Noncommutative BTZ Black Hole and Discrete Time

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ABSTRACT

We search for all Poisson brackets for the BTZ black hole which are consistent with the geometry of the commutative solution and are of lowest order in the embedding coordinates. For arbitrary values for the angular momentum we obtain two two-parameter families of contact structures. We obtain the symplectic leaves, which characterize the irreducible representations of the noncommutative theory. The requirement that they be invariant under the action of the isometry group restricts to $\mathbb{R} \times S^1$ symplectic leaves, where $\mathbb{R}$ is associated with the Schwarzschild time. Quantization may then lead to a discrete spectrum for the time operator.

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1

Most approaches to noncommutative gravity have involved deforming the commutative Einstein equations.[1] -[17] These approaches range from simply replacing point-wise products by Moyal star products in the Einstein-Hilbert action to the approach of Aschieri, et. al.[13] which preserves the diffeomorphism invariance of general relativity. Given the ambiguities of the different schemes, it may be useful to look at other strategies towards noncommutative gravity. The one we have in mind here starts with solutions to the commutative Einstein equations, with the goal of finding their noncommutative analogues. As a first step, one can try writing down Poisson brackets which are consistent with the geometry of some classical solution. The noncommutative counterpart of the solution is then obtained by 'quantization'. (Here, of course, the deformation parameter is the noncommutativity parameter θ and not $\hbar$.) The Poisson brackets and resulting noncommutative algebra may not be unique. In this regard, it may be desirable to impose the restriction that the isometry of the classical solution survives quantization and is implementable in any irreducible representation of the noncommutative algebra. An example of this possibility is studied here.

The example is the three-dimensional (BTZ) black hole solution, which is characterized by its mass M and angular momentum J, and is asymptotically AdS^3.[18],[19] The BTZ black hole geometry is known to be the quotient space of the universal covering space of AdS^3 by some elements of its isometry group $SO(2,2)$. The quotienting breaks the isometry group to a two-dimensional subgroup $\mathcal{G}_{BTZ}$. Poisson structures (or more precisely, contact structures) can be obtained for the BTZ black hole which respect this quotienting and are invariant under $\mathcal{G}_{BTZ}$. (Some contact structures have already been previously suggested in [20].) Here we search for all such Poisson bracket which are of lowest order with respect to the four-dimensional embedding coordinates for AdS^3. For generic values of the angular momentum, the allowable Poisson brackets form two two-parameter families, and are quadratic with respect to four-dimensional embedding coordinates. Depending on the values of the parameters, the symplectic leaves (surfaces on which a symplectic two-form can be defined) are topologically either a) $\mathbb{R}^2$, b) $\mathbb{R} \times \mathbb{R}_+$ or c) $\mathbb{R} \times S^1$. The symplectic leaves characterize the irreducible representations of the corresponding noncommutative algebra. $\mathcal{G}_{BTZ}$ acts nontrivially in the case of a) and b), in general inducing a map between different symplectic leaves. It thus transforms between different irreducible representations of the noncommutative theory, and $\mathcal{G}_{BTZ}$ cannot be implemented as inner transformations. On the other hand, $\mathcal{G}_{BTZ}$ leaves c) invariant and hence also the corresponding irreducible representations. So only case c) remains if we impose the restriction that the isometry of the classical solution survives quantization. Moreover, quantization of the commutative algebra on $\mathbb{R} \times S^1$, where $\mathbb{R}$ corresponds to the time, is known to lead to a discrete spectrum for the time operator. We speculate that a similar conclusion can be drawn for c).* For us, not all cylinders need to have a space-time signature, and the coordinate associated with $\mathbb{R}$ may or may not be a time-like coordinate. Quantum theories on the noncommutative space-time cylinder have been previously studied.[21],[22] Other novel results were shown in addition to the discrete time spectrum, which may also

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*The possibility of a discrete spectrum for the time in this setting was first suggested to us by A.P. Balachandran.

apply here. Among them is the result that time-independent Hamiltonians are conserved only up to modulo $2\pi/\theta$.

