ALGEBRAIC K-THEORY AND PARTITION FUNCTIONS IN CONFORMAL FIELD THEORY

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Contents

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Notation

$\mathfrak{g}$	–	Lie algebra of rank r
$\alpha_1, \ldots, \alpha_r$	–	Simple roots of $\mathfrak{g}$
$\omega_1, \ldots, \omega_r$	–	Fundamental weights of $\mathfrak{g}$
Δ^+	–	Set of positive roots of $\mathfrak{g}$
ρ	–	Weyl vector
$C(\mathfrak{g})$	–	Cartan matrix of $\mathfrak{g}$
$W(\mathfrak{g})$	–	Weyl group of $\mathfrak{g}$
g	–	Coxeter number of $\mathfrak{g}$
h	–	Dual Coxeter number of $\mathfrak{g}$
$V(\lambda)$	–	Irreducible representation of $\mathfrak{g}$, of highest weight λ
$\chi_\lambda = \mathrm{ch}(\lambda)$	–	Character of the representation $V(\lambda)$
$Y(\mathfrak{g})$	–	Yangian associated with $\mathfrak{g}$
W_i^j	–	Irreducible representation of $Y(\mathfrak{g})$
Q_j^i	–	Character of the representation $W_i^j\|_{\mathfrak{g}}$
Q_0	–	Weyl denominator

Chapter 1

Introduction

Quantum field theory (QFT) plays a major role in modern theoretical physics. From its beginnings in the late 1920s, it has grown to be the most successful physical theory in existence today. It provides the best working description of the fundamental laws of physics and is an extremely useful tool for investigating the behaviour of complex systems. Its unrivalled ability to accurately calculate physical quantities only adds to its reputation. Therefore it is hardly surprising that quantum field theory plays such a central role in our description of nature.

In addition to being a significant theory in its own right, QFT provides essential tools to many other branches of physics, for example to condensed matter physics. The influence of QFT is far-reaching, with its very mathematical nature helping to build new bridges between physics and mathematics. Despite its many great successes, very little is known about the deep mathematical structure underlying this theory. Progress in this area would be beneficial to both mathematics and physics.

With any quantum field theory, the main aim is to find an exact solution, by no means an easy task. In fact the solution of any non-trivial QFT, whether or not it is physically relevant, would be a major step forward in the search for a better understanding of the subject. By an exact solution we mean the explicit calculation of the n-point functions (or correlation functions) of the theory. This is sufficient since Wightman's reconstruction theorem [1] states that with this knowledge the entire field content and physical state space of the theory can be computed.

The attempt to solve any QFT exactly is a very ambitious undertaking. The most likely candidates for success are those QFTs whose symmetries give rise to a large number of conservation laws. These laws might impose enough restrictions on the theory to allow it to be solved exactly.

Unfortunately this approach is useless in 3+1 dimensions because of the Coleman-Mandula theorem [2]. This states that besides Poincaré invariance and an internal gauge group describing the degeneracy of the particle spectrum, any additional symmetries cause the scattering matrix of each massive QFT to be trivial. There is a similar result due to Lochlainn O'Raifeartaigh [3], which states that it is impossible to combine internal and relativistic symmetries other than in a trivial way.

1+1 dimensional integrable models

This motivates us to consider the 1+1 dimensional case, in which the Coleman-Mandula theorem no longer holds. Here there is nothing to rule out the existence of a theory with an infinite number of conservation laws, meaning that the hope of finding exact solutions is much more realistic. Such theories are called integrable. Integrable models are hugely relevant in physics, having found applications in offcritical descriptions of statistical mechanics and condensed matter systems reduced to two dimensions.

Restrictions to this low-dimensional case may seem unrealistic. However, by facilitating the construction of exact solutions, 1+1 dimensional theories can be used to shed new light on the structure of more general QFTs. This easily justifies the study of QFT in 1+1 dimensions.

Links to conformal field theory

The recent wave of interest in integrable models lies in their interpretation as deformed conformal field theories (CFTs) [4]. CFTs form a particular class of integrable field theories. They are characterised by scale invariance and describe massless relativistic particles or statistical mechanical systems at a critical point. As with other theories, they become extremely powerful in two dimensions, being characterised by an infinite number of conserved currents that ensure their solvability. Here the infinite set of conservation laws corresponds to conformal spacetime symmetry. Interest in this topic was rekindled when Belavin, Polyakov and Zamolodchikov [5] showed that the so-called minimal models are particular examples of solvable massless QFTs.

Given a CFT, what happens to the infinite set of conservation laws arising from conformal invariance when the system moves away from the critical point and scale invariance is lost? When the action of the critical point theory is perturbed by particular fields, the conformally invariant structure is in general destroyed. However, Zamolodchikov showed that for particular deformations of the CFT, an infinite set of the conserved charges may survive the breaking of conformal symmetry. This results in the corresponding massive field theory being integrable.

Scattering matrices

The scattering matrix (or S-matrix) is central to the study of quantum field theory. It determines the on-shell structure of the model and describes the collision of quantum particles. It must obey certain constraints inspired by the physics of the system. These include crossing symmetry, Lorentz invariance, analyticity in the energy variables, and the conservation of probability. In 1+1 dimensions these constraints are often restrictive enough that they enable one to conjecture the S-matrix and hence determine the theory completely. An essential feature of any 1+1 dimensional integrable field theory is an infinite set of conserved charges. This strongly restricts the dynamics of the system, imposing the following conditions on the scattering process: conservation of the total number of particles, conservation of individual particle momenta, and factorisation of the S-matrix into 2-particle scattering amplitudes.

Consistency of different ways of decomposing an amplitude into two-particle amplitudes leads to cubic relations between the two-particle amplitudes. These cubic relations are essentially the Yang-Baxter (or star-triangle) relations [6, 7].

In principle, exact construction of the scattering matrix serves as a first step in the calculation of the n-point or correlation functions of a system, for example via the form factor programme [8, 9, 10]. Although the relevant calculations are highly non-trivial and have been carried out in only the simplest cases, this nevertheless demonstrates the importance of the S-matrix in the search for a complete solution of any quantum field theory.

The thermodynamic Bethe ansatz

On the other hand it is possible to start with a massive integrable field theory and proceed in the opposite direction to recover conformal invariance. In the high-energy limit the masses of particles become negligible, with the result that scale and therefore conformal invariance are approximately restored. The high energy limit of an integrable QFT is studied by means of the thermodynamic Bethe ansatz (TBA).

The TBA was developed over twenty years ago by Yang and Yang [11, 12], as a technique to calculate thermodynamic quantities for a system of bosons interacting dynamically through factorisable scattering. The method was later generalised [13, 14, 15] to a system of relativistic particles interacting dynamically through the scattering matrix of an integrable QFT. It has become one of the most effective techniques for exploring the close relationship between conformal and integrable field theories.

Using the TBA approach, information can be extracted from a massive integrable quantum field theory once its scattering matrix is known. In particular, the effective central charge of the corresponding UV conformal field theory can be calculated. The information gained in this way is often sufficient to determine the field content at the critical point. Once the perturbing operator has been determined, the integrable field theory can then be formally described in terms of a classical Lagrangian. The TBA also has applications in scattering theory, where it can be used to test conjectured S-matrices for consistency.

This brief account of quantum field theory sets the scene for the work in this thesis.

Outline of PhD thesis

By a mixture of analytical and numerical techniques, physicists have formulated many beautiful and well-supported hypotheses for the integrable massive perturbations of conformally invariant theories, though mathematical proofs are not yet available. In particular it has been found that conformal dimensions are given in terms of the dilogarithm formulas [16], by finite order elements of the Bloch group, a basic tool in algebraic K-theory. This shows a deep connection between physics and a very active domain of number theory.

Among the integrable models for which this relationship holds are those described by pairs (X, Y) of ADET Dynkin diagrams [17]. Such models have equations of the form $AU = V$, where $A = C(X)^{-1} \otimes C(Y)$, C denotes a Cartan matrix, and X and Y are the Dynkin diagrams of simple Lie algebras of ranks m and n respectively. Moreover $U = \log(x)$ and $V = \log(1 - x)$ satisfy $e^U + e^V = 1$, the vector x is given by $x = (x_{11}, \ldots, x_{mn})$, and $f(x)$ is to be interpreted as $(f(x_{11}), \ldots, f(x_{mn}))$. The relationship between the matrix A and the scattering matrix of the integrable quantum field theory is described in chapter 3.

It seems that the resulting algebraic equations for x are solvable in terms of roots of unity. The case $(X, Y) = (A_m, A_n)$ has already been studied [17]. In this thesis we consider mainly the case $(X, Y) = (D_m, A_n)$. We study the algebraic equations of the model and find all solutions in explicit form using the representation theory of Lie algebras and related quantum groups. The solutions seem to be torsion elements of the Bloch group, allowing the effective central charge of the corresponding CFT to be calculated using the dilog formula mentioned above.

On a more mathematical note, we are interested in a closely related problem involving certain q-hypergeometric series. We investigate the overlap between series of this type and modular functions. We study a particular class of 2-fold q-hypergeometric series, denoted $f_{A,B,C}$, depending on some 2×2 matrix A. It turns out that for certain choices of the matrix A, the function f can be made modular. We calculate the corresponding values of B and C. It is expected that functions f arising in this way are characters of some rational conformal field theory. We show that this is true in at least one case. The content of the thesis is arranged as follows:

Chapter 2 introduces the mathematical and physical concepts important for subsequent chapters. The first section presents the theory of simple Lie algebras, as well as a number of other useful mathematical definitions. The second section is a short introduction to conformal field theory, concentrating on the aspects most relevant to this thesis.

Chapter 3 is the core of the thesis. We study the integrable models described by pairs of Dynkin diagrams (D_m, A_n). The equations of these models are solved in the general case, and their relation to Yangian representation theory is discussed. Results of the effective central charge calculations for many different models are summarised. The realisation of such models as coset models is also discussed.

Chapter 4 places our results in a more general context. We examine the relationship between certain q-hypergeometric series and modular functions. This is very closely related to the conformal field theory of the previous chapter.

Chapter 5 contains initial steps for the solutions of the remaining cases. A number of integrable models described by pairs of 'non-classical' Dynkin diagrams is studied. In particular the pairs (E_6, T_1), (E_7, T_1), and (E_8, T_1) are considered. In each case the equations of the models are solved, and corresponding values of the effective central charge are calculated. All calculations in this chapter are carried out using only elementary algebra.

Chapter 2

Overview

2.1 Mathematical Overview

2.1.1 Lie algebras

This section gives a brief introduction to simple Lie algebras, focusing mainly on concepts that arise in the study of conformal field theory. As far as possible the discussion is self-contained. There are numerous good books on the subject, and for a more detailed account of the material presented here see e.g. [18, 19].

Simple Lie algebras, generators, and roots

A vector space $\mathfrak{g}$ is called a **Lie algebra** if it is equipped with an anti-symmetric bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the **Lie bracket**, satisfying the Jacobi identity

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0 \ ,$$

for all $x, y, z \in \mathfrak{g}$.

We consider only finite-dimensional Lie algebras, meaning that $\mathfrak{g}$ is finite-dimensional when viewed as a vector space. Moreover we restrict ourselves to **simple Lie algebras**. These are non-abelian, i.e. $[\mathfrak{g}, \mathfrak{g}] \neq 0$, and contain no proper ideal, i.e. there is no subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ such that $[\mathfrak{k}, \mathfrak{g}] \subset \mathfrak{k}$. This particular class of Lie algebras is special in that each of its members is classified uniquely by its Dynkin diagrams.

Below are some examples of simple Lie algebras that can be realised in terms of matrix algebras.

$$\begin{aligned} A_n &= \mathfrak{su}(n+1) \ , \\ B_n &= \mathfrak{so}(2n+1) \ , \\ C_n &= \mathfrak{sp}(2n) \ , \\ D_n &= \mathfrak{so}(2n) \ . \end{aligned}$$

Here A_n denotes the set of anti-hermitian complex $(n+1) \times (n+1)$ matrices of trace zero. Elements of B_n and D_n are anti-symmetric complex matrices of trace zero. Elements of C_n are $2n \times 2n$ matrices of form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where B and C are symmetric and $D = -A^T$.

A Lie algebra can be specified by a set of generators $\{I^a\}$ and their commutation relations $[I^a, I^b] = \sum_c f^{abc} I^c$, where f^{abc} denote the structure constants of $\mathfrak{g}$. There are dim($\mathfrak{g}$) generators in total.

In the **Cartan-Weyl basis** generators are constructed as follows:

- Choose a maximal commuting subspace $\mathfrak{h}$ of $\mathfrak{g}$ (called a **Cartan subalgebra**). Choose the first r generators $\{H^1, \ldots, H^r\}$ to be a basis of $\mathfrak{h}$. The dimension of $\mathfrak{h}$, denoted r, is called the **rank** of $\mathfrak{g}$.
- One can simultaneously diagonalise the actions $x \to [H^i, x]$ of the H^i on x. The remaining generators are chosen to be eigenvectors X_β of these maps, so that $[H^i, X^\beta] = \beta^i X^\beta$.

A **root** $\beta = (\beta^1, \ldots, \beta^r)$ is an element of $\mathfrak{h}^\star$ which is a non-zero eigenvector of the action of X^β on $\mathfrak{h}$. The set of all roots is denoted by Δ. Δ spans $\mathfrak{h}^\star$ but is not a linearly independent set, therefore it is natural to choose a subset Π of Δ that forms a basis of $\mathfrak{h}^\star$. There exists a subset $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ of Δ, called the set of **simple roots**, such that Π is a linearly independent set and each root β can be written as an integral linear combination of simple roots

$$\beta = \sum_{i=1}^{r} n_i \alpha_i \ ,$$

where $n_i \in \mathbb{Z}$, and either all $n_i \geq 0$ or all $n_i \leq 0$. A root β is called a **positive root** if all $n_i \geq 0$ and a **negative root** if all $n_i \leq 0$. The sets of positive and negative roots are denoted by Δ^+ and Δ^- respectively. Clearly $\Delta^- = -\Delta^+$ and $\Delta = \Delta^+ \cup \Delta^-$.

