

$d = 4 + 1$ gravitating nonabelian solutions with bi-azimuthal symmetry

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Abstract

We construct static, asymptotically flat solutions of $SU(2)$ Einstein-Yang-Mills theory in $4 + 1$ dimensions, subject to bi-azimuthal symmetry. Both particle-like and black hole solutions are considered for two different sets of boundary conditions in the Yang-Mills sector, corresponding to multisolitons and soliton-antisoliton pairs. For gravitating multi-soliton solutions, we find that their mass per unit charge is lower than the mass of the corresponding unit charge, spherically symmetric soliton.

1 Introduction

The last years have seen an increasing interest in the solutions of Einstein equations involving more than four dimensions. The results in the literature indicate that the physics in higher-dimensional general relativity is far richer and complex than in the standard four-dimensional theory.

Naturally, most of the studies in the literature were carried out for vacuum solutions or to configurations with an Abelian matter content. At the same time, a number of results in the literature clearly indicate that solutions to the Einstein equations coupled to non Abelian matter fields possess a much richer structure than in the $U(1)$ case (see [1] for a survey of the situation in four dimensions and the more recent review [2] for $d > 4$), most notably in that they are not restricted to black holes, but can also be regular.

Physically reasonable stationary vacuum solutions in higher dimensional spacetimes, $d \geq 4$, fall in two categories, distinguished by their asymptotic behaviours. In the first category, there are the static spherically symmetric solutions generalising the $d = 4$ Schwarzschild black hole, found by Tangherlini a long time ago [3], the rotating Myers-Perry solution [4] generalising the four dimensional Kerr black hole, and more recently the black ring solutions [5, 6]. In all these cases, the d -dimensional spacetime approaches asymptotically the \mathcal{M}^d Minkowski background. The second category are the black string solutions, and the corresponding black p -branes generalizations [7]. The black strings approach asymptotically $d - 1$ dimensional Minkowski-spacetime times a circle, $\mathcal{M}^{d-1} \times S^1$, and in the simplest case present translational symmetry along the extra-coordinate direction. (Such configurations are important if one supposes the existence of extra dimensions in the universe, which are likely to be compact and described by a Kaluza-Klein (KK) theory.)

As is the case with the usual Schwarzschild black hole, all these vacuum solutions can be extended to describe configurations with an Abelian matter content. The inclusion of non Abelian matter fields is less systematic and is complicated by the fact that all known such solutions can only be evaluated numerically, starting from the earliest found Einstein-Yang-Mills (EYM) solution in four spacetime dimensions discovered by Bartnik and McKinnon [8].

Spherically symmetric solutions to EYM systems in d -spacetime dimensions, approaching asymptotically the \mathcal{M}^d Minkowski background, were constructed systematically in [9]-[15]. The Yang–Mills (YM) sector of the systems studied there consisted of all needed terms belonging to the YM hierarchy [16, 17], which are higher order in the YM curvature in the manner of the Skyrme model. Such terms may arise in the low energy **effective** action of string theory [18, 19, 20]. It has been established that only in the presence of these higher order in the YM curvature terms, does the EYM solution lead to a finite mass. In the absence of such Skyrme-like terms, for example in [21, 22] (in $d = 5$), the mass of the solution diverges. Both particle like and black hole solutions were constructed. The properties of these configurations are rather different from the familiar Bartnik-McKinnon solutions [8] in $d = 4$, and are somewhat more akin to the gravitating monopole solutions to EYM-Higgs system [23], which is not surprising since the latter features the dimensionful vacuum expectation value (VEV) of the Higgs field, while the former contain additional dimensionful terms entering as the couplings of the higher order YM terms.

As for solutions to the EYM system in d dimensional spacetime whose vacuum has the structure of $\mathcal{M}^{d-1} \times S^1$ like the black string solutions, these are only constructed if one of the spacelike dimensions is supposed to be compact, and a Kaluza-Klein descent is performed, essentially eliminating that coordinate. Such solutions are given for $d = 5$ in [21]-[27]. However, in the present work we will not be concerned with this type of solutions.

Our aim in the present work is to extend the construction of asymptotically flat finite mass EYM solutions vacuum, relaxing the constraint of spherical symmetry in the $d - 1$ dimensional spacelike subspace as in [9]-[15].

The simplest possibility is to consider the imposition of a symmetry which leads to two a dimensional reduced effective system, rather than the one dimensional one in the previous examples. This is the first such attempt in the literature, and the numerical work of solving a two dimensional EYM boundary value problem is a task of considerable complexity. To achieve a two dimensional subsystem, we have found that the simplest option is to impose bi-azimuthal symmetry on the $d = 5$ static EYM system. This is why we have restricted to $d = 5$, for otherwise a similar application of azimuthal symmetries in each plane would result in multi-azimuthal ¹ subsystems, with higher dimensional boundary value problems to be solved, technically beyond the scope of this work. Indeed, as a warmup for the task at hand, we have carried out the same program in [29] recently, with the dilaton replacing gravity.

