

Spinning $U(1)$ gauged Skyrmions

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Abstract

We construct axially symmetric solutions of $U(1)$ gauged Skyrme model. Possessing a nonvanishing magnetic moment, these solitons have also a nonzero angular momentum proportional to the electric charge.

Introduction.— Many nonlinear classical field theories on flat spacetime backgrounds admit soliton solutions. These nonsingular solutions describe particle-like, localised configurations with finite energy. There has been some interest in recent years in the issue of globally regular spinning soliton solutions. However, to the best of our knowledge, to date no *stationary* and *spinning* solitons were found. (We describe single lumps with angular momentum as spinning, and reserve rotating for more general (gravitating-) solutions, including multilumps.) Notably, it is known that that finite energy solutions of the Yang-Mills-Higgs (YMH) system with a nonvanishing magnetic charge have zero angular momentum [1, 2]¹. Moreover, as found in [5], none of the known gauge field solitons with gauge group $SU(2)$ (*e.g.* dyons, sphalerons, vortices) admit spinning generalizations within the stationary, axially symmetric, one-soliton sector.

To date two types of spinning solitons have been found in the literature, *a*) Q -balls solitons in a complex scalar field theory with a non-renormalizable self-interaction [6], which are nontopological solitons so their stability is not guaranteed by a topological charge, and *b*) the electrically charged dipole monopole–antimonopole pair [7] of the YMH system with vanishing topological charge, which is not topologically stable even in the limit of vanishing angular momentum.

It is our purpose here to construct a soliton which has intrinsic angular momentum and presents a *topologically stable limit*². Our definition for a ‘soliton presenting a topologically stable limit’ is, a finite energy spinning lump which is topologically stable in the limit of vanishing angular momentum. This configuration corresponds to axially symmetric, electrically charged solutions of the $U(1)$ gauged Skyrme model.

Concerning the question of the existence of any given topologically stable solution, this is quite an intricate matter that deserves a brief description. To start with, there must be a valid topological lower bound on the energy, which may or may not be saturated, and for the Skyrme it is not. Then there is the question whether any given field configuration (the solution) does minimise the energy? For the Skyrme model, this is a difficult problem for two reasons: *a*) because the sigma model fields are constrained, and *b*) because in addition to the quadratic kinetic term there is also a quartic kinetic term. Thus for the 1-Skyrmion, the existence proof is given by [9] and, [10], while for axially symmetric case, to the best of our knowledge, there is no rigorous existence proof. So axially symmetric Skyrmions and their magnetically gauged counterparts are supported only numerically.

In addition, when a nonvanishing electric field is present, as it is in the present work, the functional misminimised is not the positive definite energy but the indefinite action. The proof of existence of such solutions, namely that for YMH dyons, is given by [11], but again it is too hard to adapt this proof for the gauged (and ungauged) Skyrme model. Thus the existence of the $U(1)$ gauged axially symmetric solutions of the present paper, and those of [8], are supported only numerically.

The model.— The Skyrme model has been proposed a long time ago [12] as an effective theory for nucleons in the large N limit of QCD at low energy [13, 14, 15], the baryon number being identified with the topological charge. The classical as well as the quantum properties are in relatively good agreement with the observed

¹Also the axially symmetric spinning Einstein–Yang–Mills sphalerons, although predicted perturbatively [3], are unlikely to exist [1, 4], but these are in anycase not topologically stable.

²An axially symmetric, spinning soliton of the ungauged Skyrme model, similarly presenting a *topologically stable limit*, has been recently constructed in [8]. However, this is a Q -ball type of solution featuring time-dependent fields.

features of small nuclei. The $U(1)$ gauged Skyrme model was originally proposed by Callan and Witten to study the decay of the nucleons in the vicinity of a monopole [16]. Axially symmetric solutions of this model were constructed previously in [17], but the emphasis there was on the static properties of nucleons and not the calculation of its classical spin.