In what follows, after briefly reviewing the geometry of the commutative BTZ solution and the quotient space construction, we write down the two-parameter families of Poisson brackets and map them to the Schwarzschild-like coordinates and obtain the symplectic leaves. There is an analogous quotient space construction in the noncommutative theory which will be discussed in a later article.[23]

In terms of Schwarzschild-like coordinates (r, t, ϕ) the invariant measure for the BTZ black hole is expressed as[18],[19]

$$ds^2 = \left(M - \frac{r^2}{\ell^2} - \frac{J^2}{4r^2}\right)dt^2 + \left(-M + \frac{r^2}{\ell^2} + \frac{J^2}{4r^2}\right)^{-1}dr^2 + r^2\left(d\phi - \frac{J}{2r^2}dt\right)^2 , \qquad (1)$$

$$0 \le r < \infty , \ -\infty < t < \infty , \ 0 \le \phi < 2\pi ,$$

where M and J are the mass and spin, respectively, and $\Lambda = -1/\ell^2$ is the cosmological constant. For $0 < |J| < M\ell$, there are two horizons, the outer and inner horizons, corresponding respectively to $r = r_+$ and $r = r_-$, where

$$r_\pm^2 = \frac{M\ell^2}{2}\left\{1 \pm \left[1 - \left(\frac{J}{M\ell}\right)^2\right]^{\frac{1}{2}}\right\} \qquad (2)$$

The two horizons coincide in the extremal case $|J| = M\ell > 0$, while the inner one disappears for $J = 0$, $M > 0$. The metric is diagonal in the coordinates (χ_+, χ_-, r), where

$$\chi_\pm = \frac{r_\pm}{\ell}t - r_\mp\phi , \qquad (3)$$

$$ds^2 = \frac{-(r^2 - r_+^2)d\chi_+^2 + (r^2 - r_-^2)d\chi_-^2}{r_+^2 - r_-^2} + \frac{\ell^2 r^2 dr^2}{(r^2 - r_+^2)(r^2 - r_-^2)} , \qquad (4)$$

which shows that χ_+ is the time-like coordinate in the region I) $r \ge r_+$, r is the time-like coordinate in the region II) $r_- \le r \le r_+$ and χ_- is the time-like coordinate in the region III) and $0 \le r \le r_-$.

It was shown that the manifold of the BTZ black hole solution is the quotient space of the universal covering space of AdS^3 by some elements of the group of isometries of AdS^3. The connected component of the latter is $SO(2,2)$. Say AdS^3 is spanned by coordinates (t_1, t_2, x_1, x_2) parameterizing $\mathbb{R}^4$, satisfying

$$-t_1^2 - t_2^2 + x_1^2 + x_2^2 = -\ell^2 \qquad (5)$$

Alternatively, one can introduce 2×2 real unimodular matrices

$$g = \frac{1}{\ell}\begin{pmatrix} t_1 + x_1 & t_2 + x_2 \\ -t_2 + x_2 & t_1 - x_1 \end{pmatrix} , \qquad \det g = 1 , \qquad (6)$$

belonging to the defining representation of $SL(2,R)$. The isometries correspond to the left and right actions on g,

$$g \to h_L g h_R , \qquad h_L, h_R \in SL(2,R) \qquad (7)$$

3

Since (h_L, h_R) and $(-h_L, -h_R)$ give the same action, the connected component of the isometry group for AdS^3 is $SL(2, R) \times SL(2, R)/Z_2 \approx SO(2, 2)$.