While we have restricted to five dimensional EYM solutions for technical reasons, this example is of considerable physical relevance since it enters all $d = 5$ gauged supergravities as the basic building block and one can expect the basic features of its solutions to be generic. Also special about $d = 5$ gravitating YM is the particular critical properties of the solutions present in all $d = 4p + 1$ analysed in [12], and first discovered in [10]. Indeed in the $d = 5$ YM-dilaton (YMd) system, studied in [29], these critical properties were present, providing yet another confirmation that dilaton interactions with YM, mimic [30] those with Einstein gravity.

The purpose of this paper is to present numerical arguments for the existence of a class of static $d = 5$ solutions to the EYM equations of the model studied in [10], but now, subject to bi-azimuthal symmetry. These configurations present a spacetime symmetry group $R \times U(1) \times U(1)$, where R denotes time translation symmetry and the $U(1)$ factors the rotation symmetry in two orthogonal planes. We present both regular and black hole solutions. In the particle like case we find solutions with many similar properties to those of the four dimensional $SU(2)$ YM multi-instantons and composite instanton-antiinstanton bound states with $U(1) \times U(1)$ symmetry, reported in Ref. [31]. Dilatonic generalizations of these solutions have been considered in [29], in which higher order gauge curvature terms were included in the action to enable the existence of finite mass solutions.

¹If one applied instead, spherical symmetry in the $d - 2$ dimensional subspace of the $d - 1$ spacelike dimensions, then the residual subsystems will always be *two dimensional* irrespective of the value of d . For example in $d = 5$ this would be the $SO(3)$ symmetry in the 3 dimensional subspace of the 3 dimensional subspace of the 4 spacelike dimensions, exactly as for the axially symmetric instantons [28]. While this may appear to be an attractive alternative, we have found that tackling the boundary value problem in that case is a considerably harder task, even in $d = 4$.

2 The model

2.1 The ansatz and field equations

We consider the five dimensional SU(2) EYM action

$$S = \int d^5x \sqrt{-g} \left(\frac{R}{16\pi G} - \mathcal{L}_m \right), \quad (1)$$

where \mathcal{L}_m is given by the superposition of the $p = 1$ and $p = 2$ terms in the YM hierarchy [10]

$$\mathcal{L}_m = \frac{\tau_1}{2 \cdot 2!} \text{Tr} \mathcal{F}_{\mu\nu}^2 + \frac{\tau_2}{2 \cdot 4!} \text{Tr} \mathcal{F}_{\mu\nu\rho\sigma}^2, \quad (2)$$

with $\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + i[\mathcal{A}_\mu, \mathcal{A}_\nu]$ the 2-form YM curvature and $\mathcal{F}_{\mu\nu\rho\sigma} = \{\mathcal{F}_{\mu[\nu}, \mathcal{F}_{\rho\sigma]}\}$ the 4-form YM curvature consisting of the totally antisymmetrised product of two YM 2-form curvatures (the bracket $[\nu\rho\sigma]$ implies cyclic symmetry). τ_1 and τ_2 are dimensionful coupling strengths.

Variation of the action (1) with respect to the metric $g^{\mu\nu}$ and gauge potential A_μ leads to the EYM equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G (T_{\mu\nu}^{(1)} + T_{\mu\nu}^{(2)}), \quad (3)$$

$$\tau_1 D_\mu \mathcal{F}^{\mu\nu} + \frac{1}{2} \tau_2 \{\mathcal{F}_{\rho\sigma}, D_\mu \mathcal{F}^{\mu\nu\rho\sigma}\} = 0, \quad (4)$$

where

$$T_{\mu\nu}^{(p)} = \text{Tr} \{ \mathcal{F}(2p)_{\mu\lambda_1\lambda_2\dots\lambda_{2p-1}} \mathcal{F}(2p)_\nu^{\lambda_1\lambda_2\dots\lambda_{2p-1}} - \frac{1}{4p} g_{\mu\nu} \mathcal{F}(2p)_{\lambda_1\lambda_2\dots\lambda_{2p}} \mathcal{F}(2p)^{\lambda_1\lambda_2\dots\lambda_{2p}} \}, \quad (5)$$

is the energy-momentum tensor for the p -th YM term in (2), $p = 1, 2$.

We consider a $d = 5$ static metric form with two orthogonal commuting rotational Killing vectors

$$ds^2 = -f(r, \theta) dt^2 + \frac{s(r, \theta)}{f(r, \theta)} (dr^2 + r^2 d\theta^2) + \frac{l(r, \theta)}{f(r, \theta)} \sin^2 \theta d\varphi^2 + \frac{p(r, \theta)}{f(r, \theta)} \cos^2 \theta d\psi^2, \quad (6)$$

where r is the radial coordinate, and θ, φ, ψ are Hopf coordinates in S^3 , with $0 \leq \theta \leq \pi/2$ and $0 \leq \varphi, \psi \leq 2\pi$.