We define our model in terms of the $O(4)$ sigma model field $\phi^a = (\phi^\alpha, \phi^A)$, $\alpha = 1, 2$; $A = 3, 4$, satisfying the constraint $|\phi^\alpha|^2 + |\phi^A|^2 = 1$, the Lagrangean of the Maxwell gauged Skyrme model is (up to an overall factor which we set equal to one)

$$\mathcal{L} = -\frac{1}{2}|F_{\mu\nu}|^2 + \frac{1}{2}|D_\mu\phi^a|^2 - \frac{\kappa^2}{8}|D_{[\mu}\phi^a D_{\nu]}\phi^b|^2 \quad (1)$$

in terms of the Maxwell field strength $F_{\mu\nu}$, and the covariant derivatives defined by the gauging prescription

$$D_\mu\phi^\alpha = \partial_\mu\phi^\alpha + A_\mu(\varepsilon\phi)^\alpha \quad , \quad D_\mu\phi^A = \partial_\mu\phi^A. \quad (2)$$

The energy-momentum tensor which follows from (1) is

$$\begin{aligned} T_{\mu\nu} = & \left\{ -2 \left(F_{\mu\lambda} F_\nu^\lambda - \frac{1}{4} g_{\mu\nu} F_{\tau\lambda} F^{\tau\lambda} \right) + \left(D_\mu\phi^a D_\nu\phi^a - \frac{1}{2} g_{\mu\nu} D_\lambda\phi^a D^\lambda\phi^a \right) \right. \\ & \left. - 2 \cdot \frac{\kappa^2}{4} \left[(D_{[\mu}\phi^a D_{\lambda]}\phi^b) (D_{[\nu}\phi^a D^{\lambda]}\phi_b) - \frac{1}{4} g_{\mu\nu} (D_{[\tau}\phi^a D_{\lambda]}\phi^a) (D^{[\tau}\phi_a D^{\lambda]}\phi_b) \right] \right\}. \quad (3) \end{aligned}$$

Here we note that the Skyrmeion gauged with the purely magnetic $U(1)$ field is a topologically stable soliton. This is stated in terms of topological lower bound on the static energy density functional of the purely magnetically gauged system, namely the T_{tt} component of (3) with $A_t = 0$,

$$T_{tt} = \mathcal{E} = |F_{ij}|^2 + |D_i\phi^a|^2 + \frac{\kappa^2}{4}|D_{[i}\phi^a D_{j]}\phi^b|^2. \quad (4)$$

Defining the gauge invariant topological charge density as

$$\varrho = \frac{1}{4\pi}\varepsilon_{ijk}\varepsilon^{abcd} D_i\phi^a D_j\phi^b D_k\phi^c \phi^d + \frac{3}{8\pi}\varepsilon_{ijk} F_{ij} \varepsilon^{AB} \phi^B \partial_k\phi^A \quad (5)$$

$$= \frac{1}{4\pi}\varepsilon_{ijk}\varepsilon^{abcd} \partial_i\phi^a \partial_j\phi^b \partial_k\phi^c \phi^d - \frac{3}{4\pi}\varepsilon_{ijk}\partial_k (A_i \varepsilon^{AB} \partial_j\phi^A \phi^B) \quad (6)$$

the gauge invariance of ϱ is manifest from (5), while it is easily checked that the finite energy conditions lead to the vanishing of the surface integral term in (6), as a result of which the topological is simply the volume integral of the first term, namely the winding number n or, Baryon charge.