The **length** of a root β is defined by

$$l(\beta) = \sum_{i=1}^{r} n_i \ .$$

The length functional $l : \Delta \to \mathbb{Z}$ allows a partial ordering, $<$, to be defined on the set of roots as follows. For $\alpha, \beta \in \Delta$ we write

$$\alpha > \beta \Leftrightarrow l(\alpha) > l(\beta) \ . \tag{2.1}$$

The adjoint representation and the Killing form

The most natural representation of a Lie algebra is the **adjoint representation**, in which $\mathfrak{g}$ is represented as an operator algebra acting on itself. Every element $x \in \mathfrak{g}$ can be viewed as an operator by means of the adjoint action, $x \mapsto \mathrm{ad}(x)$, which acts on $\mathfrak{g}$ as

$$\mathrm{ad}(x)(y) = [x, y] \ .$$

This leads in a natural way to the introduction of a symmetric bilinear form on $\mathfrak{g}$. This is the so-called **Killing form** $K : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$, defined by

$$(x, y) \mapsto K(x, y) = \frac{1}{2h} \mathrm{Tr}\left(\mathrm{ad}(x) \circ \mathrm{ad}(y)\right) , \tag{2.2}$$

for all $x, y \in \mathfrak{g}$. The factor h in the normalisation constant is the dual Coxeter number of $\mathfrak{g}$, see (2.3).

The Killing form establishes an isomorphism between $\mathfrak{h}$ and $\mathfrak{h}^\star$, and can be used to induce a positive definite scalar product on $\mathfrak{h}^\star$ through $(\alpha, \beta) = K(H^\alpha, H^\beta)$. Here the element $H^\alpha \in \mathfrak{h}$ is uniquely defined by the relation $\alpha(H) = K(H^\alpha, H)$. Since all roots of $\mathfrak{g}$ sit in $\mathfrak{h}^\star$, this defines a scalar product on the root space. From now on this is the scalar product we use between roots.

The partial ordering introduced in (2.1) identifies a unique **highest root** θ for which

$$l(\theta) > l(\alpha) ,$$

for all other roots α. The particular normalisation (2.2) of the Killing form was chosen to ensure that

$$(\theta, \theta) = 2 .$$

This follows the standard convention in which the square length of a long root is 2.

The highest root can be written as

$$\theta = \sum_{i=1}^{r} a_i \alpha_i = \sum_{i=1}^{r} a_i^\vee \alpha_i^\vee ,$$

where the a_i and $a_i^\vee$ are natural numbers, called the **Kac labels** and **dual Kac labels** respectively. Here $\alpha_i^\vee$ is the **simple coroot** corresponding to the simple root α_i. It is defined as

$$\alpha_i^\vee := \frac{2\alpha_i}{|\alpha_i^2|} .$$

The sums of the Kac labels and dual Kac labels define two important constants in the theory of simple Lie algebras. These are the **Coxeter number**

$$g = \sum_{i=1}^{r} a_i + 1 ,$$

and the **dual Coxeter number**

$$h = \sum_{i=1}^{r} a_i^\vee + 1 . \tag{2.3}$$

Cartan matrices and Dynkin diagrams

To each simple Lie algebra $\mathfrak{g}$ we can associate a unique **Cartan matrix** $C(\mathfrak{g})$. This is an positive definite $r \times r$ matrix, whose elements C_{ij} are defined in terms of the simple roots by

$$C_{ij} = (\alpha_i, \alpha_j^\vee) . \tag{2.4}$$

The Cartan matrix has the following properties:

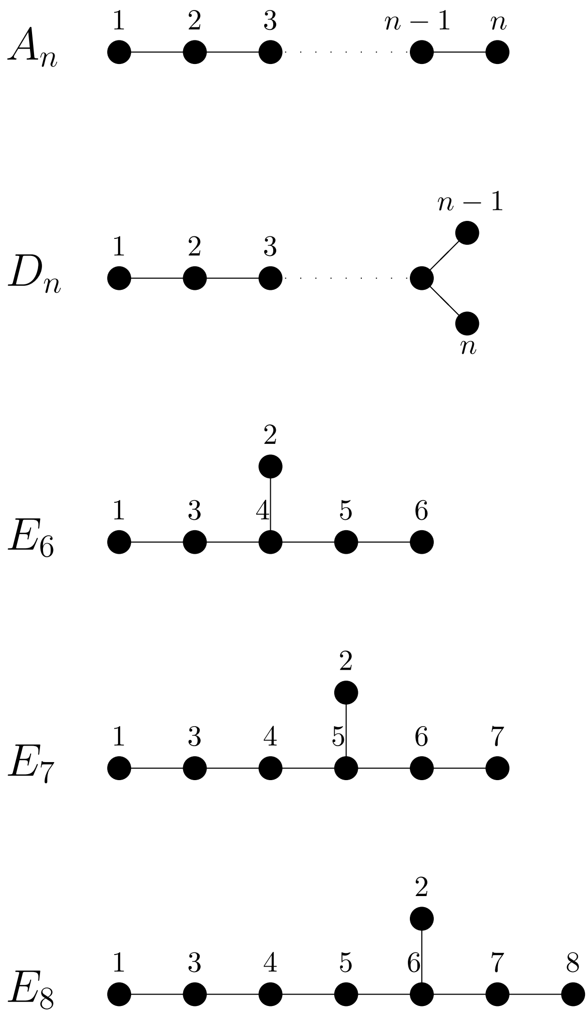
- $C_{ij} \in \mathbb{Z}$,
- The diagonal entries satisfy $C_{ii} = 2$,

- The off-diagonal entries C_{ij} can take values 0, -1, -2, or -3 ,
- $C_{ij} = 0 \Leftrightarrow C_{ji} = 0$,
- $\det(C) > 0$.

The **Dynkin diagram** of a simple Lie algebra is a connected planar diagram encoding the Cartan matrix in the following way. To every simple root α_i we associate a vertex, and vertices i and j are joined by $C_{ij}C_{ji}$ lines. Let θ_{ij} be the angle between the simple roots α_i and α_j. It can easily be shown that $C_{ij}C_{ji} = 4\cos^2\theta_{ij}$, and hence that any two vertices can be joined by either 0, 1, 2, or 3 links. Where there is only one link joining two vertices, the corresponding simple roots are both of the same length. Where there is more than one link joining two vertices, an arrow is drawn pointing from the longer to the shorter root. The Dynkin diagrams of simple Lie algebras are separated into two classes, those in which only single links occur (the ADE series), and those where multiple links are allowed (the BCFG series). For obvious reasons these are called **simply-laced** and **non simply-laced** respectively. Clearly in the former case all roots must have the same length, while in the latter they can be different. In the set of all roots of a given Lie algebra, at most two different lengths are possible.

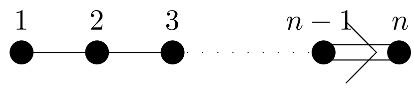
A simple Lie algebra uniquely determines its Dynkin diagram, and hence the classification of simple Lie algebras corresponds exactly to the classification of Dynkin diagrams. The Dynkin diagram contains all of the information necessary to reconstruct the entire root system of the algebra. For simple Lie algebras the set of all allowed Dynkin diagrams is as follows.

Simply-laced:

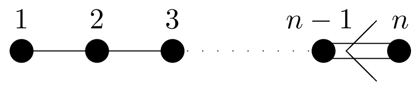


Non simply-laced:

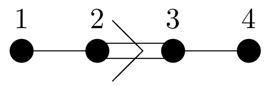
B_n



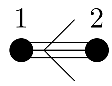
C_n



F_4



G_2



The algebras of types A, B, C, and D are known as **classical Lie algebras**, while E_6, E_7, E_8, F_4, and G_2 are the **exceptional Lie algebras**.

Weights and the Weyl vector

Let $\{\omega_1, \ldots, \omega_r\}$ be the basis dual to the basis of simple coroots, i.e.

$$(\omega_i, \alpha_j^\vee) = \delta_{ij} \ .$$

Then $\omega_1, \ldots, \omega_r$ are called the **fundamental weights** of $\mathfrak{g}$.

For an arbitrary representation of $\mathfrak{g}$, a basis $\{|\lambda\rangle\}$ can always be found such that

$$H^i|\lambda\rangle = \lambda^i|\lambda\rangle \ .$$

The eigenvalues λ^i form a vector $\lambda = (\lambda^1, \ldots, \lambda^r)$ called a **weight**. As in the case of roots, weights live in the space $\mathfrak{h}^\star$; $\lambda(H^i) = \lambda^i$. Therefore, the scalar product between weights is also fixed by the Killing form. That roots and weights occupy the same space makes perfect sense, since roots are just a special name given to the weights of the adjoint representation.

Every weight can be written as an integral linear combination of the fundamental weights

$$\lambda = \sum_{i=1}^{r} \lambda_i \omega_i \ ,$$

where $\lambda_i \in \mathbb{Z}$. The expansion coefficients λ_i of a weight λ in the fundamental weight basis are called **Dynkin labels**. From now on when a weight is written in component form

$$\lambda = (\lambda_1, \ldots, \lambda_r) \ ,$$

it is assumed that the components are the Dynkin labels.

One weight of particular importance is the **Weyl vector**, ρ. This is written in terms of fundamental weights as

$$\rho = \sum_{i=1}^{r} \omega_i = (1, 1, \ldots, 1) \ .$$

Alternatively it can be expressed in terms of positive roots as

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha \ .$$

The Weyl vector is uniquely determined by the property that $(\rho, \alpha_i^\vee) = 1$. This follows immediately from the definition of ρ in terms of fundamental weights.

Weyl reflections and the Weyl group

Given roots α and β of $\mathfrak{g}$, it can be shown that the quantity $\beta - (\alpha^\vee, \beta)\,\alpha$ is also a root. This leads to the introduction of a new operation $s_\alpha : \Delta \to \Delta$ defined by

$$s_\alpha \beta := \beta - (\alpha^\vee, \beta)\alpha \ .$$

Such a mapping is called a **Weyl reflection**. It is a reflection with respect to the hyperplane perpendicular to α. The set of all such reflections with respect to roots forms the **Weyl group** of $\mathfrak{g}$, denoted $W(\mathfrak{g})$. The r **simple Weyl reflections**

$$s_i \equiv s_{\alpha_i} \ ,$$

generate the whole of the Weyl group via composition. Since only a limited number of composite simple Weyl reflections can differ from the identity, $W(\mathfrak{g})$ is always a finite group. The action of the Weyl group on the simple roots yields the set of all roots of the algebra

$$\Delta = W(\mathfrak{g})\{\alpha_1, \ldots, \alpha_r\} \ .$$

The Weyl character formula

A character is a useful way of encoding all of the information about a representation. Suppose $\pi : \mathfrak{g} \to \mathrm{End}(V)$ is a representation of $\mathfrak{g}$. Then the **character** of π is the function

$$\begin{aligned} \chi : \mathfrak{g} &\rightarrow \mathbb{C} \\ x &\mapsto \mathrm{Tr}(\pi(x)) \ . \end{aligned}$$

The character is independent of the choice of basis for V.

The **Weyl character formula** allows us to calculate the character of an irreducible representation given its highest weight. The character of the irreducible representation of highest weight λ is given by

$$\chi_\lambda = \frac{\sum_{w \in W(\mathfrak{g})} \det(w) e^{w(\lambda+\rho)}}{\sum_{w \in W(\mathfrak{g})} \det(w) e^{w\rho}} \ .$$

The Weyl denominator can be written as

$$\sum_{w \in W(\mathfrak{g})} \det(w) e^{w\rho} = \prod_{\alpha \in \Delta^+} \left(e^{\alpha/2} - e^{-\alpha/2} \right) \ .$$

In many calculations it is more convenient to use the multiplicative form of this expression.

Chevalley generators

The Lie algebra $\mathfrak{g}$ can also be described as the Lie algebra with generators H_i, $X_i^{\pm}$ for $i = 1, \ldots, r$, and defining relations

$$\begin{aligned} [H_i, H_j] &= 0 \ , \\ \left[H_i, X_j^{\pm}\right] &= C_{ij} X_j^{\pm} \ , \\ \left[X_i^{+}, X_j^{-}\right] &= \delta_{ij} H_i \ , \\ \left(ad_{X_i^{\pm}}\right)^{1-C_{ij}} \left(X_j^{\pm}\right) &= 0 \ \text{ for i} \neq \text{j} \ . \end{aligned}$$

The $H_i,\ X_i^{\pm}$ are called the **Chevalley generators** of $\mathfrak{g}$.

Universal enveloping algebra

For a finite-dimensional complex Lie algebra $\mathfrak{g}$, its tensor algebra $T(\mathfrak{g})$ is given by

$$T(\mathfrak{g}) = \mathbb{C} \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}) \oplus \ldots \ .$$

Let I be the two-sided ideal of $T(\mathfrak{g})$ generated by all elements of the form

$$x \otimes y - y \otimes x - [x, y] \ ,$$

where $x,\ y \in \mathfrak{g}$. Then the universal enveloping algebra of $\mathfrak{g}$, denoted $U\mathfrak{g}$, is defined to be the quotient

$$U\mathfrak{g} = T(\mathfrak{g})/I \ .$$

Affine Kac-Moody algebras

To every finite-dimensional Lie algebra $\mathfrak{g}$, we can associate an affine extension $\hat{\mathfrak{g}}$ by adding an extra node to the Dynkin diagram of $\mathfrak{g}$. For example if $\mathfrak{g} = E_8$ the new node is added to the first node on the left hand side of the original E_8 Dynkin diagram. The resulting algebra is called an **affine Kac-Moody algebra**. Affine Kac-Moody algebras have important applications in conformal field theory, in particular in the study of WZW models.

The fundamental concepts of roots, weights, Cartan matrices and the Weyl group extend easily from the finite to the affine case. However the addition of the extra simple root in the affine case results in both the root system and the Weyl group of $\hat{\mathfrak{g}}$ becoming infinite. Consequently all highest-weight representations are infinite-dimensional. For simplicity these representations are arranged in terms of a new parameter, k, called the **level**. The level of a weight (now described by $r + 1$ Dynkin labels) is just the sum of all of its Dynkin labels, each multiplied by the corresponding comark.

Affine Kac-Moody algebras form a large subclass of the more general Kac-Moody algebras. In particular they consist of those Kac-Moody algebras whose Cartan matrix is positive semi-definite. For a good overview of affine Kac-Moody algebras see e.g. [18, 20].