The BTZ black-hole is obtained by discrete identification of points on the universal covering space of AdS_3. This insures periodicity in ϕ, $\phi \sim \phi + 2\pi$. The condition is

$$g \sim \tilde{h}_L g \tilde{h}_R \ , \tag{8}$$

where $(\tilde{h}_L, \tilde{h}_R)$ are certain elements of $SO(2, 2)$. $\tilde{h}_L$ and $\tilde{h}_R$ can be expressed as diagonal $SL(2, R)$ matrices

$$\tilde{h}_L = \begin{pmatrix} e^{\pi(r_+ - r_-)/\ell} & \\ & e^{-\pi(r_+ - r_-)/\ell} \end{pmatrix} \ , \qquad \tilde{h}_R = \begin{pmatrix} e^{\pi(r_+ + r_-)/\ell} & \\ & e^{-\pi(r_+ + r_-)/\ell} \end{pmatrix} \tag{9}$$

For $0 < |J| < M\ell$, the universal covering space of AdS_3 is covered by three types of coordinate patches which are bounded by the two horizons at $r = r_+$ and $r = r_-$. For all three coordinate patches, g can be decomposed according to

$$g = \begin{pmatrix} e^{\frac{1}{2\ell}(\chi_+ - \chi_-)} & \\ & e^{-\frac{1}{2\ell}(\chi_+ - \chi_-)} \end{pmatrix} g^{(0)}(r) \begin{pmatrix} e^{\frac{1}{2\ell}(\chi_+ + \chi_-)} & \\ & e^{-\frac{1}{2\ell}(\chi_+ + \chi_-)} \end{pmatrix} \ , \tag{10}$$

where $g^{(0)}(r)$ is an $SO(2)$ matrix which only depends on r and the coordinate patch. The periodicity condition for ϕ easily follows from (8). The identification (8) breaks the $SO(2, 2)$ group of isometries to a two-dimensional subgroup $\mathcal{G}_{BTZ}$, consisting of only the diagonal matrices in $\{h_L\}$ and $\{h_R\}$. $\mathcal{G}_{BTZ}$ is the isometry group of the BTZ black hole, and from (10) is associated with translations in χ_+ and χ_-, or equivalently t and ϕ, on $r =$ constant surfaces.

For generic spin, $0 < |J| < M\ell$ (and $M > 0$), we shall search for Poisson brackets for the matrix elements of g which are polynomial of lowest order. They should be consistent with the quotienting (8), as well as the unimodilarity condition and, of course, the Jacobi identity. For convenience we write the $SL(2, R)$ matrix as

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \ , \qquad \alpha\delta - \beta\gamma = 1 \ , \tag{11}$$

Under the quotienting (8):

$$\begin{aligned} \alpha &\sim e^{\ 2\pi r_+/\ell} \, \alpha \\ \beta &\sim e^{-2\pi r_-/\ell} \, \beta \\ \gamma &\sim e^{\ 2\pi r_-/\ell} \, \gamma \\ \delta &\sim e^{-2\pi r_+/\ell} \, \delta \end{aligned} \tag{12}$$

All quadratic combinations of matrix elements scale differently, except for $\alpha\delta$ and $\beta\gamma$, which are invariant under (12). Lowest order polynomial expressions for the Poisson brackets of α, β, γ and δ which are preserved under (12) are quadratic and have the form

$$\begin{matrix} \{\alpha, \beta\} = c_1 \alpha\beta & \{\alpha, \gamma\} = c_2 \alpha\gamma & \{\alpha, \delta\} = f_1(\alpha\delta, \beta\gamma) \\ \{\beta, \delta\} = c_3 \beta\delta & \{\gamma, \delta\} = c_4 \gamma\delta & \{\beta, \gamma\} = f_2(\alpha\delta, \beta\gamma) \end{matrix} \ , \tag{13}$$

4

where c_{1-4} are constants and $f_{1,2}$ are functions.[†] They are constrained by

$$
\begin{aligned}
c_1 + c_2 &= c_3 + c_4 \\
f_1(\alpha\delta, \beta\gamma) &= (c_1 + c_2)\beta\gamma \\
f_2(\alpha\delta, \beta\gamma) &= (c_2 - c_4)\alpha\delta ,
\end{aligned}
\tag{14}
$$

after demanding that $\det g$ is a Casimir of the algebra. From (13) there are three independent constants c_{1-4}. Further restrictions on the constants come from the Jacobi identity, which leads to the following two possibilities:

$$
\text{A.} \quad c_2 = c_4 \qquad \text{and} \qquad \text{B.} \quad c_2 = -c_1
$$