As was shown in [17] in detail, the energy density functional (4) is bounded from below by

$$\mathcal{E} \geq \frac{\kappa}{\sqrt{1 + \frac{9}{4}\kappa}} \varrho. \quad (7)$$

The ansatz.— In a cylindrical coordinate system, we parametrise the axially symmetric Maxwell connection as

$$A_t = b(\rho, z) \quad , \quad A_\alpha = \frac{a(\rho, z) - n}{\rho} \varepsilon_{\alpha\beta} \hat{x}_\beta \quad , \quad A_z = 0, \quad (8)$$

$a(\rho, z)$ and $b(\rho, z)$ corresponding to the electric and magnetic potentials, with n a positive integer - the winding number, and the polar parametrisation of the chiral field in terms of the two functions $f(\rho, z)$ and $g(\rho, z)$ as

$$\phi^\alpha = \sin f \sin g n^\alpha, \quad \phi^3 = \sin f \cos g, \quad \phi^4 = \cos f, \quad (9)$$

where $\rho = \sqrt{|x_\alpha|^2}$, $\alpha = 1, 2$, and $z = x_3$. In the following we will find it convenient instead to work with spherical coordinates (r, θ) , i.e. $\rho = r \sin \theta$ and $z = r \cos \theta$. After replacing this ansatz in (1), one finds the reduced lagrangean

$$\begin{aligned}
L = & r^2 \sin \theta \left\{ \frac{2}{r^2 \sin^2 \theta} \left(a_{,r}^2 + \frac{1}{r^2} a_{,\theta}^2 \right) - 2 \left(b_{,r}^2 + \frac{1}{r^2} b_{,\theta}^2 \right) \right. \\
& + \left[\left(f_{,r}^2 + \frac{1}{r^2} f_{,\theta}^2 \right) + \left(g_{,r}^2 + \frac{1}{r^2} g_{,\theta}^2 \right) \sin^2 f + \frac{a^2 - r^2 b^2 \sin^2 \theta}{r^2 \sin^2 \theta} \sin^2 f \sin^2 g \right] + \\
& \left. + \kappa^2 \sin^2 f \left(\frac{1}{r^2} (f_{,r} g_{,\theta} - f_{,\theta} g_{,r})^2 + \frac{a^2 - r^2 b^2 \sin^2 \theta}{r^2 \sin^2 \theta} \left[\left(f_{,r}^2 + \frac{1}{r^2} f_{,\theta}^2 \right) + \left(g_{,r}^2 + \frac{1}{r^2} g_{,\theta}^2 \right) \sin^2 f \right] \sin^2 g \right) \right\}. \tag{10}
\end{aligned}$$

The Euler-Lagrange equations arising from the variations of this Lagrangean have been integrated by imposing the following boundary conditions, which respect finite mass-energy and finite energy density conditions as well as regularity and symmetry requirements. We impose

$$f|_{r=\infty} = 0, \quad g_{,r}|_{r=\infty} = 0, \quad a|_{r=\infty} = n, \quad b|_{r=\infty} = V, \tag{11}$$

at infinity, and

$$f|_{r=0} = \pi, \quad g_{,r}|_{r=0} = 0, \quad a|_{r=0} = n, \quad b_{,r}|_{r=0} = 0, \tag{12}$$

at the origin. For solutions with parity reflection symmetry (the case considered in this paper), the boundary conditions along the z -axis are

$$f_{,\theta}|_{\theta=0} = g|_{\theta=0} = 0, \quad a_{,\theta}|_{\theta=0} = b_{,\theta}|_{\theta=0} = 0, \tag{13}$$

and agree with the boundary conditions on the ρ -axis, except for $g(r, \theta = \pi/2) = \pi/2$.

It may appear from the boundary conditions (11)-(13) that the natural condition $a|_{\theta=0, \pi} = n$ is not imposed. This is not done since its imposition in addition to (11)-(13) would be an overdetermination. We have nonetheless checked that $a = n$ is satisfied on the z -axis by the numerical solutions.

The constant V appearing in (11) corresponds to the magnitude of the electric potential at infinity and has a direct physical relevance. In the pure Maxwell theory, one can set $V = 0$ (or any other value) without any loss of generality. In the $U(1)$ gauged Skyrme model, however, such a gauge transformation would render the whole configuration time-dependent.