2.1.2 Hopf algebras and Yangians

Let A be an associative unitary algebra over a commutative ring k, and let $\mathbf{1} \in A$ be the unit element of A. Let $\mathrm{id} : A \to A$ denote the identity transformation, given by $\mathrm{id}(a) = a$ for all $a \in A$. A is a **Hopf algebra** if it contains linear operators

- multiplication $m : A \otimes A \to A$, given by $m(a \otimes b) = ab$ for all $a, b \in A$, and satisfying $m(m \otimes \mathrm{id}) = m(\mathrm{id} \otimes m)$,
- co-multiplication $\Delta : A \to A \otimes A$,
- antipode $s : A \to A$,
- unit $\eta : k \to A$, given by $\eta(c) = c\mathbf{1}$ for all $c \in k$,
- co-unit $\epsilon : A \to k$,

subject to the following axioms for all $a, b \in A$

- Associativity of Δ : $\quad (\mathrm{id} \otimes \Delta)\Delta(a) = (\Delta \otimes \mathrm{id})\Delta(a)$,
- Definition of s : $\quad m(\mathrm{id} \otimes s)\Delta(a) = m(s \otimes \mathrm{id})\Delta(a) = \eta\epsilon(a)$,
- Definition of ϵ : $\quad (\epsilon \otimes \mathrm{id})\Delta(a) = (\mathrm{id} \otimes \epsilon)\Delta(a) = a$,
  (Note that we identify the spaces k, $k \otimes A$, and $A \otimes k$, which are naturally isomorphic) .

Δ and ϵ are homomorphisms

$$\begin{aligned}\Delta(ab) &= \Delta(a)\Delta(b) \ , \\ \epsilon(ab) &= \epsilon(a)\epsilon(b) \ ,\end{aligned}$$

and s is an anti-homomorphism of A

$$s(ab) = s(b)s(a) \ .$$

The map s^2 is an automorphism of A, and if A is commutative or co-commutative it can be shown that $s^2 = \mathrm{id}$.

For example, given a Lie algebra $\mathfrak{g}$, its universal enveloping algebra $U\mathfrak{g}$ is a Hopf algebra. Take m to be the usual formal multiplication on $U\mathfrak{g}$, and define a co-multiplication Δ, a co-unit ϵ, and an antipode s for all $x_1, \ldots, x_r \in \mathfrak{g}$ as follows

$$\begin{aligned}\epsilon(x_1 \ldots x_r) &= \delta_{r\,0} 1 \ , \\ s(x_1 \ldots x_r) &= (-1)^r x_r \ldots x_1 \ , \\ \Delta(x_1 \ldots x_r) &= \prod_{i=1}^{r} (x_i \otimes 1 + 1 \otimes x_i) \\ &= \sum_{s=0}^{r} \sum_{\mathbf{i},\mathbf{j}} x_{i_1} \ldots x_{i_s} \otimes x_{j_1} \ldots x_{j_{r-s}} \ ,\end{aligned}$$

where the second summation is over $i_1 < \ldots < i_s$, $j_1 < \ldots < j_{r-s}$, and $\{i_1, \ldots, i_s\} \cup \{j_1, \ldots, j_{r-s}\} = \{1, \ldots, r\}$.

Yangians

Throughout this discussion $\mathfrak{g}$ will denote a finite-dimensional complex simple Lie algebra of dimension d, whose generators $I^1, \dots, I^d$ are orthonormal with respect to the Killing form, and satisfy the commutation relations $[I^a, I^b] = f^{abc} I^c$. To every such $\mathfrak{g}$ we can associate an (infinite-dimensional) Hopf algebra, $Y(\mathfrak{g})$, called a Yangian. One has $U\mathfrak{g} \subset Y(\mathfrak{g})$.

Let $Y(\mathfrak{g})$ be the algebra generated by $\{I^a, J^a\}$, with the additional constraints that

$$\begin{aligned}
\left[I^a, J^b\right] &= f^{abc} J^c \,, \\
\left[J^a, \left[J^b, I^c\right]\right] - \left[I^a, \left[J^b, J^c\right]\right] &= a_{abcdeg} \left\{I^d, I^e, I^g\right\} \,, \\
\left[\left[J^a, J^b\right], \left[I^l, J^m\right]\right] + \left[\left[J^l, J^m\right], \left[I^a, J^b\right]\right] &= \left(a_{abcdeg} f^{lmc} + a_{lmcdeg} f^{abc}\right) \left\{I^d, I^e, I^g\right\} \,,
\end{aligned} \tag{2.5}$$

where

$$a_{abcdeg} = \frac{1}{24} f^{adi} f^{bej} f^{cgk} f^{ijk} \,,$$

and

$$\{x_1, x_2, x_3\} = \sum_{\{i,\,j,\,k\} = \{1,\,2,\,3\}} x_i x_j x_k \,.$$

The final two constraints in (2.5) are chosen such that the coproduct $\Delta : Y(\mathfrak{g}) \to Y(\mathfrak{g}) \otimes Y(\mathfrak{g})$, given by

$$\Delta(J^a) = J^a \otimes 1 + 1 \otimes J^a + \frac{1}{2} f^{abc} I^c \otimes I^b \,,$$

becomes a homomorphism.

The first of equations (2.5) says that the J^a's form a basis of a representation isomorphic to the adjoint representation of $\mathfrak{g}$ (on a new vector space that has no Lie algebra properties).

To see that the Yangian is in fact a Hopf algebra, define a co-unit $\epsilon : Y(\mathfrak{g}) \to \mathbb{C}$ by

$$
\begin{aligned}
\epsilon(I^a) &= 0 \,, \\
\epsilon(J^a) &= 0 \,,
\end{aligned}
$$

and an antipode $s : Y(\mathfrak{g}) \to Y(\mathfrak{g})$ by

$$
\begin{aligned}
s(I^a) &= -I^a \,, \\
s(J^a) &= -J^a + \frac{f^{abc}}{2} I^c \otimes I^b \,.
\end{aligned}
$$

Yangians were introduced in this way by Drinfeld [21] as part of his work on solutions of the Yang-Baxter equation. He later gave a second description [22] of Yangians, in terms of generators and relations, which we now show.

Let $C = (C_{ij})$, $i, j = 1, \ldots, r$, denote the Cartan matrix of $\mathfrak{g}$, where r is the rank of $\mathfrak{g}$. Let $d_1, \ldots, d_r$ be a set of coprime positive integers such that the matrix $d_i C_{ij}$ is symmetric. These d_i are uniquely determined. Then the Yangian $Y(\mathfrak{g})$ is isomorphic to the associative algebra with generators $X^{\pm}_{ik}$ and H_{ik}, $i = 1, \ldots r$ and $k \in \mathbb{N}$, and defining relations

- $[H_{ik}, H_{jl}] = 0$,

- $[H_{i0}, X^{\pm}_{jl}] = \pm d_i C_{ij} X^{\pm}_{jl}$,
- $[X^{+}_{ik}, X^{-}_{jl}] = \delta_{ij} H_{i,k+l}$,
- $[H_{i,k+1}, X^{\pm}_{jl}] - [H_{ik}, X^{\pm}_{j,l+1}] = \pm\frac{1}{2} d_i C_{ij} \left(H_{ik} X^{\pm}_{jl} + X^{\pm}_{jl} H_{ik}\right)$,
- $[X^{\pm}_{i,k+1}, X^{\pm}_{jl}] - [X^{\pm}_{ik}, X^{\pm}_{j,l+1}] = \pm\frac{1}{2} d_i C_{ij} \left(X^{\pm}_{ik} X^{\pm}_{jl} + X^{\pm}_{jl} X^{\pm}_{ik}\right)$,
- $i \neq j$ and $n = 1 - C_{ij} \Rightarrow \mathrm{Sym}\left[X^{\pm}_{i,k_1}\left[X^{\pm}_{i,k_2} \ldots \left[X^{\pm}_{i,k_n}, X^{\pm}_{jl}\right] \ldots\right]\right] = 0$,

where Sym is the sum over all permutations of $k_1, \ldots, k_n$. There are similarities between this realisation of the Yangian, and the Chevalley description of a Lie algebra $\mathfrak{g}$.

The isomorphism ϕ between the two different realisations of $Y(\mathfrak{g})$ is given by

$$\begin{aligned} \phi(H_i) &= d_i^{-1} H_{i0} , \\ \phi(J(H_i)) &= d_i^{-1} H_{i1} + \phi(v_i) , \\ \phi(X^{\pm}_i) &= X^{\pm}_{i0} , \\ \phi(J(X^{\pm}_i)) &= X^{\pm}_{i1} + \phi(w^{\pm}_i) , \end{aligned}$$

where

$$\begin{aligned} v_i &= \frac{1}{4} \sum_{\beta \in \Delta^+} \frac{d_\beta}{d_i} K(\beta,\, \alpha_i) \left(X^{+}_\beta X^{-}_\beta + X^{-}_\beta X^{+}_\beta\right) - \frac{d_i}{2} (H_i)^2 , \\ w^{\pm}_i &= \pm\frac{1}{4} \sum_{\beta \in \Delta^+} d_\beta \left(\left[X^{\pm}_i,\, X^{\pm}_\beta\right] X^{\mp}_\beta + X^{\mp}_\beta \left[X^{\pm}_i,\, X^{\pm}_\beta\right] - \frac{1}{4} d_i \left(X^{\pm}_i H_i + H_i X^{\pm}_i\right) \right) . \end{aligned}$$

Here H_i and $X^{\pm}_i$ are Lie algebra generators (i.e. I^a's), and $J(H_i)$ and $J(X^{\pm}_i)$ are the corresponding Yangian generators $J^a = J(I^a)$.

While the coalgebra structure of $Y(\mathfrak{g})$ in the second realisation can, in principle, be determined by the isomorphism ϕ and the first Yangian definition, no explicit formula for the action of the co-multiplication on the generators $X_{ik}^{\pm},\ H_{ik}$ is known.

For a more complete introduction to Yangians see [23, 24].

2.1.3 The dilogarithm and related functions

The **dilogarithm** is the function defined by the power series

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}\ , \qquad\qquad z \in \mathbb{C}\ ,\ \ |z| < 1\ .$$

It has an analytic continuation to $z \in \mathbb{C} - (1, \infty)$, given by

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-u)}{u}\, du\ .$$

There are only eight special values for which this function can be computed exactly. These are

$$0,\ \pm 1,\ \frac{1}{2},\ \frac{3-\sqrt{5}}{2},\ \frac{-1+\sqrt{5}}{2},\ \frac{1-\sqrt{5}}{2},\ \frac{-1-\sqrt{5}}{2}\ .$$

Nevertheless it satisfies many functional equations, for example

$$\begin{aligned}
\mathrm{Li}_2\left(\frac{1}{z}\right) &= -\mathrm{Li}_2(z) - \frac{\pi^2}{6} - \frac{1}{2}\log^2(-z)\ , \\
\mathrm{Li}_2(1-z) &= -\mathrm{Li}_2(z) + \frac{\pi^2}{6} - \log(z)\log(1-z)\ .
\end{aligned}$$

A more detailed discussion of the dilogarithm is found in [25, 26].

The **Bloch-Wigner function** is closely related to the dilogarithm, and is defined for all $z \in \mathbb{C}$ by

$$D(z) = \mathrm{Im}\left(\mathrm{Li}_2(z)\right) + \arg(1-z)\log|z| \ .$$

$D(z)$ is a continuous function, and is real analytic on $\mathbb{C}/\{0,1\}$. All of the functional equations satisfied by $\mathrm{Li}_2(z)$ lose the elementary correction terms when expressed in terms of $D(z)$. This function is particularly useful when studying torsion in the Bloch group.

The **Rogers dilogarithm** is defined by

$$L(x) = -\frac{1}{2}\int_0^x \left[\frac{\log|1-u|}{u} + \frac{\log|u|}{1-u}\right] du \ , \quad x \in \mathbb{R} \ .$$

On the interval $(0,1)$ it is related to the ordinary dilogarithm Li_2 by

$$L(x) = \mathrm{Li}_2(x) + \frac{1}{2}\log(x)\log(1-x) \ ,$$

and outside this interval one can set $L(0) = 0$, $L(1) = \pi^2/6$, and

$$L(x) = \begin{cases} 2L(1) - L(\frac{1}{x}) & \text{if } x > 1 \ , \\ -L\left(\frac{x}{x-1}\right) & \text{if } x < 0 \ . \end{cases} \tag{2.6}$$

This function has many intriguing properties. It appears in various branches of mathematics, including number theory, algebraic K-theory, and the geometry of hyperbolic 3-manifolds.

As well as the functional equations

$$L(x) + L(1-x) = L(1)\ , \quad \text{for all } x \in \mathbb{R}\ , \tag{2.7}$$

and

$$L(x) + L\left(\frac{1}{x}\right) = \begin{cases} 2L(1) & \text{if } x > 0\ , \\ -L(1) & \text{if } x < 0\ , \end{cases} \tag{2.8}$$

$L(x)$ satisfies the 5-term relation

$$L(x) + L(y) + L\left(\frac{1-x}{1-xy}\right) + L(1-xy) + L\left(\frac{1-y}{1-xy}\right)$$

$$= \begin{cases} -3L(1) & \text{if } x,\ y < 0, xy > 1\ , \\ +3L(1) & \text{otherwise}\ . \end{cases} \tag{2.9}$$

(Notice that this equation is cyclically symmetric in its five arguments, and that the right hand side is $-3L(1)$ when all five arguments are negative). The function $L(x)$ is not continuous at infinity, but is continuous if we consider it modulo $\pi^2/2$.

Define a Riemann surface $\hat{\mathbb{C}}$ by

$$\hat{\mathbb{C}} = \left\{(u,v) \in \mathbb{C}^2 \mid e^u + e^v = 1\right\} \cup (\infty, 0) \cup (0, \infty)\ .$$

The Rogers dilogarithm can be extended to a holomorphic function

$$\hat{L} : \hat{\mathbb{C}} \to \mathbb{C}/(2\pi i)^2\mathbb{Z}\ , \tag{2.10}$$

given by $\hat{L}(u,v) = F(v) + \frac{uv}{2}$, where

$$F(v) = \mathrm{Li}_2(1 - e^v) \in \mathbb{C}/(2\pi i)^2\mathbb{Z} ,$$

and $v \in \mathbb{C} - 2\pi i\mathbb{Z}$. F is well-defined modulo $(2\pi i)^2\mathbb{Z}$ since $\mathrm{Li}_2(1 - e^v) = \int_0^v \frac{t\,dt}{e^t - 1}$, where the integrand $\frac{t}{e^t-1}$ has a pole at every $t \in 2\pi i n$, with residue $2\pi i n$.