The construction of a YM Ansatz compatible with the symmetries of the above line element has been discussed at length in [29], [31]. The purely magnetic gauge connection has six nonvanishing components and reads

$$\begin{aligned} \mathcal{A} = & \frac{1}{2} u_3 a_r(r, \theta) dr + \frac{1}{2} u_3 a_\theta(r, \theta) d\theta \\ & + \left(\frac{1}{2} u_1 \chi^1(r, \theta) + \frac{1}{2} u_2 \chi^2(r, \theta) + \frac{n}{2} u_3 \right) d\varphi + \left(\frac{1}{2} u_1 \xi^1(r, \theta) + \frac{1}{2} u_2 \xi^2(r, \theta) + \frac{n}{2} u_3 \right) d\psi, \end{aligned} \quad (7)$$

where $u_1 = \sin n(\varphi + \psi)\tau_1 - \cos n(\varphi + \psi)\tau_2$, $u_2 = \cos n(\varphi + \psi)\tau_1 + \sin n(\varphi + \psi)\tau_2$, $u_3 = \tau_3$, τ_i being the Pauli matrices and n the winding number of the solutions, $n = 1, 2, \dots$. In the flat space limit, the reduced action density describes a $U(1)$ Higgs like model with **two** effective Higgs fields χ^A and ξ^A ($A = 1, 2$), coupled minimally to the $U(1)$ gauge connection (a_r, a_θ) [31].

To remove the $U(1)$ residual gauge freedom of the connection, we impose the usual gauge condition $\partial_r a_r + \frac{1}{r} \partial_\theta a_\theta = 0$.

2.2 Boundary conditions

In this paper we shall consider both globally regular and black hole solutions of the field equations (3), (4). The boundary conditions satisfied at infinity and at $\theta = 0, \pi/2$ is the same in both cases, and are found from the requirements of finite energy and regularity of solutions. At $r \rightarrow \infty$ one imposes

$$a_r = 0, \quad a_\theta = -2m, \quad \chi^A = (-1)^{m+1}n \begin{pmatrix} \sin 2m\theta \\ \cos 2m\theta \end{pmatrix}, \quad \xi^A = -n \begin{pmatrix} \sin 2m\theta \\ \cos 2m\theta \end{pmatrix}, \quad f = l = p = s = 1, \quad (8)$$

with m a positive integer. The following boundary conditions holds for gauge potentials at $\theta = 0$

$$a_r = \frac{1}{n}\partial_r\xi^1, \quad a_\theta = \frac{1}{n}\partial_\theta\xi^1, \quad \chi^1 = 0, \quad \xi^1 = 0, \quad \partial_\theta\chi^2 = 0, \quad \xi^2 = -n, \quad (9)$$

while for $\theta = \pi/2$ one imposes

$$a_r = \frac{1}{n}\partial_r\chi^1, \quad a_\theta = \frac{1}{n}\partial_\theta\chi^1, \quad \chi^1 = 0, \quad \xi^1 = 0, \quad \chi^2 = -n, \quad \partial_\theta\xi^2 = 0. \quad (10)$$

The boundary conditions for the metric functions at $\theta = 0$ are $\partial_\theta f = \partial_\theta s = \partial_\theta l = \partial_\theta p = 0$, and agree with the boundary conditions on the $\theta = \pi/2$ axis. There are also elementary flatness requirements which imposes for the metric functions $s = l$ at $\theta = 0$ and $s = p$ at $\theta = \pi/2$.

To obtain globally regular EYM solutions with finite energy density we impose at the origin ($r = 0$) the boundary conditions

$$a_r = 0, \quad a_\theta = 0, \quad \chi^A = \begin{pmatrix} 0 \\ -n \end{pmatrix}, \quad \xi^A = \begin{pmatrix} 0 \\ -n \end{pmatrix}, \quad \partial_r f = \partial_r s = \partial_r l = \partial_r p = 0. \quad (11)$$

The black hole configurations possess an event horizon located at some constant value of the radial coordinate $r_h > 0$, where the following boundary conditions are imposed

$$a_r = 0, \quad \partial_r a_\theta = 0, \quad \partial_r \chi^A = 0, \quad \partial_r \xi^A = 0, \quad f = s = l = p = 0. \quad (12)$$

For $m = n = 1$, these are the spherically symmetric solutions discussed in [21],[10],[22]. In this case the metric functions present no angular dependence, with $l = p = s$, while $a_\theta = w(r) - 1$, $a_r = 0$, $\chi^1 = -\xi^1 = \frac{1}{2}(w(r) - 1)\sin 2\theta$, $\chi^2 = -(w(r) - 1)\cos^2\theta - 1$, $\xi^2 = -(w(r) - 1)\sin^2\theta - 1$.