Integration over all space of the energy density \mathcal{E} yields the total mass-energy, $E = \int T_{tt} \sqrt{-g} d^3x$. The total angular momentum is given by $J = \int T_{\varphi t} \sqrt{-g} d^3x$, where

$$T_{t\varphi} = 2 \left(a_{,r} b_{,r} + \frac{a_{,\theta} b_{,\theta}}{r^2} \right) + a b \sin^2 f \sin^2 g \left(1 + \kappa^2 \left[\left(f_{,r}^2 + \frac{f_{,\theta}^2}{r^2} \right) + \left(g_{,r}^2 + \frac{g_{,\theta}^2}{r^2} \right) \sin^2 f \right] \right). \tag{14}$$

However, by using the field equations, the volume integral of the above quantity can be converted into a surface integral at infinity in terms of Maxwell potentials

$$J = 4\pi \lim_{r \rightarrow \infty} \int_0^\pi d\theta \sin \theta r^2 a b_{,r}. \tag{15}$$

The field equations imply the asymptotic behaviour of the electric potential $b \sim V - Q/(2r) + O(1/r^2)$, the parameter Q corresponding to the electric charge of the solutions. Therefore the following relation holds

$$J = 4\pi n Q, \tag{16}$$

which resembles the case of a monopole-antimonopole configuration in a YMH theory [7]. Note that the solutions discussed here possess also a magnetic dipole moment [17] which can be read from the asymptotics of the $U(1)$ magnetic potential, $A_\varphi \sim \mu \sin \theta / r^2$.

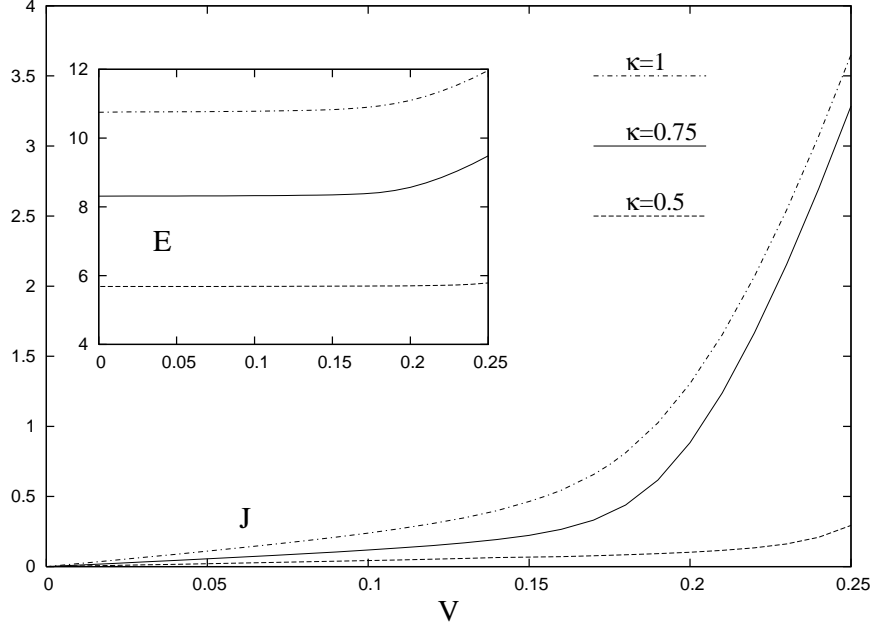


Figure 1a.