The function $\hat{L}$ has the properties

$$\begin{aligned} \hat{L}(u + 2\pi i, v) &= \hat{L}(u,v) + \pi i v , \\ \hat{L}(u, v + 2\pi i) &= \hat{L}(u,v) - \pi i u , \end{aligned}$$

of which the first characterises $\hat{L}$ up to some additive constant. Here $(u,v) \in \hat{\mathbb{C}}$. There is an embedding $(0,1) \to \hat{\mathbb{C}}$ given by

$$x \mapsto (\log(x), \log(1-x)) = (u,v) .$$

Clearly $\hat{L}(u,v) = L(x)$ for $x \in (0,1)$.

2.1.4 The Bloch group and related structures

Let $\mathbb{F}$ be a field. Consider the free abelian group with basis $[z],\ z \in \mathbb{F}^\star$. Let $\mathcal{A} = \mathcal{A}(\mathbb{F})$ be the subgroup of elements $\sum_{i=1}^n n_i[z_i]$, ($z_i \in \mathbb{F}^\star,\ n_i \in \mathbb{Z}$), satisfying

$$\sum_{i=1}^n n_i\ (z_i) \wedge (1 - z_i) = 0 ,$$

where $1 \wedge 0$ is to be interpreted as 0.

Here the sum is taken in the abelian group $\Lambda^2\mathbb{F}^\star$ (the set of all formal linear combinations of symbols $x \wedge y$, for $x, y \in \mathbb{F}^\star$, subject to the relations $x \wedge x = 0$ (and hence $x \wedge y = -y \wedge x$) and $(x_1 x_2) \wedge y = x_1 \wedge y + x_2 \wedge y$).

For example $6.[2/3] - [8/9] \in \mathcal{A}(\mathbb{Q})$ since

$$\begin{aligned}
& 6.\,(2/3) \wedge (1 - 2/3) - (8/9) \wedge (1 - 8/9) \\
= \; & 6.\,(2/3) \wedge (1/3) - (8/9) \wedge (1/9) \\
= \; & 6.(2) \wedge (1/3) - 8 \wedge (1/9) \\
= \; & -6.(2) \wedge (3) + (8) \wedge (9) \\
= \; & -6.(2) \wedge (3) + 6.(2) \wedge (3) \\
= \; & 0\,.
\end{aligned}$$

For all $x, y \in \mathbb{F}^\star - \{1\}$ with $xy \neq 1$, $\mathcal{A}$ contains the elements

$$2\left([x] + \left[\frac{1}{x}\right] + [1]\right) \;,$$

$$[x] + [1 - x] - [1] \;,$$

$$[x] + [y] + \left[\frac{1-x}{1-xy}\right] + [1 - xy] + \left[\frac{1-y}{1-xy}\right] \;.$$

Let $\mathcal{C} = \mathcal{C}(\mathbb{F})$ be the subgroup of $\mathcal{A}$ generated by all such elements. Then the **Bloch group** of $\mathbb{F}$ is defined as

$$\mathcal{B}(\mathbb{F}) = \mathcal{A}/\mathcal{C}\;.$$

The Bloch group is closely related to the Rogers dilogarithm described in the previous section. Namely the equations (2.7), (2.8), and (2.9) imply that $L : \mathbb{R} \to \mathbb{R}$ can be extended to a function

$$L : \mathcal{B}(\mathbb{R}) \to \frac{\mathbb{R}}{3L(1)\mathbb{Z}} = \mathbb{R} \mod \frac{\pi^2}{2}\mathbb{Z}\,,$$

by setting $L\left(\sum n_i[x_i]\right) = \sum n_i L(x_i) \mod L(1)\mathbb{Z}$.

The **torsion subgroup** of the Bloch group is the subgroup consisting of all elements of finite order. An element $z \in \mathcal{B}[\bar{\mathbb{Q}}]$ is torsion if and only if its Bloch-Wigner dilogarithm, $D(z)$, is zero in all complex embeddings (in which case its Rogers dilogarithm, $L(z)$, is a rational multiple of π^2 in all real embeddings.)

For example, consider the inverse golden ratio $\alpha = \frac{\sqrt{5}-1}{2}$. Setting $x = y = \alpha$ in the 5-term relation $[x] + [y] + \left[\frac{1-x}{1-xy}\right] + [1-xy] + \left[\frac{1-y}{1-xy}\right]$, it follows that the element $[\alpha] \in \mathbb{Z}\left[\mathbb{Q}\left(\sqrt{5}\right)\right]$ is killed by 5 in the Bloch group. That the corresponding value of the Rogers dilogarithm is $L(\alpha) = \frac{\pi^2}{10} \notin \frac{\pi^2}{2}\mathbb{Z}$ proves that α is indeed 5-torsion and not trivial in the Bloch group of $\mathbb{Q}(\sqrt{5})$ (or even in the Bloch group of $\mathbb{R}$).

It is interesting to note that the same element α becomes zero in $\mathcal{B}(\mathbb{C})$, and similarly in $\mathcal{B}(\mathbb{F})$ for the field $\mathbb{F} = \mathbb{Q}(\zeta)$, where ζ is a 5^{th} root of unity. In both of these cases $[\alpha]$ itself can be written as a 5-term relation.

One weakness of the Bloch group for $F = \mathbb{C}$, is that it does not take into account the multi-valued nature of the dilog function. To deal with this problem we introduce an extension $\hat{\mathcal{B}}(\mathbb{C})$ of $\mathcal{B}(\mathbb{C})$, called the **extended Bloch group**.

Let $[\hat{\mathbb{C}}]$ be the free abelian group with basis $[(u,v)]$ for $(u,v) \in \hat{\mathbb{C}}$. The extended Bloch group $\hat{\mathcal{B}}(\mathbb{C})$ can be introduced as an extension of $[\hat{\mathbb{C}}]$ as follows. There is a natural linear map $\sigma : [\hat{\mathbb{C}}] \to \Lambda^2\mathbb{C}$ induced by $\sigma(u,v) = u \wedge v$, with $\sigma(0,\infty) = \sigma(\infty,0) = 0$. Let $\mathcal{P}$ be the kernel of this map. Define

$$\hat{\mathcal{B}}(\mathbb{C}) = \mathcal{P}/\mathcal{P}_0 \ ,$$

where $\mathcal{P}_0$ is the subgroup of $\mathcal{P}$ generated by all elements of the form

$$\begin{aligned}
&(u,v) + (v,u) - (0,\infty) \ , \\
&(u - 2\pi i, v) + 2(u - v - \pi i, -v) + (u,v) \ , \\
&\sum_{i=1}^{5} (u_i, v_i) - 2(0,\infty) \ ,
\end{aligned}$$

where $u_i = v_{i-1} + v_{i+1}$ for $i = 1, \ldots, 5$, and $v_0 = v_5$, $v_1 = v_6$ for cyclic symmetry.

There is a map $\hat{\mathcal{B}}(\mathbb{C}) \to \mathcal{B}(\mathbb{C})$ given by $\sum_i n_i(u_i, v_i) \mapsto \sum_i n_i e^{u_i}$. For a more detailed explanation of the extended Bloch group see [17, 27].

One significant difference between the extended Bloch group and the ordinary Bloch group of the complex numbers is that the torsion subgroup of the extended Bloch group is non-trivial, while that of the ordinary Bloch group is trivial.

The torsion subgroup of $\hat{\mathcal{B}}(\mathbb{C})$ plays an important role in quantum field theory. On elements of this subgroup, the map $(2\pi i)^2 L$ takes values in $\mathbb{Q}/\mathbb{Z}$. These values yield the conformal dimensions (more precisely the exponents $h_i - c/24$) of the fields of the theory.

2.2 Introduction to Conformal Field Theory

Here we give a short account of some important aspects of conformal field theory. The emphasis is on the ideas relevant to subsequent discussions. For a more detailed introduction to the subject see e.g. [18, 28, 29, 30].

A conformal field theory is a special type of quantum field theory. Many quantum field theories are attained from quantisation of classical field theories. Their key ingredients are a fixed spacetime and an action, $S[\Phi]$, defined on a set of fields Φ_i, $i \in \Delta$, which are real or complex-valued functions on the spacetime. When renormalisation problems are handled properly one can calculate the vacuum expectation values of the corresponding quantum field theory by

$$\langle \Phi_1(x_1) \dots \Phi_M(x_M) \rangle = \mathcal{N}^{-1} \int \prod [\mathcal{D}\Phi_\Delta] \Phi_1(x_1) \dots \Phi_M(x_M) e^{-S[\Phi_\Delta]} \ ,$$

where

$$\mathcal{N} = \int \prod [\mathcal{D}\Phi_\Delta] e^{-S[\Phi_\Delta]} \ ,$$

is the normalisation factor. Such a vacuum expectation value is called a **correlation function**. A theory is considered to be solved once all of its correlation functions have been calculated. The vacuum expectation values have certain properties, in particular invariances and operator product expansions, which can be used for an axiomatic description. There is no general method of doing this, however in certain QFTs the existence of symmetries places sufficient constraints on the correlation functions to allow them to be calculated exactly. This approach is especially likely to be successful in the case of conformal field theories, those particular QFTs that are invariant under conformal transformations.

2.2.1 Conformal invariance

A **conformal transformation** is a restricted general coordinate transformation $\mathbf{x} \rightarrow \tilde{\mathbf{x}}$, for which the metric $g_{\mu\nu}$ is invariant up to a scale factor

$$g_{\mu\nu}(\mathbf{x}) \rightarrow \tilde{g}_{\mu\nu}(\tilde{\mathbf{x}}) = \Lambda(\mathbf{x}) g_{\mu\nu}(\mathbf{x}) \; ; \quad \Lambda(x) \equiv e^{\omega(x)} \; , \tag{2.11}$$

where ω is some function of x. The set of all conformal transformations forms the **conformal group**. In two dimensions the metric $g_{\mu\nu}$ is given by $dt^2 + dx^2$ on the torus, or by the standard metric on S^2.

Two-dimensional conformal field theories have an infinite number of conserved quantities (corresponding to local conformal symmetry), and are therefore completely solvable by symmetry considerations alone. We now take a closer look at this important special case.

Conformal group in two dimensions

In two dimensions there is an infinite number of coordinate transformations that, although not everywhere well-defined, are locally conformal. These are the analytic maps from the complex plane to itself. This set is known as the **local conformal group**, although strictly speaking it is not a group since the mappings are not necessarily one-to-one and do not map the Riemann sphere to itself. Hence the need to distinguish these local transformations from global conformal transformations, which are well-defined everywhere.

The set of all analytic maps contains the six-parameter **global conformal group**, which is the subset of mappings that are invertible and defined everywhere. The group formed by these global transformations is often called the **special conformal group**. It can be shown that the functions $f(z) = (az + b)/(cz + d)$, satisfying $ad - bc = 1$, are the only globally defined invertible analytic maps. Hence we can write the special conformal group as the set

$$\left\{ f(z) = \frac{az + b}{cz + d}; \quad ad - bc = 1, \quad a, b, c, d \in \mathbb{C} \right\} .$$

It is easy to see that this group can be parametrised by the set of complex 2×2 matrices with unit determinant, modulo the negative unit matrix, i.e. $SL(2, \mathbb{C})/\mathbb{Z}_2$.

The distinction between local and global conformal groups is unique to the two-dimensional case. In higher dimensions all local conformal transformations are global. As already stated, local properties of conformal invariance are of more immediate interest than global ones, since it is the infinite-dimensionality of the local conformal group that allows so much to be known about conformally invariant field theories in two dimensions. With this in mind we now proceed to find the algebra of generators of the local conformal group. This is the **Witt algebra**, also known as the **conformal algebra**.

Conformal generators and the Witt algebra

The infinitesimal analytic functions parametrising the conformal transformations, can be defined by restricting the plane to a finite region around the origin and assuming that all singularities of the analytic functions are outside the region chosen.

A suitable basis for the infinitesimal coordinate transformations

$$z \rightarrow \tilde{z} = z + \epsilon(z), \quad \bar{z} \rightarrow \tilde{\bar{z}} + \bar{\epsilon}(\bar{z}) \ ,$$

is generated by the operators

$$l_n = -z^{n+1}\frac{d}{dz},\ \bar{l}_n = -\bar{z}^{n+1}\frac{d}{d\bar{z}}, \quad n \in \mathbb{Z} \ .$$

The holomorphic generators $\{l_n\}$ form a Lie algebra with commutation relations

$$[l_n, l_m] = (n-m)\, l_{n+m} \ .$$

This is the so-called **Witt algebra**. The corresponding anti-holomorphic generators form an isomorphic Lie algebra with commutation relations

$$\left[\bar{l}_n, \bar{l}_m\right] = (n-m)\, \bar{l}_{n+m} \ .$$

These generators also satisfy the relation

$$\left[l_n, \bar{l}_m\right] = 0 \ .$$

Each of these infinite-dimensional algebras contains a finite subalgebra generated by $\{l_{-1},\, l_0,\, l_1\}$. This is the subalgebra associated with the global conformal group. In the quantum case the Witt algebra will be corrected to include an extra term proportional to the central charge (the so-called **central extension**). The unique central extension will be given by the **Virasoro algebra**.

Primary and quasi-primary fields

Representations of the global conformal algebra (after quantisation) assign quantum numbers to physical states. We can assume the existence of the **vacuum state** $|\,0\rangle$ among the physical states; it has vanishing quantum number and is invariant under the transformation $z \longrightarrow \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{C}$ and $ad - bc = 1$. The eigenvalues h and $\bar{h}$ of l_0 and $\bar{l}_0$ respectively, are called the **conformal weights** of a state.

Given a state with conformal dimensions h and $\bar{h}$, its scaling dimension, Δ, and planar spin, s, are defined by

$$\Delta = h + \bar{h} \quad \text{and} \quad s = h - \bar{h}\,.$$

Any field Φ that satisfies the transformation property

$$\Phi(\tilde{z}, \bar{\tilde{z}}) = \Phi(z, \bar{z}) \left(\frac{dz}{d\tilde{z}}\right)^{h} \left(\frac{d\bar{z}}{d\bar{\tilde{z}}}\right)^{\bar{h}}\,, \tag{2.12}$$

is called a **primary field**. The remaining CFT fields are called **secondary fields**. As we will see later, the importance of primary fields lies in their ability to generate the whole field content of a theory.