2.3 Physical quantities

The mass M of solutions is the conserved charge associated with the Killing vector $v = \partial/\partial t$ and can be read from the asymptotic expression of the g_{tt} -component of the metric tensor

$$-g_{tt} = f = 1 - \frac{8GM}{3\pi r^2} + O\left(\frac{1}{r^4}\right). \quad (13)$$

The mass can also be expressed as an integral [32] over the 3-sphere at spacelike infinity,

$$M = \frac{1}{16\pi G} \frac{3}{2} \oint_\infty v^{\mu;\nu} d^3\Sigma_{\mu\nu}. \quad (14)$$

The topological charge of the particle-like solutions as evaluated in [31] is

$$q = \frac{1}{2} [1 - (-1)^m] n^2, \quad (15)$$

such that the Pontryagin charge is nonzero only for odd m , being equal to n^2 . For even values of m , the solutions will describe soliton-antisoliton bound states.

To evaluate the Hawking temperature and entropy of the black hole solutions, we use the following expansions of the metric functions at the horizon

$$\begin{aligned} f(r, \theta) &= f_2(\theta) \left(\frac{r - r_h}{r_h} \right)^2 + O \left(\frac{r - r_h}{r_h} \right)^3, & p(r, \theta) &= p_2(\theta) \left(\frac{r - r_h}{r_h} \right)^2 + O \left(\frac{r - r_h}{r_h} \right)^3 \\ l(r, \theta) &= l_2(\theta) \left(\frac{r - r_h}{r_h} \right)^2 + O \left(\frac{r - r_h}{r_h} \right)^3 & s(r, \theta) &= s_2(\theta) \left(\frac{r - r_h}{r_h} \right)^2 + O \left(\frac{r - r_h}{r_h} \right)^3. \end{aligned}$$

The zeroth law of black hole physics states that the surface gravity κ is constant at the horizon of the black hole solutions, where $\kappa^2 = -(1/4)g^{tt}g^{ij}(\partial_i g_{tt})(\partial_j g_{tt})\Big|_{r=r_h}$. Since from general arguments the Hawking temperature T_H is proportional to the surface gravity κ , $T_H = \kappa/(2\pi)$, we obtain the relation

$$T_H = \frac{f_2(\theta)}{2\pi r_h \sqrt{s_2(\theta)}}. \quad (16)$$

One can show, with help of the (r, θ) -component of the Einstein equations which implies $f_2 s_{2,\theta} = 2s_2 f_{2,\theta}$, that the temperature T_H , as given in (16), is indeed constant.

For the line element (6), the area A of the event horizon is given by

$$A = 4\pi^2 r_h^3 \int_0^{\pi/2} d\theta \sin \theta \cos \theta \sqrt{\frac{l_2(\theta) p_2(\theta) s_2(\theta)}{f_2^3(\theta)}}. \quad (17)$$

According to the usual thermodynamic arguments, the entropy S is proportional to the area A , $S = A/4G$.

We mention here also the Smarr-type relation which follows from (14) together with Einstein equations

$$\frac{2}{3}M = T_H S - \frac{4\pi^2}{6} \int_{r_h}^{\infty} dr \int_0^{\pi/2} d\theta \sin \theta \cos \theta \frac{\sqrt{l p s}}{f} (T_t^t - \frac{1}{3}T). \quad (18)$$

This relation has been used in practice to verify the accuracy of the numerical computation.

3 Properties of the solutions

The numerical calculations in this paper were performed by using the software package CADSOL, based on the Newton-Raphson method [33]. In this approach, the field equations are first discretised on a nonequidistant grid and the resulting system is solved iteratively until convergence is achieved. In this scheme, a new radial variable $x = r/(1+r)$ (or $x = 1 - r_h/r$ for black hole solutions) is introduced which maps the semi-infinite region $[0, \infty)$ (or $[r_h, \infty)$) to the closed region $[0, 1]$.

For any set of boundary conditions, we have found that the numerical iteration fails to converge for $\tau_2 = 0$. Thus, similar to the spherically symmetric case, no reasonable EYM- $p = 1$ solutions with biaximuthal symmetry is likely to exist. This agrees with the physical intuition based on a heuristic Derick-type scaling argument (although a rigorous proof exists for the spherically symmetric limit only [21, 22]). It is the $p = 2$ YM term, scaling as L^{-8} , which enables the existence of configurations with finite mass and well defined asymptotics.

As in the spherically symmetric case [10], dimensionless quantities in this model are obtained by rescaling the radial coordinate $r \rightarrow (\tau_2/\tau_1)^{1/4}r$. This reveals the existence of one fundamental parameter which gives the strength of the gravitational interaction $\alpha^2 = \tau_1^{3/2}(16\pi G/\tau_2^{1/2})$. Thus without loss of generality, one can fix the values of τ_1 and τ_2 to some arbitrary positive values and construct the solutions in terms of α . We use this property to set in the numerical computation $\tau_1 = \tau_2 = 1$ for $m = 1$ solutions and $\tau_1 = \tau_2 = 1/3$ for $m = 2$ configurations.