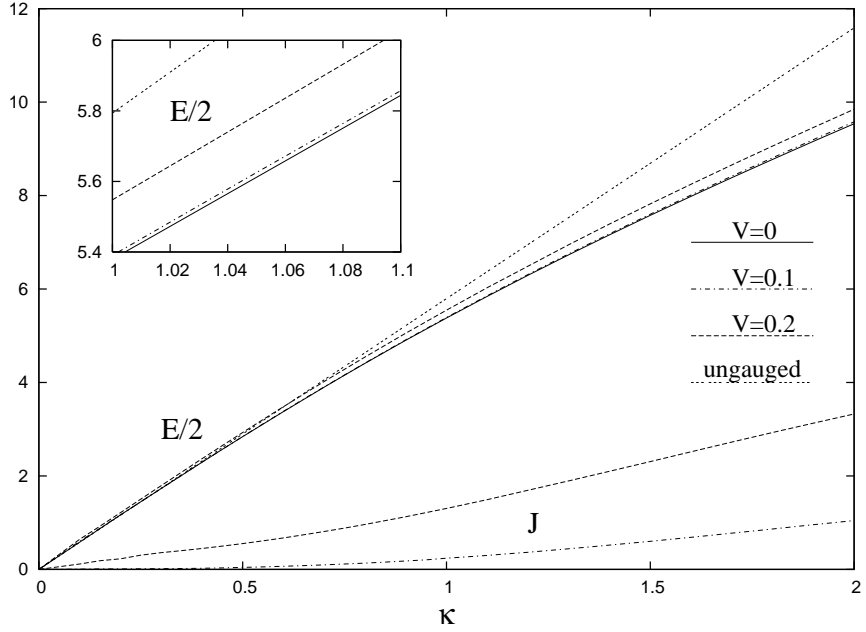


Figure 1b.

Figure 1. The energy E and the angular momentum J of $U(1)$ gauged Skyrmion are shown as a function on the parameter V (Figure 1a) and the parameter κ (Figure 1b) for a baryon number $n = 1$.

Numerical solutions.— Subject to the above boundary conditions, we solve numerically the set of four Maxwell-Skyrme equations. The numerical calculations are performed by using the program CADSOL [18], based on the iterative Newton-Raphson method. As initial guess in the iteration procedure, we use the spherically symmetric regular solutions of the pure Skyrme model. The typical relative error is estimated to be lower than 10^{-3} .

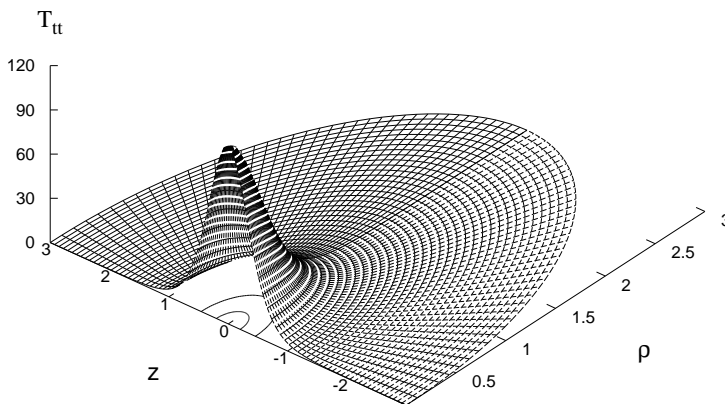


Figure 2a.

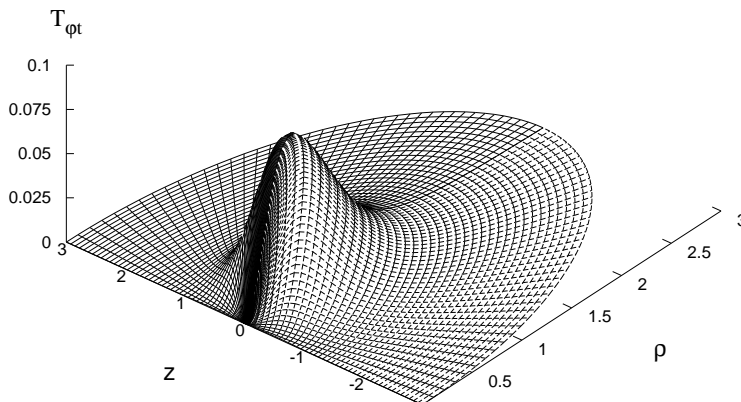


Figure 2b.