The operator product expansion should be constant under the equation (2.12). For global conformal transformations the correlation functions are conserved.

Operator product expansion, central charge, and conformal families

The **operator product expansion** (OPE) expresses a product of two operator-valued fields, at different points z and w, as an infinite sum of single fields. In two-dimensional CFTs it is a convergent expansion. Although written without brackets, it is understood that the OPE is meaningful only within correlation functions.

In general the OPE of a holomorphic field $A(z)$ with an arbitrary field $B(w)$ can be written as

$$A(z)B(w) = \sum_i C_i(z-w)\mathcal{O}_i(w) \ ,$$

where $\{\mathcal{O}_i\}$ is a complete set of local operators, and the C_i are (singular) numerical coefficients, and the OPE is understood to be meaningful only within an correlation function. Here we have

$$\langle A(z_1)B(z_2)C(z_3)\ldots\rangle = \sum_i f_i(z_1-z_2)\Phi_i(z_2)C(z_3)\ldots \ ,$$

with the functions f_i depending only on A, B, and Φ_i a basis of fields.

There exists a particular field, T, such that the expansion of T with a primary field (of conformal dimensions $h, \bar{h}$) is given by

$$T(z)\Phi(w,\bar{w}) = \frac{h}{(z-w)^2}\Phi(w,\bar{w}) + \frac{1}{z-w}\partial_w\Phi(w,\bar{w}) + \Phi^{(-2)}(w,\bar{w}) + (z-w)\Phi^{(-3)}(w,\bar{w}) + \dots ,$$

where $\dots$ represents an infinite set of regular terms depending on the new local fields, called the **descendant fields**, of the primary field Φ. T is called the **energy-momentum tensor**. The descendant fields are determined by

$$\Phi^{(-n)}(w,\bar{w}) = L_{-n}\Phi(w,\bar{w}) \equiv \oint_w \frac{dz}{2\pi i}(z-w)^{-n+1}T(z)\Phi(w,\bar{w}) .$$

The OPE of T with non-primary fields contains correction terms. For example, the general OPE of the energy-momentum tensor, T, with itself is

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \text{holomorphic terms} ,$$

where ∂_w denotes differentiation with respect to w. c is a constant that depends on the specific model under consideration. This constant is called the **central charge**. Physically the central charge describes how a specific system reacts to the introduction of a macroscopic length scale.

We have introduced the operators L_n, which appear in the formal expansion of the energy-momentum tensor $T(z)$ around a point w

$$T(z) = \sum_{n\in\mathbb{Z}} \frac{L_n}{(z-w)^{n+2}} .$$

In particular we have

$$\begin{aligned} L_0\Phi(z,\bar{z}) &= h\Phi(z,\bar{z})\ , \\ L_{-1}\Phi(z,\bar{z}) &= \partial_z\Phi(z,\bar{z})\ , \\ L_n\Phi(z,\bar{z}) &= 0, \quad n \geq 1\ , \end{aligned}$$

where $\Phi_{-n} = L_{-n}\Phi$ are the new descendant fields.

For each primary field Φ, there exists an infinite **conformal family** $[\Phi]$ of descendant fields. These are generated by the repeated use of the operators L_{-n}

$$[\Phi] := \{L_{-k_1}\dots L_{-k_n}\Phi :\ k_1 \geq k_2 \geq \dots \geq k_n > 0\}\ .$$

It can be shown that every conformal family defines a highest-weight representation of the Virasoro algebra.

Virasoro algebra

We have already seen that the classical generators of local conformal transformations obey the Witt algebra. We now show that the corresponding quantum generators obey a similar algebra with an added central extension term. This is the well-known **Virasoro algebra**.

We saw above that the energy-momentum tensor, T, has a Laurent expansion in terms of modes L_n. These modes are themselves operators, and their action on the operator $\Phi(w)$ can be written as

$$L_n\Phi(w) = \frac{1}{2\pi i}\oint_w dz\,(z-w)^{n+1}T(z)\Phi(w)\;.$$

The mode operators L_n, and their anti-holomorphic counterparts $\bar{L}_n$, are the quantum generators of the local conformal transformations on the Hilbert space. They obey the algebra

$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}\;,\\
\left[L_n, \bar{L}_m\right] &= 0\;,\\
\left[\bar{L}_n, \bar{L}_m\right] &= (n-m)\bar{L}_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}\;.
\end{aligned}$$

Each set of generators $\{L_n\}$ and $\{\bar{L}_n\}$ constitutes a copy of the so-called **Virasoro algebra**. It is worth noting that the central term is absent for the subalgebra $\{L_{-1}, L_0, L_1\}$ belonging to the global conformal group.

Proof of commutation relations:

$$\begin{aligned}
& [L_n, L_m]\\
=\;& \frac{1}{(2\pi i)^2}\oint_0 dw\,w^{m+1}\oint_w dz\,z^{n+1}\left\{\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \text{ reg.}\right\}\\
=\;& \frac{1}{2\pi i}\oint_0 dw\,w^{m+1}\left\{\frac{c}{12}(n+1)n(n-1)w^{n-2} + 2(n+1)w^nT(w) + w^{n+1}\partial T(w)\right\}\\
=\;& \frac{c}{12}n(n^2-1)\,\delta_{n+m,0} + 2(n+1)L_{m+n} - \frac{1}{2\pi i}\oint_0 dw(n+m+2)w^{n+m+1}T(w)\\
=\;& \frac{c}{12}n(n^2-1)\,\delta_{n+m,0} + (n-m)L_{m+n}\;.
\end{aligned}$$

Every conformal field theory determines a representation of the Virasoro algebra for some value of c. For $c = 0$ there is no central extension term, and the Virasoro algebra reduces to the classical Witt algebra.

Verma modules

To introduce the Hilbert space of states, we begin by defining the **vacuum state**, $|0\rangle$, of the theory by the condition

$$L_n|0\rangle = 0 \ , \quad \text{for all}\, n \geq 0 \ .$$

To each primary field $\Phi_{h,\bar{h}}$ we can associate a **highest-weight state** $|h, \bar{h}\rangle$ by

$$|\, h, \bar{h}\rangle = \lim_{z,\bar{z}\to 0}\Phi_{h,\bar{h}}(z, \bar{z})|\, 0\rangle \ .$$

It follows that

$$\begin{aligned} L_n|\, h, \bar{h}\rangle &= 0 \quad n > 0 \ , \\ L_0|\, h, \bar{h}\rangle &= h|\, h, \bar{h}\rangle \ , \\ \bar{L_0}|\, h, \bar{h}\rangle &= \bar{h}|\, h, \bar{h}\rangle \ . \end{aligned}$$

The states of the associated **Verma module** are created by acting on the primary state $|\, h, \bar{h}\rangle$ with arbitraty polynomials in

$$\left\{L_{-n},\, \bar{L}_{-m} :\ m, n \geq 1\right\} \ ,$$

and no relations between these states except those given by the Virasoro algebra. The Verma module is an infinite-dimensional representation of the Virasoro algebra, completely characterised by its central charge and the dimension of the highest-weight state. Physical representations arise from Verma modules by reducing them modulo maximal submodules. Their descendant states can be viewed as the result of the action of descendant field on the vacuum

$$L_{-n}|\,h\rangle = \Phi^{(-n)}(0)|\,0\rangle \ .$$

Note that the descendant state $L_{-k_1}\dots L_{-k_n}|\,h\rangle$ is itself an eigenvector of L_0 through $L_0 L_{-k_1}\dots L_{-k_n}|\,h\rangle = (h+l)L_{-k_1}\dots L_{-k_n}|\,h\rangle$. The integer $l = \sum_{i=1}^{n} k_i \ \ (k_i > 0)$ is called the **level** of the state.

The CFT vacuum is a trivial highest-weight state which defines the trivial module corresponding to the identity operator. The Virasoro operators L_n in a Verma module act like raising and lowering operators. Since $[L_0, L_{-n}] = nL_{-n}$, L_0 can be viewed as a grading operator measuring the conformal dimension of a state.

All of the above can equally be applied to the anti-holomorphic counterparts.

Null states and the Kac determinant

A descendant state $|\,v\rangle$ satisfying the equations

$$L_0|\,v\rangle = (h+N)|\,v\rangle, \quad L_n|\,v\rangle = 0 \quad \text{for } n > 0 \ ,$$

is called a **null state**. It is simultaneously a primary and a descendant state, and is also a highest-weight state. To get an irreducible representation of the Virasoro algebra we must eliminate all null states and their descendant states, and consider the reduced theory.

The scalar product of two states at level l is given by

$$\langle h\,|(L_{r_k}\ldots L_{r_1})(L_{-s_1}\ldots L_{-s_t})|\,h\rangle \equiv M^{(l)}_{\{r\}\{s\}}\ ,$$

where $\sum r_i = \sum s_i = l$. M is a block diagonal matrix, with blocks $M^{(l)}$ corresponding to states of level l. M is called the **Gram matrix**. The matrices M related to the lowest levels of a generic Verma module can easily be calculated. For example, the states of level 2 are $L_{-1}^2|h\rangle$ and $L_{-2}|h\rangle$. Therefore

$$\begin{aligned} M^{(2)}_{12} &= \langle h|L_1L_1L_{-2}|h\rangle \\ &= \langle h|L_1(L_{-2}L_1+3L_{-1})|h\rangle \\ &= 3\langle h|L_1L_{-1}|h\rangle \\ &= 6h\langle h|h\rangle\ . \end{aligned}$$

Similar calculations give the other three entries, resulting in the matrix

$$M^{(2)} = \begin{pmatrix} 4h(2h+1) & 6h \\ 6h & 4h+c/2 \end{pmatrix}\ .$$

The determinant of this matrix is known as the **Kac determinant**, and null states in the Verma module correspond to zeros of the Kac determinant. In the above example we have

$$\det M^{(2)} = 32h^3 + (4c - 20)h^2 + 2ch \ .$$

Writing

$$\det M = 32(h - h_{11})(h - h_{12})(h - h_{21}) \ ,$$

we can see that the roots of the Kac determinant are given by

$$\begin{aligned} h_{1,1} &= 0 \ , \\ h_{1,2} &= \frac{1}{16}\left(5 - c - \sqrt{(1-c)(25-c)}\right) \ , \\ h_{2,1} &= \frac{1}{16}\left(5 - c + \sqrt{(1-c)(25-c)}\right) \ . \end{aligned}$$

There exists a general formula for calculating the Kac determinant. It is given by

$$\det M^{(l)}(c,h) = \prod_{k=1}^{l} \prod_{rs=k} [h - h_{r,s}]^{p(l-k)} \ ,$$

where r and s are positive integers, and $p\,(l-k)$ denotes the number of partitions of the integer $l - k$.

There are many ways to express the roots of the Kac determinant. One way is to write

$$h_{r,s}(m) = \frac{[(m+1)r - ms]^2 - 1}{4m(m+1)} \ , \tag{2.13}$$

where

$$m = -\frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{25-c}{1-c}} \ .$$

2.2.2 Models in conformal field theory

Minimal models

CFTs that have a finite number of primary fields are called rational conformal field theories. The minimal models are particular rational CFTs of central charge $c < 1$. They are called minimal since they are based on a finite number of scalar primary fields, they have no multiplicities in their spectra of conformal dimensions, and they contain no additional symmetries except for conformal symmetry. These models are characterised by

$$\begin{aligned} c &= 1 - 6\frac{(p-p')^2}{pp'} , \\ h_{r,s} &= \frac{(pr-p's)^2-(p-p')^2}{4pp'} , \end{aligned}$$

where p and p' are positive integers having no non-trivial common divisors. Notice that setting $m = p/p'$ in (2.13), with $\gcd(p,p') = 1$, is equivalent to the above expressions for c and $h_{r,s}$. We can restrict values of r, s to the rectangle $0 < r < p'$, $0 < s < p$. This rectangle in the (r,s) plane is called the **Kac table**. The symmetry $h_{r,s} = h_{p'-r,p-s}$ makes half of this table redundant. Minimal models usually describe discrete statistical models at their critical points, and their simplicity allows in principle for a complete solution.

WZW models

A **Wess-Zumino-Witten model** is a simple model of conformal field theory whose solutions are realised by affine Kac-Moody algebras. It has holomorphic fields with $h = 1$, the so-called currents J^a. Using the J^a_m instead of L_m one can repeat most of the preceeding discussion. Given a Kac-Moody algebra $\hat{g}_k$, of level k, the central charge of the corresponding WZW model is given by

$$c(\hat{\mathfrak{g}}_k) = \frac{k\,\dim(\mathfrak{g})}{k + h(\mathfrak{g})} \,. \tag{2.14}$$

Every WZW model has central charge $c > 1$.

Coset models

We now introduce a third class of models by means of the **coset construction**. This greatly increases the number of known solvable models. A **coset model** is a quotient of two WZW models, with the central charge of the coset being the difference of the central charges of the two WZW components. It is expected that the coset construction will provide a framework for the complete classification of all rational conformal field theories. Coset models incorporate the two classes of models that we have already looked at

- WZW models are represented by trivial cosets,
- Models with $c < 1$ can be represented by the coset construction, since the central charge of a coset is the difference of the central charges of the two

WZW components. However all RCFTs with $c < 1$ are known to be minimal models. Hence any coset with $c < 1$ must provide a new representation of a minimal model.

Characters

Let $c, h \in \mathbb{Q}$ denote central charge and conformal dimension respectively. Let $V_{c,h}$ denote the Verma module, (before excluding the null submodules), generated by the Virasoro generators L_{-n} $(n > 0)$ acting on the highest-weight state $|\, h, \bar{h}\rangle$. To each such Verma module we can associate a generating function $\chi_{c,h}(\tau)$, called the character of the module. This is defined by

$$\begin{aligned} \chi_{c,h}(\tau) &= \mathrm{Tr}\, q^{L_0 - c/24} \qquad (q \equiv e^{2\pi i \tau}) \\ &= \sum_{n=0}^{\infty} \mathrm{d}(h+n)\, q^{n+h-c/24} \,. \end{aligned}$$

Here $\mathrm{d}(n+h)$ denotes the number of linearly independent states at level n in the module and τ is a complex variable. Characters can be viewed as generating functions for the number of states at any given level.