For any set (m, n) , the limit $\alpha \rightarrow 0$ can be approached in two ways and two different branches of solutions may exist. The first limit corresponds to a pure $p = 1$ YM theory in a flat background

Figure 1: The mass M is shown as a function of α for globally regular EYM solutions (a) for $m = 1$, and (b) for $m = 2$. Note that for the $m = 1$ configurations, M is the value of the mass divided by n^2 , the value of the corresponding topological charge.

(*i.e.* no gravity and no $p = 2$ YM terms), the solutions here replicating the (multi-)instantons and composite instanton-antiinstanton bound states discussed in [31]. The other possibility corresponds to a finite value of G as $\tau_1 \rightarrow 0$. Thus, the second limiting configuration is a solution of the truncated system consisting of $p = 2$ YM interacting with gravity, with no $p = 1$ YM term.

3.1 Particle-like solutions

3.1.1 $m = 1$ configurations

The $m = 1$ configurations carry a topological charge n^2 and describe (multi-)solitons. The $n = 1$ spherically symmetric case was discussed in [10] in a Schwarzschild coordinate system. We repeated the numerical analysis of [10] using the isotropic coordinate system (6). In the spherically symmetric limit only two of the functions in (6) are independent, f and $s = l = p$. The dominant term at the gravity decoupling limit $\alpha \rightarrow 0$ is the $F(2)$ term, the YM solution being the well known BPST instanton [34]. When α increases, these solutions get deformed by gravity and the mass M decreases (see Figure 1a). At the same time, the values of the metric functions f and s at the origin decrease, as indicated in Figure 2. This branch of solutions exists up to a maximal value α_{\max} of the parameter α . Another branch of solutions is found on the interval $\alpha \in [\alpha_{cr(1)}, \alpha_{\max}]$. On this second branch of solutions, both $f(0)$ and $s(0)$ continue to decrease but stay finite. However, a third branch of solutions exists for $\alpha \in [\alpha_{cr(1)}, \alpha_{cr(2)}]$, on which the two quantities decrease further. A fourth branch of solutions has also been found, with a corresponding $\alpha_{cr(3)}$ close to $\alpha_{cr(2)}$. Along this succession of branches, the values of the metric functions f and s at the origin continue to decrease.

On the other hand, the mass parameters do not increase significantly along these secondary branches. This behaviour with respect to the parameter α is the same as that which was found in [10], for the metric function $\sigma(r)$ at $r = 0$. An analytic explanation of these results was given in [12], where the observed oscillatory behaviour of these functions at $r = 0$ was characterised as a *conical fixed point*.

The $n > 1$ non-spherically symmetric solutions are constructed by starting with the known

spherically symmetric configuration and increasing the winding number n in small steps. The iterations converge, and repeating the procedure one obtains in this way solutions for arbitrary n . The physical values of n are integers. We have studied $m = 1$ solutions with $n = 2, 3$. As expected, the general features of the spherically symmetric solutions are the same for all $n > 1$ multi-solitons. Like for the Yang–Mills dilaton (YMd) model discussed in [29], when α is increased

Figure 2: The value at the origin of the metric functions f and s are shown as a function of α for $m = 1$ particle-like solutions with $n = 1, 2$.

from zero, a branch of gravitating solutions with winding number n emerges smoothly from the corresponding $F(2)$ flat space multi-instanton solution.

This branch extends up to a maximal value $\alpha_{max}(n)$ of the coupling constant α , beyond which the numerical iteration fails to converge. The value of $\alpha_{max}(n > 1)$ is smaller than the corresponding value in the spherically symmetric case. For example, we find numerically $\alpha_{max}(n = 2) \approx 0.412$ while the corresponding value for $n = 1$ is $\alpha_{max} \approx 0.571$. For all values $n \geq 1$ we considered, the limiting solutions at $\alpha_{max}(n)$ has no special features. A secondary branch, extending backward in α emerges at $\alpha_{max}(n)$. However, the numerical accuracy deteriorates drastically for the secondary branch of solutions around some critical value $\alpha_{cr} \sim 0.38$. Our numerical results in this case are less conclusive, the properties of these configurations requiring further work. We notice, however, that the value at the origin of all metric functions decreases along these branches, as seen in Figure 2 ². We expect that the oscillatory pattern of $g_{tt}(0)$ arising from the *conical fixed point* observed for the spherically symmetric $m = 1, n = 1$ solutions, will also be discovered for the $n > 1$ solutions here. However, the construction of the secondary branches of solutions is a difficult numerical problem beyond the scope of the present work.

In all cases we have studied, the metric functions f, l, p, s are completely regular and show no sign of an apparent horizon, while l and p have rather similar shapes. The angular dependence of the metric functions is rather small, although it increases somewhat with n . The gauge functions $a_r, a_\theta, \chi^A, \xi^A$ look very similar to those of the YMd solutions presented in [29]. Both $|\chi| = ((\chi^1)^2 + (\chi^2)^2)^{1/2}$ and $|\xi| = ((\xi^1)^2 + (\xi^2)^2)^{1/2}$ possess one node on the $\theta = 0$ and $\theta = \pi/2$ axis, respectively. The positions of these nodes move inward along the branches.

