attractive fixed points in the UV direction. There are semi-circular trajectories linking some of the IR attractive fixed points with odd monopole charge.

Note that, since the scaling function (5) is symmetric under $u \to -u$, which is equivalent to $\tau \to \tau + 1$, the full symmetry of the scaling flow is slightly larger than $\Gamma(2)$, it is generated by $\mathcal{F}^2$ and $\mathcal{T}$ and corresponds to matrices γ such that c is even but b can be either even or odd. This group is often denoted by $\Gamma_0(2)$.

When matter in the fundamental representation is included the picture changes in detail, but is similar in structure.[4,5]

2. The Quantum Hall Effect

Modular symmetry manifests itself in the quantum Hall effect (QHE) in a manner remarkably similar to $\mathcal{N} = 2$ supersymmetric Yang-Mills. The conductivity tensor for an isotropic 2-dimensional medium can be described by a single complex conductivity

$$\sigma := \sigma_H + i\sigma_L, \qquad (5)$$

where σ_L is the longitudinal, or Ohmic, conductivity and σ_H is the Hall conductivity (in units with $e^2/h = 1$). Note that Ohmic conductivities must be positive for stability reasons, so σ is restricted to the upper half-complex plane.

The response functions (*i.e.* the conductivities) in a low temperature 2-dimensional system can be obtained from a 2+1-dimensional field theory. Ohmic conductivity can be incorporated by working in Fourier space $(\omega, \mathbf{p})$ and introducing a frequency dependent electric permittivity. In a conductor the low frequency electric permittivity diverges, in the long wavelength limit $\mathbf{p} \to 0$, as

$$\epsilon(\omega) = -i\frac{\sigma_L}{\omega}, \tag{6}$$

so, working in Fourier space, the effective dynamics of the electromagnetic field are governed by

$$\begin{aligned}\widetilde{\mathcal{L}}_{eff}[A] &= -\frac{\epsilon}{4}F^2 - \frac{\sigma_H}{4}\epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho} + A_\mu J^\mu \\ &\approx \frac{i\sigma_L}{4\omega}F^2 - \frac{\sigma_H}{4}\epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho} + A_\mu J^\mu.\end{aligned}$$

Note that the effective action is not real, an indication of the dissipative nature of Ohmic resistance, and non-local in time, again a feature of a conducting medium. A version of the Lagrangian (7) was used in the analysis of[6] in which the following transformations

$$\mathcal{T} : \sigma \;\to\; \sigma + 1 \text{ and } \mathcal{F}^2 : \sigma \;\to\; \frac{\sigma}{1-2\sigma} \tag{7}$$

were derived, which generate $\Gamma_0(2)$ and map between different quantum Hall phases of a spin polarised sample. The $\mathcal{T}$ transformation is interpreted as being due to shifting Landau levels by one and the $\mathcal{F}^2$ transformation, known as flux-attachment, was anticipated in[7] for $\sigma_L = 0$ as a mapping between ground state wave-functions. It is related to the composite fermion picture of the QHE [8]. That $\Gamma_0(2)$ should be a symmetry of the complex quantum Hall conductivity effect was originally suggested, without a microscopic justification, in [9].

Of course $\Gamma_0(2)$ is not a symmetry of all of the physics, after all the conductivities differ on different plateaux, nevertheless it is a symmetry of some physical properties, in particular it should be a symmetry of the scaling flow[11,12]. Physically the scaling flow of the QHE can be viewed as arising from changing the electron coherence length l, *e.g.* by varying the temperature T with $l(T)$ a monotonic function of T.[13] Define a scaling function by

$$\Sigma(\sigma,\bar{\sigma}) := l\frac{d\sigma}{dl}. \tag{8}$$

Then, for any $\gamma \in \Gamma_0(2)$, $\gamma(\sigma) = \frac{a\sigma+b}{c\sigma+d}$ with $ad - bc = 1$, so

$$\Sigma\big(\gamma(\sigma),\gamma(\bar{\sigma})\big) = \frac{1}{(c\sigma+d)^2}\Sigma(\sigma,\bar{\sigma}). \tag{9}$$

In general one expects σ to depend on various parameters, such as the temperature T, the external field B, the charge carrier density $\mathtt{n}$ and the impurity density $\mathtt{n}_i$. If $\mathtt{n}$ and $\mathtt{n}_i$ are fixed then $\sigma(B,T)$ becomes a function of B and T only.

If we further assume that the *only* fixed points of the scaling flow are the fixed points of $\Gamma_0(2)$ then, with a few extra reasonable assumptions, the topology of the flow diagram is completely determined and it is exactly the same as the flow diagram in figure 1 for $\mathcal{N} = 2$ SUSY.[10] IR fixed points have $\tau = \sigma_H = q/p$ corresponding to fermionic charge carriers which are composite objects consisting of bosons with p units of statistical flux attached, with p odd. Experimental data are shown in figure 2 and the similarity with the $\Gamma_0(2)$ in figure 1 is remarkable (the conductivities are doubled in the figure because of spin degeneracy, the spins in the sample are degenerate).

A second striking consequence of $\Gamma_0(2)$ symmetry of the scaling flow is a selection rule for quantum Hall transitions [11]

$$q_1p_2 - q_2p_1 = 1. \tag{10}$$

This selection rule is well borne out by the experimental data. In spin-split samples any two adjacent well-formed plateaux, with no unresolved sub-structure between them, obey this rule.

The group $\Gamma_0(2)$ is the symmetry group relevant to 2-dimensional samples in strong magnetic fields when the charge carriers are fermions. When the charge carriers are bosons, as in 2-d superconductors, a different group emerges, but a similar hierarchy of quantum states is predicted that differs in essential details[14] that would provide a clear experimental signal if high mobility samples that can sustain a large magnetic field are ever manufactured.

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Figure 1

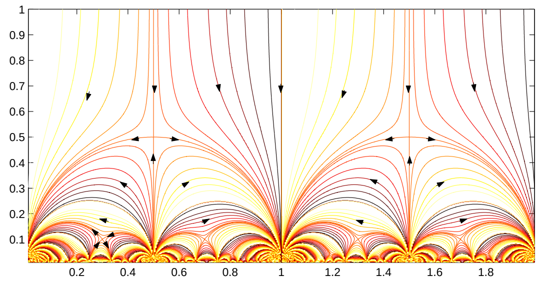

Figure 2