Now let $V_{r,s}$ denote the Verma module $V(c(p,p'), h_{r,s}(p,p'))$, built on the highest weight $h_{r,s}$ appearing in the Kac table. This reducible Verma module, with highest weight $h_{r,s}$, contains null states that must be eliminated in order to get the corresponding irreducible Verma module, denoted $M_{r,s}$.

Define a new function by

$$K^{(p,p')}_{r,s}(q) = \frac{q^{-1/24}}{\phi(q)} \sum_{n\in\mathbb{Z}} q^{(2pp'n+pr-p's)^2/4pp'} \ .$$

Then the character of the irreducible Verma module $M_{r,s}$ is given by

$$\chi_{r,s}(q) = K^{(p,p')}_{r,s}(q) - K^{(p,p')}_{r,-s}(q) \ .$$

2.2.3 Partition functions and modular invariance

We have so far assumed conformal field theories to be defined on the whole complex plane. Physically this is not a very realistic situation as the holomorphic and antiholomorphic parts of the theory decouple completely. To impose more realistic physical constraints on a conformal field theory, we look to couple the holomorphic and antiholomorphic sectors through the geometry of the space on which the theory is defined. For this purpose we consider conformal field theory on a torus. The interaction of the holomorphic and antiholomorphic sectors in this case is given by **modular transformations**.

Conformal field theory on the torus

Define a torus on the complex plane by specifying two linearly independent lattice vectors. These vectors are represented by complex numbers ω_1 and ω_2 respectively, called the **periods** of the lattice. Identify points that differ by an integer combination of these vectors. The properties of the conformal field theory defined on the torus do not depend on the overall scale of the lattice or on the absolute orientation of the lattice vectors. The relevant scale parameter is $\tau = \omega_2/\omega_1$, called the **modular parameter**.

The partition function

The **partition function** Z of a CFT can be expressed in terms of the Virasoro characters (of the Verma modules forming the Hilbert space of the theory) as

$$Z = \sum_{h,\bar{h}} n_{h,\bar{h}} \chi_h(\tau) \chi_{\bar{h}}(\bar{\tau}) \ ,$$

where h and $\bar{h}$ label a certain highest-weight state $|h, \bar{h}\rangle$, and $n_{h,\bar{h}}$ is the multiplicity of such a state. For rational conformal field theories this sum[1] is always finite.

Modular invariance

The partition function must be independent of the particular periods $\omega_{1,2}$ chosen for a given torus. This has the advantage of imposing certain constraints on the conformal field theory defined on the torus.

Suppose $\omega'_{1,2}$ are two periods describing the same lattice as $\omega_{1,2}$. Since they belong to the same lattice, the points ω'_1 and ω'_2 must be integer combinations of ω_1 and ω_2, and vice versa. Moreover, the unit cell of the lattice should have the same area no

[1]For simplicity we will usually write this sum as $Z = \sum_{i,j} n_{ij} \chi_i(\tau) \chi_j(\bar{\tau})$.

matter what periods are chosen. Writing

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} ,$$

the conditions above restrict the choice of entries to $a, b, c, d \in \mathbb{Z}$ with $ad - bc = 1$. This directs us to consider the group of matrices $SL(2, \mathbb{Z})$.

Under the change of period above, the modular parameter transforms as

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d} , \quad ad - bc = 1 .$$

Clearly the signs of all parameters a, b, c, d can be simultaneously changed without affecting the overall transformation. Hence the symmetry of interest here is the **modular group** $SL(2, \mathbb{Z})/\mathbb{Z}_2$. It can be shown that the modular transformations $T : \tau \rightarrow \tau + 1$ and $S : \tau \rightarrow -\frac{1}{\tau}$ generate the whole of the modular group.

Conformal invariance requires the partition function Z to be invariant under the modular group. This places some restrictions on the characters χ_i; in particular, the space generated by the characters must be invariant under the modular transformation $\tau \rightarrow -1/\tau$.

The connection to algebraic K-theory

For certain conformal field theories with integrable perturbations, their characters χ_i can be described combinatorially as

$$\chi_i(\tau) = \sum_{m} \frac{q^{Q_i(m)}}{(q)_{m_1} \cdots (q)_{m_r}} \ , \tag{2.15}$$

where $Q_i(\mathbf{m}) = \frac{\mathbf{m}A\mathbf{m}}{2} + b_i\mathbf{m} + h_i - \frac{c}{24}$, $(q)_n = \prod_{i=1}^{n}(1-q^i)$, $\mathbf{m} = (m_1, \ldots, m_r)$, and r is the rank of the matrix A. Here the matrix A is the same for all characters χ_i of a given CFT.

It is expected [17] that a general sum of the form (2.15), with rational coefficients, can only be modular when all solutions of the equations

$$\sum_j A_{ij} \log(x_j) = \log(1 - x_i) \ ,$$

give finite order elements $(\log(x_i), \log(1 - x_i))$ of the extended Bloch group. That the elements $\sum_i [x_i]$ belong to the Bloch group at all is shown at the beginning of Chapter 3.

There are many matrices A for which $\sum_j A_{ij} \log(x_j) = \log(1 - x_i)$ yield torsion elements of the extended Bloch group. The best known examples are related to Dynkin diagrams. Given a pair of Dynkin diagrams (X, Y), the matrix A is given by $A(X, Y) = C(X) \otimes C(Y)^{-1}$. In chapter 3 we will consider the case $(X, Y) = (D_m, A_n)$.

It is expected that whenever $\sum_j A_{ij} \log(x_j) = \log(1 - x_i)$ yields finite order elements of the Bloch group, the resulting x_i should be rational linear combinations of roots of unity, possibly apart from some $\mathbb{Z}_2$ extension. This has turned out to be true for all examples considered so far. Moreover there is a map from finite order elements of the Bloch group to the central charges and scaling dimensions of conformal field theories. This mapping is given by the dilogarithm function and will be described in chapter 3.

This suggests a clear, and very interesting, relationship between certain integrable quantum field theories in two dimensions and the algebraic K-theory of the complex numbers.

Chapter 3

Yangians and the (D_m, A_n) Models

It is clear from the previous chapter that a study of integrable models described by pairs of Dynkin diagrams should yield interesting results. In this chapter we consider the particular models described by the pairs (D_m, A_n). Using the representation theory of Yangians we solve the equations of these models in the general case. We demonstrate how to calculate the effective central charge using the dilogarithm formula, and finally we relate these models to the coset models described in chapter 2.

3.1 Overview

Equations of the model (X, Y)

Consider the integrable model described by the pair of Dynkin diagrams (X, Y). The equations of this model are of the form $AU = V$, where the matrix A is given by

$$A(X,Y) = C(X)^{-1} \otimes C(Y) \,, \tag{3.1}$$

and X and Y are Dynkin diagrams of ranks m and n respectively. Here $U = \log(x)$ and $V = \log(1-x)$, such that the equation $e^U + e^V = 1$ is satisfied. x denotes the vector $(x_{11}, \ldots, x_{mn})$. The matrix A, defined in (3.1), is positive definite and symmetric.

Exponentiation of $AU = V$ leads to a set of purely algebraic equations.

$$\begin{aligned} AU = V \quad &\equiv \quad \sum_j A_{ij} \log(x_j) = \log(1 - x_i) && (3.2) \\ &\Rightarrow \quad \prod_j x_j^{A_{ij}} = 1 - x_i \,. && (3.3) \end{aligned}$$

Notice that this exponentiation transforms a set of equations with infinitely many solutions (3.2) into a set with a finite number of solutions (3.3). For simplicity we choose to solve the equations in the form (3.3); however, care must be taken to choose logarithms in such a way that the original equations (3.2) of the model are satisfied.

For any solution $(x_{11}, \ldots, x_{mn})$ of (3.3), the element $[x_{11}] + \ldots + [x_{mn}]$ belongs to the Bloch group:

$$\begin{aligned}\sum_i x_i \wedge (1-x_i) &= \sum_i x_i \wedge \left(\prod_j x_j^{A_{ij}}\right) ,\\ &= \sum_i \sum_j x_i \wedge A_{ij} x_j ,\\ &= \sum_i \sum_j A_{ij} x_i \wedge x_j ,\\ &= 0 ,\end{aligned}$$

since A_{ij} is symmetric and $x_i \wedge x_j$ is anti-symmetric in i and j.

In some cases the algebraic equations (3.3) can be solved using nothing more than elementary algebra. However as the matrix A grows in size this becomes more difficult. By a suitable change of variables, the equations (3.3) can be written in a form that allows them to be solved relatively easily using the representation theory of Lie algebras and related quantum groups.

For this purpose we introduce the new variable $z = (z_{11}, \ldots, z_{mn})$, where

$$x = z^{-C(X)\otimes I_Y} . \tag{3.4}$$

The algebraic equations (3.3) can be rewritten in terms of z.

$$\begin{aligned}x^A = 1-x \quad &\Rightarrow \quad x^{C(X)^{-1}\otimes C(Y)} = 1-x\\ &\Rightarrow \quad z^{-I_X\otimes C(Y)} = 1 - z^{-C(X)\otimes I_Y}\\ &\Rightarrow \quad z^{2-C(Y)} + z^{2-C(X)} = z^2 . \end{aligned}\tag{3.5}$$

Equation (3.5) has many solutions for which some components of z vanish. These are called non-admissable since they do not yield solutions of (3.2). We discard these solutions immediately.

We impose the boundary condition $z_{i,n+1} = 1$ on the equations (3.5), because with these boundary conditions the solutions z_{ij} arise naturally in representation theory, see [17] and references therein. (After imposing $z_{i,n+1} = 1$, equations (3.5) are exactly the equations discussed by Kirillov and Reshetikhin [31], for the Lie algebra X, whose solutions arise as characters of the Yangian $Y(X)$). In particular, for the model (D_m, A_n), the components z_{ij} of z are the characters Q^i_j of the Yangian $Y(D_m)$, that satisfy $Q^i_{n+1} = 1$ for $i = 1, 2, \ldots, m$. Hence a solution of (3.5) amounts to finding a matrix $g \in SO(2m)$, whose Yangian characters satisfy $Q^i_{n+1}(g) = 1$ for $i = 1, 2, \ldots, m$. Once such a matrix has been found, the equations (3.5) can be solved using the relation

$$z_{ij} = Q^i_j(g) \ .$$

These z_{ij} can easily be transformed into solutions x_{ij} of (3.3) using equation (3.4).

Effective Central Charge Calculations

Let $x^i = (x^i_{11}, \ldots, x^i_{mn})$ denote a solution of the system of equations (3.3). Here i is an index to distinguish between solutions, so $0 \leq i \leq I$, where I is the number of solutions.

If A is any positive definite matrix then the system of equations (3.3) has a unique solution with all x_{ij} real and between 0 and 1. This fact is proved in [27]. Denote this solution by $x^0 = (x^0_{11}, \ldots, x^0_{mn})$. Then $0 < x^0_{jk} \in \mathbb{R} < 1$ for all j and k. For future reference we refer to x^0 as the 'minimal solution'.

We are interested in the values taken by the solutions x^i under the mapping

$$\frac{6}{\pi^2} \sum_{jk=1,\ldots,mn} L(u^i_{jk}, v^i_{jk}) = c - 24h_i \mod 24\mathbb{Z} \; . \tag{3.6}$$

Here $u^i_{jk} = \log(x^i_{jk})$ and $v^i_{jk} = \log(1 - x^i_{jk})$ form solutions of the equations $AU = V$, provided logarithms of the complex numbers are chosen appropriately. Here L is the analytic continuation (2.10) of the Rogers dilogarithm, so that for the real solution x^0 one has simply

$$L(u^0_{jk}, v^0_{jk}) = L(x^0_{jk}) \; .$$

The **effective central charge** is defined as the particular value of $c-24h_i$ that arises from the minimal solution x^0. In this case we have

$$c_{\text{eff}} = \frac{6}{\pi^2} \sum_{jk=1,\ldots,mn} L(u^0_{jk}, v^0_{jk}) \; . \tag{3.7}$$

The factor of $24\mathbb{Z}$ in (3.6) essentially arises because the dilogarithm function is multi-valued outside its region of convergence $|z| < 1$. The only value of $c - 24h_i$ that can be calculated exactly is c_{eff}, since it corresponds to the minimal solution (all of whose components are real and between 0 and 1). Although it will not always be mentioned, this mod $24\mathbb{Z}$ term of course applies throughout the thesis.

The matrix A

The matrix A defined in (3.1) is in fact related to scattering matrices as follows. Suppose we take a system containing r different species of particles. Consider the scattering of two particles of types i and j, and rapidities θ_i and θ_j respectively. For the type of system of interest to us, the particles merely pass through each other with some time delay (i.e. there is no exchange of particle quantum number). This time delay is described by the scattering matrix, with the scattering being described by an energy-dependent phase

$$S_{ij} = e^{if_{ij}(\theta_j - \theta_i)} \ .$$

The scattering matrix takes particular values at $\pm\infty$. In terms of these values A is defined as

$$A_{ij} = \frac{f_{ij}(-\infty) - f_{ij}(+\infty)}{2\pi} \ .$$

3.2 Quantum Groups and Yangians

Quantum groups were first introduced by Fadeev and collaborators. They arose from the quantum inverse scattering method [32], developed to construct and solve quantum integrable systems. In their original form quantum groups are associative algebras whose defining relations are expressed in terms of a matrix of constants called a quantum R-matrix. This matrix depends on the particular integrable system under consideration. Quantum groups facilitate the understanding of solutions

(R-matrices) of the quantum Yang-Baxter equation associated with such integrable systems. Furthermore they provide a general framework for finding new solutions. Of special importance are those solutions that depend on a spectral parameter. In particular those which are rational functions of this parameter arise from the family of quantum groups called Yangians. More recently quantum groups have arisen in connection with 1+1 dimensional integrable quantum field theories, as the algebras satisfied by certain non-local conserved currents. For example, Yangians appear as 'quantum symmetry algebras' in G-invariant Wess-Zumino-Witten models [33].

The term Yangian was introduced by V.G. Drinfeld [21] to specify those quantum groups related to rational solutions of the quantum Yang-Baxter equation. In fact Yangians are named after C.N. Yang who found the simplest such solution [6]. It is worth noting that Lüscher [34] effectively found much of the Yangian $Y(\mathfrak{so}_n)$ well in advance of the general construction.