²Note that the values at the origin of all metric functions exhibited in this paper correspond to $f(r = 0, \theta = 0)$, $s(r = 0, \theta = 0)$. This restriction is reasonable since for all solutions with bi-azimuthal symmetry that we have found, the metric functions at $r = 0$ present almost no dependence on the angle θ .

It is also interesting to note that for the $m = 1$ solutions, the mass per unit charge of the gravitating multisoliton solutions is lower than the mass of a single particle, see Figure 1a. Thus these multisolitons are gravitationally bound states. This case resembles the situation found for $d = 4$ gravitating EYMH monopoles with a vanishing or small Higgs selfcoupling [35].

Figure 3: The same as Figure 2 for $m = 2$ particle-like solutions with $n = 1, 2$.

3.1.2 $m = 2$ configurations

The $m = 2$ configurations reside in the topologically trivial sector. These solutions can be thought of as composite systems consisting of two components which are pseudoparticles of Chern-Pontryagin topological charges $\pm n^2$. This type of solutions have no spherically symmetric limit. The position of each constituent can be identified according to the location of the maxima of the energy density. Also, the structure and location of the nodes of the (effective Higgs) scalar fields nicely reveal the evolution and the types of the solutions present at the respective values of the gravitational strength.

As in the case of the $m = 1$ configurations, coupling with gravity yields various branches of gravitating solutions which, however, have different limits depending on the values of the topological charge n^2 of the constituents. Also, their behaviours as functions of the gravitational coupling α differ from those with $m = 1$ presented above.

$n = 1$

There is a certain similarity between the properties of the $4 + 1$ dimensional YM d model studied in [29], and the model under consideration here. As in the former case, we find that in the limit $\alpha \rightarrow 0$ resulting from $G \rightarrow 0$, no solution with $n = 1$ exists, *i.e.* that in the gravity decoupling limit no such solution exists. On the other hand, we know from the work of [10] that in the flat space limit the EYM solution of this model reduces to the BPST instanton [34] of the $p = 1$ (usual) YM model, so that in this limit the $p = 1$ YM term dominates over the $p = 2$ term. Thus the nonexistence of a $m = 2$, $n = 1$ solution here in the gravity decoupling limit implies that there should exist no such solution in the $4 + 0$ dimensional $p = 1$ YM model on flat space. This is precisely what was found in [31].

In the other limit of $\alpha \rightarrow 0$ however, when both $\tau_1 \rightarrow 0$ and the gravitational coupling G remain finite, such solutions exist. It turns out that in this limit, it is the $p = 2$ term which dominates over the $p = 1$ YM term. The characteristic feature of this configuration is that both nodes of the effective Higgs fields $|\chi|$ and $|\xi|$ merge on the $\theta = \pi/4$ hypersurface. From this limiting configuration, a branch evolves as α increases. Along this branch the nodes move towards the symmetry axes, ρ and σ , respectively (with $\rho = r \sin \theta$, $\sigma = \rho \cos \theta$), forming two identical vortex rings whose radii slowly decrease, while the separation of both rings from the origin also decreases. The evolution of the solution along this branch can be associated with the increase of the coupling τ_1 , while τ_2 and the gravitational coupling G remain fixed. This reproduces the corresponding pattern in the YMd system [29]. Note that there is a difference between the evolutions of the configurations we are considering here in this $4 + 1$ dimensional theory, and the behaviour of the gravitating multimonoles or the monopole-antimonopole solutions of the gravitating YMH system $3 + 1$ theory [35, 36]. Although the latter also feature different branches, the evolution along those branches is usually associated with the increasing of the gravitational coupling G on the lower mass branch, and, the decreasing of the VEV of the Higgs field on the upper mass branch. More importantly, the $m = 2$, $n = 1$ solution in that case does have a gravity decoupling limit. Thus, the gravitating solutions of the $3 + 1$ YMH theory usually are linked to flat space

Figure 4: The position of nodes $(z_0^{(1)}, z_0^{(2)})$ is presented for $m = 2$ particle-like solutions with $n = 1, 2$.

configurations, while the solutions discussed here clearly do not have a flat space limit.

On the $p = 2$ branch (where the $p = 2$ term $\mathcal{F}_{MNR S}^2$ dominates) of five dimensional EYM $m = 2$, $n = 1$ solutions, the gauge functions a_r, a_θ as well the metric functions f and s are almost θ -independent, whereas the metric functions l and p possess reflection symmetry with respect to $\theta = \pi/4$ axis. As seen in Figures 1 and 2, the mass of the gravitating solutions on this branch decreases, as well as the values at the origin of the metric functions.