Both cases define two-parameter families of Poisson brackets. Say we call c_2 and c_3 the two independent parameters. The two cases are connected by an $SO(2,2)$ transformation. Case A goes to case B when

$$
g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \to g' = \begin{pmatrix} \beta & -\alpha \\ \delta & -\gamma \end{pmatrix} = g h_R^{(0)} , \qquad h_R^{(0)} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} ,
\tag{15}
$$

along with

$$
c_3 \to c_2 \qquad c_2 \to c_3
\tag{16}
$$

In terms of the embedding coordinates, this corresponds to $(t_1, t_2, x_1, x_2) \to (t_2, -t_1, x_2, -x_1)$.

There are three types of coordinate patches in the generic case of $M > 0$ and $0 < |J| < M\ell$, and their boundaries are the two horizons. Denote them again by: I) $r \geq r_+$, II) $r_- \leq r \leq r_+$ and III) $0 \leq r \leq r_-$. The corresponding maps to $SL(2, R)$ are given by (10), with

I) $r \geq r_+$,

$$
g^{(0)}(r) = g_I^{(0)}(r) = \frac{1}{\sqrt{r_+^2 - r_-^2}} \begin{pmatrix} \sqrt{r^2 - r_-^2} & \sqrt{r^2 - r_+^2} \\ \sqrt{r^2 - r_+^2} & \sqrt{r^2 - r_-^2} \end{pmatrix}
\tag{17}
$$

II) $r_- \leq r \leq r_+$,

$$
g^{(0)}(r) = g_{II}^{(0)}(r) = \frac{1}{\sqrt{r_+^2 - r_-^2}} \begin{pmatrix} \sqrt{r^2 - r_-^2} & -\sqrt{r_+^2 - r^2} \\ \sqrt{r_+^2 - r^2} & \sqrt{r^2 - r_-^2} \end{pmatrix}
\tag{18}
$$

III) $0 \leq r \leq r_-$,

$$
g^{(0)}(r) = g_{III}^{(0)}(r) = \frac{1}{\sqrt{r_+^2 - r_-^2}} \begin{pmatrix} \sqrt{r_-^2 - r^2} & -\sqrt{r_+^2 - r^2} \\ \sqrt{r_+^2 - r^2} & -\sqrt{r_-^2 - r^2} \end{pmatrix}
\tag{19}
$$

[†]More generally, if we drop the assumption that the Poisson brackets are polynomial of lowest order we can replace the constants c_i by functions of $\alpha\delta$ and $\beta\gamma$. So for example, $\{\alpha, \beta\} = p_1(\alpha\delta, \beta\gamma)\alpha\beta$, where p_1 is an arbitrary function. These brackets, in general, will have more complicated transformation properties under the action of $SO(2,2)$.

Using the maps (17-19), we can write the Poisson brackets for the various cases in terms of the Schwarzschild-like coordinates (r, t, ϕ). The results are the same in all three coordinate patches. For the two-parameter families A and B one gets:

A.

$$
\begin{aligned}
\{\phi, t\} &= \frac{\ell^3}{2} \frac{c_3 - c_2}{r_+^2 - r_-^2} \\
\{r, \phi\} &= -\frac{\ell r_+ (c_3 + c_2)}{2r} \frac{r^2 - r_+^2}{r_+^2 - r_-^2} \\
\{r, t\} &= -\frac{\ell^2 r_- (c_3 + c_2)}{2r} \frac{r^2 - r_+^2}{r_+^2 - r_-^2} \quad ,
\end{aligned}
\tag{20}
$$