Although quantum groups first appeared in the physics literature and many of the fundamental papers are written in the language of integrable systems, their properties are still accessible through more mainstream mathematical techniques. There are many unexpected connections between quantum groups and other seemingly unrelated areas of mathematics (for example knot theory and the representation theory of algebraic groups in characteristic p). In recent years these connections have sparked considerable interest in quantum groups.

3.2.1 Representation theory of Yangians

As mentioned above, the importance of Yangians stems from the fact that their finite-dimensional representations can be used to construct rational solutions of the quantum Yang-Baxter equation. The problem of describing all finite-dimensional irreducible representations of $Y(\mathfrak{g})$ was solved by Drinfeld himself. He gave a classification of such representations, similar to that for the Lie algebra $\mathfrak{g}$ in terms of highest weights, but without giving an 'explicit' realisation of these representations. Such a realisation was given for $\mathfrak{g} = \mathfrak{sl}_2$ by V. Chari and A. Pressley in [35].

An alternative approach to obtaining a better understanding of such representations would be to find a Yangian character formula, analogous to the Weyl character formula for Lie algebras. In [36, 37] Chari and Pressley give examples of such a formula for $Y(\mathfrak{sl}_2)$. Unfortunately their proof does not extend to other cases, and at present no general formula is known.

Irreducible Yangian representations

Suppose $\mathfrak{g}$ is a Lie algebra of rank r. Then $\mathfrak{g}$ has r fundamental weights, denoted $\omega_1, \ldots, \omega_r$, one corresponding to each node on its Dynkin diagram. The **fundamental representations** of $\mathfrak{g}$ are the r irreducible representations of highest weights ω_i $(i = 1, 2, \ldots, r)$. Similarly $Y(\mathfrak{g})$ has r fundamental (finite-dimensional) irreducible representations.

Since $\mathfrak{g} \subset Y(\mathfrak{g})$, any representation of $Y(\mathfrak{g})$ is automatically a representation of $\mathfrak{g}$. However, a representation which is $Y(\mathfrak{g})$-irreducible may become reducible when restricted to $\mathfrak{g}$. In fact this is typically the case for the fundamental representations of $Y(\mathfrak{g})$ (whose $\mathfrak{g}$-components are the corresponding fundamental irreducible representations of $\mathfrak{g}$ + some other representations).

Nevertheless, in some cases an irreducible $Y(\mathfrak{g})$-representation remains $\mathfrak{g}$-irreducible. In the simplest situation, given an irreducible representation ρ of $\mathfrak{g}$, in certain cases we can construct a representation $\tilde{\rho}$ of $Y(\mathfrak{g})$ by

$$\tilde{\rho}(I_a) = \rho(I_a)\,, \quad \tilde{\rho}(J_a) = 0\,. \tag{3.8}$$

These cases are described as follows.

Let a_i be the coefficient of the simple root α_i in the expansion of the highest root θ of $\mathfrak{g}$. Put $k_i = (\theta,\theta)/(\alpha_i,\alpha_i)$, and let ω_i be the corresponding fundamental weight of $\mathfrak{g}$. Then the irreducible representation of $\mathfrak{g}$, of highest weight λ, can be extended to an irreducible representation of $Y(\mathfrak{g})$ using (3.8) in the following cases:

1. $\lambda = \omega_i$, when $a_i = k_i$,
2. $\lambda = t\omega_i$ ($t \in \mathbb{N}$), when $a_i = 1$.

Included in these cases are all fundamental representations of A_n and C_n, and the vector and (half)-spinor representations of B_n and D_n.

The more general case, in which an irreducible $Y(\mathfrak{g})$-representation is $\mathfrak{g}$-reducible is significantly more complicated. For more details see [24].

3.3 The Lie Algebra D_r

3.3.1 Some properties of D_r

The Lie algebra D_r has r simple roots $\alpha_1, \ldots, \alpha_r$ given by

$$\begin{aligned}
\alpha_1 &= e_1 - e_2 \ , \\
\alpha_2 &= e_2 - e_3 \ , \\
&\vdots \\
\alpha_{r-1} &= e_{r-1} - e_r \ , \\
\alpha_r &= e_{r-1} + e_r \ ,
\end{aligned}$$

and r fundamental weights $\omega_1, \ldots, \omega_r$ given by

$$\begin{aligned}
\omega_1 &= e_1 \ , \\
\omega_2 &= e_1 + e_2 \ , \\
&\vdots \\
\omega_{r-2} &= e_1 + \ldots + e_{r-2} \ , \\
\omega_{r-1} &= \frac{1}{2}(e_1 + e_2 + \ldots + e_{r-1} - e_r) \ , \\
\omega_r &= \frac{1}{2}(e_1 + e_2 + \ldots + e_{r-1} + e_r) \ .
\end{aligned} \tag{3.9}$$

The Weyl vector ρ is given by

$$\rho = (r-1)e_1 + (r-2)e_2 + \ldots + 2e_{r-2} + e_{r-1} \ .$$

The quantity e_i is defined by its action on a diagonal $n \times n$ matrix as

$$e_i\left(\mathrm{diag}(a_1, \ldots, a_n)\right) = a_i \ .$$

The Weyl group of D_r is denoted $W(D_r)$. It is an extension of S_r by $(\mathbb{Z}_2)^{r-1}$. Its elements act on $(e_1, \ldots, e_r)$ as

$$(e_1, \ldots, e_r) \to \left(\epsilon_1 e_{s(1)}, \ldots, \epsilon_r e_{s(r)}\right) \ ,$$

where $s \in S_r$, $\epsilon_i \in \{\pm 1\}$, and $\prod_{i=1}^{r} \epsilon_i = 1$. The order of $W(D_r)$ is $r!\, 2^{r-1}$.

Representations of $Y(D_r)$

In this thesis we follow the notation of Kirillov and Reshetikhin [31]. Again $\mathfrak{g}$ is a simple Lie algebra of rank r, with fundamental weights $\omega_1, \ldots, \omega_r$. $Y(\mathfrak{g})$ is the corresponding Yangian.

Let W_i^j denote the irreducible representation of $Y(\mathfrak{g})$, of highest weight $j\omega_i$. Here $j \in \mathbb{N}$ and $i = 1, 2, \ldots, r$. As mentioned in the previous section, W_i^j may become reducible when restricted to the Lie algebra $\mathfrak{g}$.

For $\mathfrak{g} = D_r$, the representations $\left. W_i^j \right|_{\mathfrak{g}}$ are irreducible in the cases

$$\begin{aligned}
(i, j) &= (1, j) \ , \\
(i, j) &= (r-1, j) \ , \\
(i, j) &= (r, j) \ .
\end{aligned}$$

These correspond to the vector and half-spinor representations.

Define polynomials Q^i_j by

$$Q^i_j = \mathrm{ch}\left(W^j_i\big|_{\mathfrak{g}} \right) ,$$

where ch denotes the character of a representation.

It is claimed in [31] that for $\mathfrak{g} = D_r$, the functions Q^i_j form the unique solution of the system of recurrence relations

$$\begin{aligned} (Q^i_j)^2 - Q^i_{j-1}Q^i_{j+1} &= Q^{i-1}_j Q^{i+1}_j, \qquad 1 \le i \le r-3 , \\ (Q^{r-2}_j)^2 - Q^{r-2}_{j-1}Q^{r-2}_{j+1} &= Q^{r-3}_j Q^{r-1}_j Q^r_j , \\ (Q^{r-1}_j)^2 - Q^{r-1}_{j-1}Q^{r-1}_{j+1} &= Q^{r-2}_j , \\ (Q^r_j)^2 - Q^r_{j-1}Q^r_{j+1} &= Q^{r-2}_j , \end{aligned} \tag{3.10}$$

with initial data given by

$$\begin{aligned} Q^0_j &= 1 , \\ Q^i_1 &= \mathrm{ch}\left(V(\omega_i) + V(\omega_{i-2}) + \ldots\right) , \quad i = 1, \ldots, r-2 , \\ Q^{r-1}_1 &= \mathrm{ch}\left(V(\omega_{r-1})\right) , \\ Q^r_1 &= \mathrm{ch}\left(V(\omega_r)\right) . \end{aligned} \tag{3.11}$$

Here $V(\omega_i)$ denotes the i^{th} fundamental representation of D_r.

In future we refer to the equations (3.10) as the Kirillov-Reshetikhin (KR) equations. In the following section we prove a formula for the quantities Q^j_1 that agrees with the Yangian interpretation.

3.3.2 Solutions of the Kirillov-Reshetikhin equations

The paper [38] studies a class of 2-dimensional Toda equations on discrete space-time. These arise as functional relations for commuting families of transfer matrices in solvable lattice models associated with any classical Lie algebra X_r. For D_r ($r \geq 4$) the relevant system of Toda equations is

$$T_k^a(u-1)T_k^a(u+1) - T_{k+1}^a(u)T_{k-1}^a(u) = \begin{cases} T_k^{a-1}(u)T_k^{a+1}(u) & 1 \leq a \leq r-3\,, \\ T_k^{r-3}(u)T_k^{r-1}(u)T_k^r(u) & a = r-2\,, \\ T_k^{r-2}(u) & a = r-1, r\,. \end{cases} \tag{3.12}$$

Here $k \in \mathbb{Z}_{\geq 0}$, $u \in \mathbb{C}$, and $a \in \{1, 2, \ldots, r\}$, and the $T_k^a(u)$ are complex numbers (depending on a, k, and u). The system is considered with the initial conditions $T_0^a(u) = 1$ for any $1 \leq a \leq r$. (Note that we also take $T_k^0 = 1$). Later it will be useful to impose the further initial conditions (3.11), however that is not necessary at this stage.

The system of equations (3.12) can be solved iteratively to express an arbitrary $T_k^a(u)$ ($k \geq 1$) as a determinant or a Pfaffian of a matrix with entries 0 and $\pm T_1^b(u + \text{const.})$ for $0 \leq b \leq r$. Such solutions are given explicitly for the cases B_r, C_r, and D_r in [38]. The proof is shown only for the case C_r, with the claim that it extends to the other cases. We carry out the proof for the D_r case and it indeed works nicely. Notation and method follow exactly that of [38].

Notation

For any $l \in \mathbb{C}$, put

$$x_l^a = \begin{cases} T_1^a(u+l) & \text{if } 1 \le a \le r \,, \\ \\ 1 & \text{if } a = 0 \,. \end{cases} \tag{3.13}$$

Introduce the infinite-dimensional matrices $\mathcal{T} = (\mathcal{T}_{ij})_{i,j\in\mathbb{Z}}$ and $\mathcal{E} = (\mathcal{E}_{ij})_{i,j\in\mathbb{Z}}$ by

$$\mathcal{T}_{ij} \;=\; \begin{cases} x^{\frac{j-i}{2}+1}_{\frac{i+j}{2}-1} & \text{if } i \in 2\mathbb{Z}+1 \text{ and } \frac{i-j}{2} \in \{1,0,\ldots,3-r\} \,, \\ -x^{r-1}_{\frac{i+j-1}{2}} & \text{if } i \in 2\mathbb{Z}+1 \text{ and } \frac{i-j}{2} = \frac{5}{2}-r \,, \\ -x^{r}_{\frac{i+j-3}{2}} & \text{if } i \in 2\mathbb{Z}+1 \text{ and } \frac{i-j}{2} = \frac{3}{2}-r \,, \\ -x^{\frac{i-j}{2}+2r-3}_{\frac{i+j}{2}-1} & \text{if } i \in 2\mathbb{Z}+1 \text{ and } \frac{i-j}{2} \in \{1-r,-r,\ldots,3-2r\} \,, \\ 0 & \text{otherwise} \,. \end{cases}$$

$$\mathcal{E}_{ij} \;=\; \begin{cases} \pm 1 & \text{if } i = j-2\pm 2 \text{ and } i \in 2\mathbb{Z} \,, \\ \\ x_i^{r-1} & \text{if } i = j-3 \text{ and } i \in 2\mathbb{Z} \,, \\ \\ x_{i-2}^{r} & \text{if } i = j-1 \text{ and } i \in 2\mathbb{Z} \,, \\ \\ 0 & \text{otherwise} \,. \end{cases}$$

For any $1 \le a \le r$ and any l, the element $\pm x_l^a$ appears exactly once in the matrix $\mathcal{T}_{u\to u+\xi}$. Here $u \to u+\xi$ means the overall shift of lower indices according to (3.13). Let $\mathcal{T}_k(i,j,\pm x_l^a)$ denote the $k \times k$ sub-matrix of $\mathcal{T}$ whose (i,j) element is exactly $\pm x_l^a$. We use a similar notation for $\mathcal{E}(i,j,\pm x_l^a)$. These definitions are unambiguous; for a more detailed explanation refer to the original paper.

Theorem 1 *For $k \in \mathbb{Z}_{\geq 1}$,*

$$\begin{aligned} T_k^a(u) &= \det\left(\mathcal{T}_{2k-1}(1,1,x^a_{-k+1}) + \mathcal{E}_{2k-1}(2,3,x^r_{-k-r+a+4})\right) \qquad (3.14)\\ & \qquad\qquad 1 \leq a \leq r-2\,, \\ T_k^{r-1}(u) &= \mathrm{pf}\left(\mathcal{T}_{2k}(2,1,-x^{r-1}_{-k+1}) + \mathcal{E}_{2k}(1,2,x^{r-1}_{-k+1})\right)\,, \qquad (3.15)\\ T_k^r(u) &= (-1)^k \mathrm{pf}\left(\mathcal{T}_{2k}(1,2,-x^r_{-k+1}) + \mathcal{E}_{2k}(2,1,x^r_{-k+1})\right)\,, \qquad (3.16)\end{aligned}$$

solve the D_r system of equations (3.12).