At the critical value $\alpha \simeq 0.672$, the node structure of the configuration changes and both vortex rings shrink to zero size, two isolated nodes appearing on each symmetry axis. This transition means that the $p = 1$ term \mathcal{F}_{MN}^2 becomes dominant. This secondary branch has a small extension in α up to the maximal value $\alpha_{max} \simeq 0.6765$, beyond which we could not find regular gravitating solutions. We found instead that this branch merges here with the second, $p = 1$ branch, which

evolves backwards in α as the value of the metric function $f(0)$ continues to decrease.

The evolution along this short branch can be associated with the decrease of the coupling constant τ_2 relative to τ_1 , as the gravitational coupling G remains fixed. For this branch the relative distance between the nodes increases, one lump slowly moving towards the origin and the other one moving in the opposite direction. This branch persists up to a value of the coupling constant $\alpha_{cr} \simeq 0.6665$, where a critical solution is approached. Due to severe numerical difficulties encountered here, we could not clarify the properties of this critical solution further. As $\alpha \rightarrow \alpha_{cr}$, the metric function $f(0)$ takes a very small value, $f(0) \simeq 10^{-3}$, while $l(0)$ remains one order of magnitude larger. At the same time, the Lagrangian density and the mass of the configuration remain finite at that point. The critical behaviour observed here resembles the case of the gravitating $4 + 1$ EYM vortices in the model consisting only of the $p = 1$ YM term [21]. It is tempting to speculate that, similar to case in [21], the solution splits into two parts: a non-singular interior region with a special geometry (so-called throat) and an exterior asymptotically flat region where two pseudoparticles are located. However, another parametrisation of the metric, differing from (6) (and possibly even a different numerical approach) appears to be necessary to clarify these aspects.

$n = 2$

This configuration also resides in the topologically trivial sector and can be considered as consisting of two pseudoparticles of charges $\pm 2^2$. In this case the interaction between the non Abelian matter fields becomes stronger than in the case of ± 1 constituents, resulting in a different pattern of possible branches of solutions.

Indeed, as in the case of the $4 + 1$ dimensional YMd system [29], we observe two different branches of gravitating solutions, both linked to the $\alpha \rightarrow 0$ limit. The lower branch, on which the

Figure 5: The energy density $\epsilon = -T_t^t$ is shown as a function of the coordinates $\rho = r \sin \theta$, $\sigma = \rho \cos \theta$ for a $m = 1$, $n = 2$ EYM black hole solutions with $\alpha = 0.2$, $r_h = 0.5$.

$p = 1$ YM term dominates, emerges from the corresponding flat space solution of the pure YM theory with vanishing $p = 2$ term. Varying α along this branch is associated with the decrease of τ_1 , at fixed τ_2 and fixed gravitational coupling G .

For small values of α the corresponding $m = 2, n = 2$ solutions possess two (double) nodes of the fields $|\chi|$ and $|\xi|$ on the ρ and σ symmetry axes, respectively. The locations of nodes correspond to the locations of the two individual constituents and the action density distribution possesses two distinct maxima on the $\theta = \pi/4$ axis. As α increases the mass of the solution increases and both pseudoparticles move from spatial infinity towards the origin. For values of α smaller than $\alpha_{cr} \simeq 0.635$ along this branch, the energy of interaction between the individual pseudoparticles is relatively small and both constituents remain individual. We observe that, as the coupling constant approaches this critical value from below, the energy of interaction rapidly increases and both pseudoparticles form a bound state, as seen from Figures 3,4.

This branch extends further up to a maximal value $\alpha_{max} \simeq 0.7265$ where it bifurcates with an upper $p = 2$ branch which extends all the way back to $\alpha = 0$. Varying α along this branch is associated with the increase of τ_2 relative to τ_1 , as gravitational constant G remains fixed.

Along the upper branch, as α slightly decreases below α_{max} , the inner node inverts direction of its movement toward the outer node which still moves inwards. Thus, both nodes on the symmetry axis rapidly approach each other and merge forming a two vortex ring solution at $\alpha \simeq 0.708$. The action density then has a single maximum on $\theta = \pi/4$ axis. As α decreases further both nodes move away from the symmetry axis and their positions do not coincide with the location of the maximum of the action density. Further decreasing α results in the increase of the radii of the two rings around the symmetry axis, and in the limit $\alpha \rightarrow 0$ the rings touch each other on the $\theta = \pi/4$ hyperplane.

3.2 Black hole solutions

According to the standard arguments, one can expect black hole generalisations of the regular configurations to exist at least for small values of the horizon radius r_h . This is confirmed by the numerical analysis for $m = 1, n = 2$. Several black hole solutions with $m = 2, n = 1$ have been also constructed, with a lower numerical accuracy, however.

As discussed in [10] spherically symmetric $m = 1, n = 1$ black hole counterparts exist for any regular solution with the same amount of symmetry. Starting for a given $\alpha_0 < \alpha_{max}$ from a $r_h = 0$ first branch regular solution, one finds a branch of black hole solutions extending up to a maximal value of the event horizon radius $r_h = r_h^{max}$. When r_h increases, both the mass and the Hawking temperature increase. The value of $r_{h(max)}$ depends on α . The Hawking temperature decreases on this branch, while the mass parameter increases; however, the variation of mass is relatively small. The corresponding picture for secondary branches is more complicated and will not be discussed here.