Figure 2. The components $T_{\phi t}$ and T_{tt} of the energy momentum tensor are shown for a typical $n = 1$ solution, with $\kappa = 0.72$, $V = 0.067$.

For a given Baryon number, the solutions depend on two continuous parameters, the values V of the electric potential at infinity and the Skyrme coupling constant κ . Here we consider solutions in the one baryon sector only, although similar results have been found for $n > 1$. The solutions with $V = 0$ have $b = 0$ and correspond to static dipoles discussed in [17]. A nonvanishing V leads to rotating regular configurations, with nontrivial functions f , g , a and b . Rotating solutions appear to exist for any value of κ . As we increase V from zero while keeping κ fixed, a branch of solutions forms. Along this branch, the total energy and the angular momentum increase continuously with V . The ratio J/E increases also, but remains always smaller than one. At the same time, the numerical errors start to increase and we obtain large values for both E and J , and for some V_{max} the numerical iterations fail to converge. An accurate value of V_{max} is rather difficult to obtain, especially for large values of κ . Alternatively, we may keep fixed the magnitude of the electric potential at infinity and vary the parameter κ .

In Figure 1 we present the properties of typical branches of solutions. In Figure 1a, the angular momentum and the energy are parametrised by V for several fixed value of κ , while in Figure 1b these quantities are

parametrised with κ for several fixed values of V , including $V = 0$ corresponding to the non-spinning soliton. The energy bound in the purely magnetically gauged case with $V = 0$ is not saturated, as is the case also for the ungauged Skyrminion. We expect likewise that this numerically constructed solution is topologically stable, but cannot estimate the energy excess above the lower bound analytically.

One can see from Figure 1b that, for a given value of κ , the energy of the spinning soliton is always smaller than the energy of the ungauged Skyrminion, but is larger than the energy of the corresponding non-spinning static gauged solution. The latter is gauged only with the magnetic field and minimises the energy functional, while the spinning system gauged with both the magnetic and the electric fields minimises the non-positive definite Lagrangian density, and the additional electric field does not feature in the topological lower bound. As a result, the spinning, electrically charged, solutions have higher energies than the static ones. The situation here is identical with that of the Julia-Zee dyon, in this respect.

In Figure 2a we plot the energy density $\mathcal{E} = T_{tt}$, and in Figure 2b the angular momentum density $T_{\varphi t}$ of a typical $n = 1$ solution as a function of the coordinates ρ, z , for $\kappa = 0.72$, $V = 0.067$. We notice that the energy density $\epsilon = T_{tt}$ does not exhibit any distinctly localised individual components, a surface of constant energy density being topologically a sphere. However, this is a deformed sphere such that the profiles of $\mathcal{E} = T_{tt}(r, \theta)$ versus r for each value of θ are distinct and non overlapping. It presents a peak on the symmetry axis, and the density profiles decrease monotonically with r .

Also, the electrically charged $U(1)$ gauged Skyrminion rotates as a single object and the $T_{\varphi t}$ -component of the energy momentum tensor associated with rotation presents a maximum in the $z = 0$ plane and no local extrema (see Figure 2b).

Conclusions.— We have presented here the first example of spinning solution residing in the one-soliton sector of the theory which has a topologically stable limit. These solutions of the $U(1)$ -gauged Skyrme model carry mass, angular momentum, electric charge and a magnetic dipole momentum. The electric charge is induced by rotation and equals the total angular momentum.

Similar qualitative results have been found by adding to the Lagrangean (1), a self-interaction potential of the $O(4)$ scalar field representing the pion mass. Nonzero pion masses lead to larger values for the energy and angular momentum.

Also, we have found that similar to the ungauged case, the spinning Skyrminions admit gravitating generalisations, which are currently under study. These solutions satisfy also the generic relation (16).

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