B.

$$
\begin{aligned}
\{\phi, t\} &= \frac{\ell^3}{2} \frac{c_3 - c_2}{r_+^2 - r_-^2} \\
\{r, \phi\} &= -\frac{\ell r_- (c_2 + c_3)}{2r} \frac{r^2 - r_-^2}{r_+^2 - r_-^2} \\
\{r, t\} &= -\frac{\ell^2 r_+ (c_2 + c_3)}{2r} \frac{r^2 - r_-^2}{r_+^2 - r_-^2}
\end{aligned}
\tag{21}
$$

These Poisson brackets are invariant under the action of the isometry group $\mathcal{G}_{BTZ}$ of the BTZ black hole. The first bracket agrees in both cases. The latter two brackets vanish at the outer horizon $r = r_+$ for case A, and the inner horizon $r = r_-$ for case B. A central element of the Poisson algebra can be constructed out of the Schwarzschild coordinates for both cases. It is given by

$$
\rho_\pm = (r^2 - r_\pm^2) \exp\left\{ -\frac{2\kappa \chi_\pm}{\ell} \right\} , \qquad c_2 \neq c_3 ,
\tag{22}
$$

where the upper and lower sign correspond to case A and B, respectively,

$$
\kappa = \frac{c_3 + c_2}{c_3 - c_2} ,
\tag{23}
$$

and $\chi_\pm$ were defined in (3). The $\rho_\pm =$constant surfaces define symplectic leaves, which are topologically $\mathbb{R}^2$ for generic values of the parameters (more specifically, $c_2 \neq \pm c_3$). We can coordinatize them by χ_+ and χ_-. One then has a trivial Poisson algebra in the coordinates $(\chi_+, \chi_-, \rho_\pm)$:

$$
\{\chi_+, \chi_-\} = \frac{\ell^2}{2}(c_3 - c_2) \qquad \{\rho_\pm, \chi_+\} = \{\rho_\pm, \chi_-\} = 0
\tag{24}
$$

The action of the $\mathcal{G}_{BTZ}$ transforms one symplectic leaf to another, except for the case $c_2 = -c_3$ which we discuss later.

The above can be readily extended to the case of zero angular momentum by simply setting $r_- = 0$. The region III is then absent in this case.[‡] On the other hand, the Poisson brackets (20) and (21) are undefined in the extremum case $J = M\ell$, or $r_+ = r_-$, for finite coefficients

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[‡]Zero angular momentum also allows for Poisson brackets which are linear with respect to the four dimensional embedding coordinates and consistent with (8). This will be discussed in a later article.[23]

c_i.[§] The brackets, however, may be rendered finite by first considering $J < M\ell$ with the coefficients c_i proportional to $r_+^2 - r_-^2$ and then taking the limit $J \to M\ell$.

In passing to the noncommutative theory, the operator associated with $\rho_\pm$ is central in the quantum algebra and proportional to the identity in any irreducible representation. Irreducible representations then select $\rho_\pm$ =constant surfaces and the isometry group $\mathcal{G}_{BTZ}$ thus maps between different irreducible representations, and thus cannot be implemented as inner transformations. In any irreducible representation the algebra is generated by the noncommutative analogues of χ_+ and χ_-. The resulting noncommutative theory differs from the Grönewald-Moyal plane since χ_+ and χ_- are not cartesian coordinates. After re-writing the commutative metric (4) in terms of coordinates $(\chi_+, \chi_-, \rho_\pm)$ and restricting to the $\rho_\pm$ =constant surface one gets by

$$ds^2|_{\rho_\pm} = \frac{-(r^2 - r_+^2)d\chi_+^2 + (r^2 - r_-^2)d\chi_-^2}{r_+^2 - r_-^2} + \frac{r^2 - r_\pm^2}{r^2 - r_\mp^2}\kappa^2 d\chi_\pm^2 \tag{25}$$