The numerical construction of nonspherically symmetric black hole solutions appears to be more difficult than in the globally regular case. However, our numerical results indicate that the $m = 1, n > 1$ black hole solutions with bi-azimuthal symmetry follow this general pattern. First, black hole solutions seem to exist for all values of α or which regular configurations could be constructed (here we restrict again to first branch solutions). Also, it appears that black hole solutions exist only for a limited region of the (r_h, α) space. However, for a given value of α , it is very difficult to find an accurate value of r_h^{max} . An approach to this problem with a different method appears to be necessary.

These solutions possess a regular deformed S^3 horizon. The energy density has a pronounced angle-dependence, with a maximum on the $\theta = \pi/2$ hypersurface. Figure 5 shows a three dimensional plot of the energy density of a $m = 1, n = 2$ black hole with $\alpha = 0.2, r_h = 0.5$ as a function of the coordinates $\rho = r \sin \theta, \sigma = r \cos \theta$. With increasing the winding number n , the absolute maximum of the energy density residing on the $\rho = \sigma$ axis, shifts inward. The metric and gauge functions possess a nontrivial angular dependence at the horizon.

Outside their event horizon, these black holes possess nontrivial non Abelian fields. Therefore they represent a further counterexample to the $d = 5$ no-hair conjecture. Also, these bi-azimuthally symmetric black holes clearly show that the higher dimensional static black hole solutions need not be spherically symmetric.

4 Conclusions

Motivated by the recent interest in gravitating solutions in higher dimensional spacetime, we have studied static, bi-azimuthally symmetric solutions with non Abelian fields in $d = 4 + 1$ spacetime dimensions. Our solutions are akin to the static, axially symmetric EYM configurations in $d = 4$, studied exhaustively in [37], [38, 39]. Our choice of bi-azimuthal symmetry is motivated by our desire to reduce the boundary value problem to a two dimensional one. An alternative symmetry imposition resulting in a two dimensional residual system would be imposition of $SO(3)$ spherical symmetry in the 3 dimensional spacelike dimensions like in [28]. We have eschewed this alternative for purely technical reasons (see footnote ¹).

The regular and black hole solutions presented are natural generalisations of the known [10] $d = 5$ EYM spherically symmetric globally regular and black hole solutions. Like the former they are asymptotically flat, finite mass solutions, that describe nontrivial gravitating magnetic gauge field configurations. Our $d = 5$ EYM configurations are the first $d \geq 5$ dimensional static solutions in the literature, which are not spherically symmetric.

In the case of particle like solutions, which we have studied much more intensively than their black hole counterparts, their dependence on the effective gravity coupling α is analysed numerically in some detail. By and large this is qualitatively very similar to that for the YMd solutions [29] in $4 + 1$ dimensions, except that here we have four metric functions to keep track of, as opposed to the single dilaton field in the previous case [29]. We have studied regular solutions with $m = 2$, $n = 1$ and $m = 2$, $n = 2$ in detail, numerically.

Just as in the YMd case, here too there exists a $m = 2$, $n = 1$ solution on the branch where the $p = 2$ YM term dominates, while on the other branch, where the $p = 1$ YM term dominates, such a solution is absent. As it turns out the $p = 1$ YM term dominates in the gravity decoupling limit, which is consistent with our knowledge that this model in $4 + 0$ dimensions does not support [31] a $m = 2$, $n = 1$ solution.

Another qualitative feature of 5 dimensional EYM solutions that is confirmed here is the occurrence of a *conical* singular behaviour with respect to the dependence of the metric functions on α . This features the oscillatory picture first discovered for the $m = 1$, $n = 1$ spherically symmetric solutions in [10] and analysed in [12], which are found also here for the $m = 1$, $n > 1$ case.

As compared to the $d = 4$ case [38, 39], we expect the existence of a much richer set of nonspherically symmetric EYM solutions in $d = 5$. The configurations studied here represent only the simplest, asymptotically flat type of $d = 5$ nonspherically symmetric gravitating nonabelian solutions. For example, it is known that $d = 5$ Einstein gravity coupled to Abelian fields presents black ring [40] solutions. These solutions have an horizon topology $S^2 \times S^1$ and approach at infinity the flat \mathcal{M}^5 background, as is the case with our solutions. It would be interesting to construct non Abelian versions of the $U(1)$ black ring solutions. A black ring can be constructed in a heuristic way by taking a black string, bending the extra dimension and spinning it along the circle direction just enough so that the gravitational attraction is balanced by the centrifugal force. In this framework the (putative) nonabelian black ring would behave locally as a boosted black string, e.g. that in [26], with very similar charges and fields. The numerical work involved in the construction of non spherically symmetric higher dimensional EYM solutions is, however, a considerably challenging task.

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