As a result, these metric components will not in general be fixed by the irreducible representation. In attempting to write down a noncommutative field theory in this case, one cannot expect to treat the metric as a background. Neither is the signature of the metric fixed by the irreducible representation, as it can differ in different regions on the surface, which is evident from the determinant of the commutative metric $\mathbf{g}$ for fixed $\rho_\pm$

$$\det \mathbf{g}|_{\rho_\pm} = -(r^2 - r_\pm^2)\left(\frac{r^2 - r_\mp^2}{r_+^2 - r_-^2} \mp \kappa^2\right) \tag{26}$$

The surface has a Minkowski signature for r sufficiently large, and space-time noncommutativity results in the noncommutative theory. On the other hand, there may be regions where (26) is positive which is then associated with space-space noncommutativity.

We next discuss the two exceptional cases: $c_2 = c_3$ and $c_2 = -c_3$.

The above results cannot be applied when $c_2 = c_3$ since κ, and hence $\rho_\pm$, are ill-defined. Instead, $\chi_\pm$ is central in the Poisson algebra in this case, where the upper and lower sign again correspond to case A and B, respectively. The $\chi_\pm$ =constant surfaces define the symplectic leaves, which are topologically $\mathbb{R} \times \mathbb{R}_+$, parametrized by r and

$$\xi_\mp = \pm \frac{r\chi_\mp}{r^2 - r_\pm^2} \tag{27}$$

In terms of these variables, the Poisson brackets are

$$\{r, \xi_\mp\} = \ell c_2 \qquad \{\chi_\mp, r\} = \{\chi_\pm, \xi_\mp\} = 0 \tag{28}$$

[§]In the extremal case, h_L reduces to the identity, and at first glance it appears that more general Poisson brackets than (13) and (14), and consequently (20) and (21), are admissible. This is because the products $\alpha\beta$, $\alpha\delta$, $\gamma\delta$ and $\beta\gamma$ are unaffected by the quotienting. Thus the quotienting conditions allows one to generalize (13) such that $\{\alpha, \beta\}$, $\{\alpha, \delta\}$, $\{\gamma, \delta\}$ and $\{\beta, \gamma\}$ depend on those four products. But the system reduces to (13) and (14) after demanding that $\det g$ is a Casimir of the algebra.

Irreducible representations now select the $\chi_\pm$ =constant symplectic leaves. As before, the isometry group $\mathcal{G}_{BTZ}$ is a map between different symplectic leaves and hence different irreducible representations in the quantum theory. The $\chi_\pm$ =constant surfaces are characterized by the metric

$$ds^2|_{\chi_\pm} = \pm \frac{r^2 - r_\mp^2}{r_+^2 - r_-^2} \left(\frac{1}{r}(r^2 - r_\pm^2) d\xi_\mp + \frac{1}{r^2}(r^2 + r_\pm^2)\xi_\mp dr \right)^2 + \frac{\ell^2 r^2 dr^2}{(r^2 - r_+^2)(r^2 - r_-^2)} , \qquad (29)$$

whose determinant is simply

$$\det \mathbf{g}|_{\chi_\pm} = \pm \ell^2 \, \frac{r^2 - r_\mp^2}{r_+^2 - r_-^2} \qquad (30)$$

For case A, the surface has a Euclidean signature for $r > r_+$ and Minkowski signature for $r < r_+$. For case B, the surface has a Minkowski signature for $r > r_-$ and Euclidean signature for $r < r_+$.

The case of $c_2 = -c_3$ is the intersection of case A and B. Here κ vanishes and from (22), the radial coordinate is in the center of, the algebra. r =constant define $\mathbb{R} \times S^1$ symplectic leaves, and they are invariant under the action of $\mathcal{G}_{BTZ}$. The coordinates ϕ and t parametrizing any such surface are canonically conjugate:

$$\{\phi, t\} = \frac{c_3 \ell^3}{r_+^2 - r_-^2} \qquad \{\phi_\pm, r\} = \{t, r\} = 0 \qquad (31)$$

The Poisson brackets can be interpreted in terms of a twist[24] in the decomposition of g given in (10), where the twist is with respect to the first and third matrices. In passing to the noncommutative theory, we need to define a deformation of the commutative algebra generated by $t, e^{i\phi}$ and r. Call the corresponding quantum operators $\hat{t}, e^{i\hat{\phi}}$ and $\hat{r}$, respectively. Their commutation relations are[¶]

$$[e^{i\hat{\phi}}, \hat{t}] = \theta e^{i\hat{\phi}} \qquad [\hat{r}, \hat{t}] = [\hat{r}, e^{i\hat{\phi}}] = 0 , \qquad (32)$$

where from (31) the constant θ is linearly related to $\ell^3/(r_+^2 - r_-^2)$. There are now two central elements in the algebra: i) $\hat{r}$ and ii) $e^{-2\pi i \hat{t}/\theta}$. From i), irreducible representations select the $\mathbb{R} \times S^1$ symplectic leaves. Unlike in all the previous cases, the action of $\mathcal{G}_{BTZ}$ does not take you out of any particular irreducible representation, and in this sense we can say that the isometry of the classical solution survives quantization. The action of $\mathcal{G}_{BTZ}$ can be implemented with inner transformations. Say X_t and X_ϕ are Killing vectors generating translations in t and ϕ, respectively. They act on functions $\hat{A}$ on the noncommutative space according to

$$X_t \hat{A} = -\frac{1}{\theta}[\hat{\phi}, \hat{A}] \qquad X_\phi \hat{A} = \frac{1}{\theta}[\hat{t}, \hat{A}] \qquad (33)$$

[¶]It should be noted that there exits a quantization ambiguity associated with the set of allowed canonical transformations on the cylinder. So for example, the commutation relations are unchanged under the redefinitions

$$\hat{t} \to \hat{t}' = \hat{t} + F_1(\hat{r}) \qquad e^{i\hat{\phi}} \to e^{i\hat{\phi}'} = e^{i\{\hat{\phi} + F_2(\hat{r})\hat{t}\}} ,$$

for arbitrary functions F_1 and F_2. More physical input is needed to resolve this quantization ambiguity.

The determinant of the metric for any symplectic leaf is a function of r,

$$\det \mathbf{g}|_r = -\frac{1}{\ell^2}\left(r^2 - r_+^2\right)(r^2 - r_-^2)\,, \tag{34}$$

and thus the signature, as well as the commutative metric, are fixed by the irreducible representation. The cylinders have a Minkowski signature for regions I and III, and a Euclidean signature for region II. With regard to the central element ii) $e^{-2\pi i\hat{t}/\theta}$, one can identify it with $e^{i\chi}\mathbb{1}$ in an irreducible representation. The spectrum of $\hat{t}$ is then discrete[21],[22]

$$n\theta - \frac{\chi\theta}{2\pi}\,, \quad n \in \mathbb{Z} \tag{35}$$

In associating $\hat{t}$ with the Schwarzschild coordinate t, we recall that the latter is the time for the exterior of the black hole, but not for the interior. More precisely, X_t is time-like provided $r > r_{erg}$, where r_{erg} is the radius of the ergosphere (or ergocircle), $r_{erg}^2 = r_+^2 + r_-^2$.

Although the Poisson brackets (20) and (21) are invariant under the action of the isometry group of the black hole, they are not invariant under the larger group of $SO(2,2)$ transformations (7). On the other hand, Poisson structures can be consistently assigned to $SO(2,2)$ such that it defines a Lie-Poisson group and (7) defines a Poisson map. The $SO(2,2)$ group get q-deformed upon quantization. The noncommmutative BTZ black hole can be obtained from this quantum group by quotenting in a manner analogous to (8). This, along with an attempt at field theory on the noncommutative background, will be pursued in later articles.[23]

Acknowledgements

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