Proof

Using equations (3.14), (3.15) and (3.16) we can show that

$$T_k^{r-1}(u)T_k^r(u) = (-1)^k \det\left(\mathcal{T}_{2k}(1,1,-x^{r-1}_{-k+1}) + \mathcal{E}_{2k}(2,2,x^r_{-k+1})\right)\,, \qquad (3.17)$$

$$T_k^{r-1}(u+1)T_k^r(u-1) = (-1)^k \det\left(\mathcal{T}_{2k}(2,1,x^{r-2}_{1-k}) + \mathcal{E}_{2k}(1,1,x^r_{-k})\right)\,, \qquad (3.18)$$

$$T_{k+1}^{r-1}(u)T_k^r(u-1) = (-1)^{k+1} \det\left(\mathcal{T}_{2k+1}(1,1,-x^{r-1}_{-k}) + \mathcal{E}_{2k+1}(2,2,x^r_{-k})\right), \qquad (3.19)$$

$$T_k^{r-1}(u+1)T_{k+1}^r(u) = (-1)^k \det\left(\mathcal{T}_{2k+1}(2,1,x^{r-2}_{-k+1}) + \mathcal{E}_{2k+1}(1,1,x^r_{-k})\right)\,. \qquad (3.20)$$

This is done by taking matrices

$$\begin{aligned} M &= \mathcal{T}_{2k+1}(2,1,-x^{r-1}_{-k+1}) + \mathcal{E}_{2k+1}(1,2,x^{r-1}_{-k+1})\,,\\ M &= \mathcal{T}_{2k+1}(1,2,-x^r_{-k}) + \mathcal{E}_{2k+1}(2,1,x^r_{-k})\,,\\ M &= \mathcal{T}_{2k+2}(2,1,-x^{r-1}_{-k}) + \mathcal{E}_{2k+2}(1,2,x^{r-1}_{-k})\,,\\ M &= \mathcal{T}_{2k+2}(1,2,-x^r_{-k}) + \mathcal{E}_{2k+2}(2,1,x^r_{-k})\,,\end{aligned}$$

respectively, and using Jacobi's identity as in [38].

Jacobi's Identity:

$$D_M \begin{bmatrix} 1 \\ 1 \end{bmatrix} D_M \begin{bmatrix} n \\ n \end{bmatrix} = D_M D_M \begin{bmatrix} 1 & , & n \\ 1 & , & n \end{bmatrix} + D_M \begin{bmatrix} 1 \\ n \end{bmatrix} D_M \begin{bmatrix} n \\ 1 \end{bmatrix} .$$

Here D_M is the determinant of any $n \times n$ matrix M, and $D_M \begin{bmatrix} i_1 & , & i_2 & , & \dots \\ j_i & , & j_2 & , & \dots \end{bmatrix}$ denotes its minor removing the i_k's rows and the j_k's columns.

The following facts are used in the proof:

1. If M is an odd anti-symmetric matrix then $\det(M) = 0$.
2. If M is an odd anti-symmetric matrix then $D_M \begin{bmatrix} 1 \\ n \end{bmatrix} = D_M \begin{bmatrix} n \\ 1 \end{bmatrix}$.
3. If M is an even anti-symmetric matrix then $D_M \begin{bmatrix} 1 \\ n \end{bmatrix} = -D_M \begin{bmatrix} n \\ 1 \end{bmatrix}$.

Having done this, (3.12) can be proved by again using the Jacobi identity, this time setting

$$\begin{aligned} D_M &= T^a_{k+1}(u) \ , \\ D_M &= T^{r-2}_{k+1}(u) \ , \\ D_M &= T^{r-1}_{k+1}(u) T^r_k(u-1) \ , \\ D_M &= T^{r-1}_k(u+1) T^r_{k+1}(u) \ , \end{aligned}$$

for $1 \leq a \leq r$, $a = r-2$, $a = r-1$, and $a = r$ respectively. M is taken as in the right hand sides of (3.17-3.20). This proves the theorem.

A special case

Of particular interest to us is the special case in which T is a constant, i.e. there is no u-dependence. In this case the structure of the T-system is exactly that of the Kirillov-Reshetikhin equations (3.10). In the case where each T is u-independent, the subindex l in the quantity x_l^a can be omitted, as this refers to the shift in the u argument, which is now irrelevant . In fact from now on we will abandon the x_l^a notation altogether as it is no longer necessary in this simpler special case. We revert to using T_1^a in place of x^a. We get the following corollary of theorem 1.

Corollary 1 *For $k \in \mathbb{Z}_{\geq 1}$,*

$$\begin{aligned}
T_k^a &= \det\left(\mathcal{T}_{2k-1}(1,1,T_1^a) + \mathcal{E}_{2k-1}(2,3,T_1^r)\right) \; ; \; 1 \leq a \leq r-2 \,, \\
T_k^{r-1} &= \mathrm{pf}\left(\mathcal{T}_{2k}(2,1,-T_1^{r-1}) + \mathcal{E}_{2k}(1,2,T_1^{r-1})\right) \,, \\
T_k^r &= (-1)^k \mathrm{pf}\left(\mathcal{T}_{2k}(1,2,-T_1^r) + \mathcal{E}_{2k}(2,1,T_1^r)\right) \,,
\end{aligned} \tag{3.21}$$

solve the D_r Kirillov-Reshetikhin equations

$$(T_k^a)^2 - T_{k+1}^a T_{k-1}^a = \begin{cases} T_k^{a-1} T_k^{a+1} & 1 \leq a \leq r-3 \,, \\ T_k^{r-3} T_k^{r-1} T_k^r & a = r-2 \,, \\ T_k^{r-2} & a = r-1,\, r \,. \end{cases}$$

We now proceed to show that, under the initial conditions (3.11) used by Kirillov and Reshetikhin, the quantities $T_1^1, \ldots, T_1^r$ are exactly the Yangian characters $Q_1^1, \ldots, Q_1^r$.

Recursion relation for T_n^1

(3.21) is a solution of the Kirillov-Reshetikhin equations in terms of determinants and Pfaffians of certain matrices with entries 0 and $\pm T_1^1, \ldots, \pm T_1^r$. In particular, for $n \geq 1$, T_n^1 is given by the determinant

$$T_n^1 = \det\left(\mathcal{T}_{2n-1}(1,1,T_1^1) + \mathcal{E}_{2n-1}(2,3,T_1^r)\right) \ .$$

Expansion of this determinant leads to the following recursion relation for the quantity T_n^1 ($n \geq 1$) in terms of T_{n-1}^1, $T_{n-2}^1, \ldots, T_1^1$ and $T_1^1, \ldots, T_1^r$:

$$\begin{aligned} T_n^1 &= T_1^1 T_{n-1}^1 - T_1^2 T_{n-2}^1 + T_1^3 T_{n-3}^1 - \ldots - T_1^{r-2} T_{n-r+2}^1 \\ &+ T_1^{r-1} T_1^r T_{n-r+1}^1 + T_1^{r-2} T_{n-r}^1 - \ldots - T_1^1 T_{n-2r+3}^1 + T_{n-2r+2}^1 \\ &- ((T_1^{r-1})^2 + (T_1^r)^2)(T_{n-r}^1 + T_{n-r-2}^1 + \ldots) \\ &+ 2T_1^{r-1} T_1^r (T_{n-r-1}^1 + T_{n-r-3}^1 + \ldots) \ . \end{aligned} \tag{3.22}$$

For simplicity we define the quantity R_{n-k} by

$$R_{n-k} = T_{n-k}^1 + T_{n-k-2}^1 + T_{n-k-4}^1 + \ldots$$

Then $R_n = T_n^1 + R_{n-2}$.

Theorem 2 *The recursion equation (3.22) can be written in matrix form as*

$$\begin{pmatrix} T_n^1 \\ T_{n-1}^1 \\ T_{n-2}^1 \\ \vdots \\ T_{n-2r+3}^1 \\ R_{n-r+1} \\ R_{n-r} \end{pmatrix} = \mathcal{M} \begin{pmatrix} T_{n-1}^1 \\ T_{n-2}^1 \\ T_{n-3}^1 \\ \vdots \\ T_{n-2r+2}^1 \\ R_{n-r} \\ R_{n-r-1} \end{pmatrix}$$

where $\mathcal{M}$ is the $2r \times 2r$ matrix

$$\begin{pmatrix}
T_1^1 & -T_1^2 & \dots & -T_1^{r-2} & T_1^{r-1}T_1^r & T_1^{r-2} & \dots & T_1^2 & -T_1^1 & 1 & -(T_1^{r-1})^2-(T_1^r)^2 & 2T_1^{r-1}T_1^r \\
1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & 0
\end{pmatrix}$$

with initial values $T_0^1 = T_{-2r+2}^1 = 1$ and $T_{-j}^1 = 0$ for $j = 1, 2, \dots, 2r-3$.

Note: This is the matrix $\mathcal{M}$ for r even. For r odd the matrix has the same structure with slight changes of sign. This does not affect subsequent calculations, and results are identical.

The eigenvalues of the matrix $\mathcal{M}$ are the solutions of the equation

$$
\begin{aligned}
0 &= 1+\lambda^{2r}-T_1^1(\lambda^{2r-1}+\lambda)+(T_1^2-1)(\lambda^{2r-2}+\lambda^2) \\
&- (T_1^3-T_1^1)(\lambda^{2r-3}+\lambda^3)+\ldots+(T_1^{r-2}-T_1^{r-4})(\lambda^{r+2}+\lambda^{r-2}) \\
&- (T_1^{r-1}T_1^r-T_1^{r-3})(\lambda^{r+1}+\lambda^{r-1})+((T_1^{r-1})^2+(T_1^r)^2-2T_1^{r-2})\lambda^r \ . \qquad (3.23)
\end{aligned}
$$

Suppose the equation (3.23) has roots $a_1,\ldots,a_{2r}$. Notice that $0,\pm 1$ are not roots. By the symmetry of equation (3.23), and the fact that $\prod_{i=1}^{2r} a_i = 1$, the roots can be written as $a_1,\ldots a_1^{-1}$, $a_r,\ldots,a_r^{-1}$, where for the purposes of ordering we identify $a_{r+1}=a_1^{-1},\ldots,a_{2r}=a_r^{-1}$. Then the following equations must be satisfied:

$$
\begin{aligned}
T_1^1 &= \sum_{i=1}^{2r} a_i \ , \\
T_1^2-1 &= \sum_{i<j} a_i a_j \ , \\
T_1^k - T_1^{k-2} &= \sum_{i_1<\ldots<i_k} a_{i_1}\ldots a_{i_k} \quad \text{for } k=3,\ldots,r-2 \ , \\
T_1^{r-1}T_1^r - T_1^{r-3} &= \sum_{i_1<\ldots<i_{r-1}} a_{i_1}\ldots a_{i_{r-1}} \ , \\
(T_1^{r-1})^2+(T_1^r)^2-2T_1^{r-2} &= \sum_{i_1<\ldots<i_r} a_{i_1}\ldots a_{i_r} \ .
\end{aligned}
$$

We consider the orthogonal group, $SO(2r)$, of linear transformations of $\mathbb{C}^{2r}$ which preserve the form $z_1z_2+z_3z_4+\ldots+z_{2r-1}z_{2r}$. A maximal torus consists of elements $g=\mathrm{diag}(a_1,a_1^{-1},\ldots,a_r,a_r^{-1}),\ a_i\in\mathbb{C}$.

Theorem 3 *For the choice (3.11) of initial conditions, and* $g = \mathrm{diag}(a_1, a_1^{-1}, \ldots, a_r, a_r^{-1})$*, the quantities* T_1^i *are exactly the Yangian characters* $Q_1^i(g)$ *for* $i = 1, 2, \ldots, r$*.*

Proof

It is a well known fact that for $1 \leq k \leq r-2$, the characters of the Lie algebra D_r have the structure

$$\chi(\omega_k) = \sum_{i_1 < \ldots < i_k} a_{i_1} \ldots a_{i_k} ,$$

where $a \in \{a_1, a_1^{-1}, \ldots a_r, a_r^{-1}\}$, as before. The final two characters have a slightly different structure: $\chi(\omega_{r-1}) = \sum a_1{}^{\pm\frac{1}{2}} \ldots a_r{}^{\pm\frac{1}{2}}$, where the sum is taken over all possible combinations with an odd number of negative exponents, and $\chi(\omega_r)$ is defined similarly but with an even number of negative exponents. This follows immediately from the structure of the fundamental weights of D_r, and the action of the Weyl group, see (3.9).

Now suppose we take initial conditions as in equations (3.11). We can then conclude the following:

$$T_1^1 = \sum_{i=1}^{2r} a_i \quad \Rightarrow \quad T_1^1 = \chi(\omega_1) = Q_1^1\ ,$$

$$\begin{aligned} T_1^2 - 1 = \sum_{i<j} a_i a_j \quad &\Rightarrow \quad T_1^2 - 1 = \chi(\omega_2) \\ &\Rightarrow \quad T_1^2 = \chi(\omega_2) + 1 \\ &\Rightarrow \quad T_1^2 = Q_1^2\ , \end{aligned}$$

$$\begin{aligned} T_1^3 - T_1^1 = \sum_{i<j<k} a_i a_j a_k \quad &\Rightarrow \quad T_1^3 - T_1^1 = \chi(\omega_3) \\ &\Rightarrow \quad T_1^3 = \chi(\omega_3) + T_1^1 \\ &\Rightarrow \quad T_1^3 = \chi(\omega_3) + \chi(\omega_1) \\ &\Rightarrow \quad T_1^3 = Q_1^3\ , \end{aligned}$$

$$\begin{aligned} T_1^k - T_1^{k-2} = \sum_{i_1<\ldots<i_k} a_{i_1} \ldots a_{i_k} \quad &\Rightarrow \quad T_1^k - T_1^{k-2} = \chi(\omega_k) \\ &\Rightarrow \quad T_1^k = T_1^{k-2} + \chi(\omega_k) \\ &\Rightarrow \quad T_1^k = Q_1^{k-2} + \chi(\omega_k) \quad \text{by induction on k} \\ &\Rightarrow \quad T_1^k = Q_1^k \quad \text{for } 4 \leq k \leq r-2\ . \end{aligned}$$

The final two equations

$$\begin{aligned} T_1^{r-1}T_1^r - T_1^{r-3} &= \sum_{i_1<\ldots<i_{r-1}} a_{i_1}\ldots a_{i_{r-1}} \\ \Rightarrow T_1^{r-1}T_1^r &= \sum_{i_1<\ldots<i_{r-1}} a_{i_1}\ldots a_{i_{r-1}} + Q_1^{r-3} , \\ &\text{and} \\ (T_1^{r-1})^2 + (T_1^r)^2 - 2T_1^{r-2} &= \sum_{i_1<\ldots<i_r} a_{i_1}\ldots a_{i_r} \\ \Rightarrow (T_1^{r-1})^2 + (T_1^r)^2 &= 2Q_1^{r-2} + \sum_{i_1<\ldots<i_r} a_{i_1}\ldots a_{i_r} , \end{aligned}$$

are satisfied by

$$T_1^{r-1} = \chi(\omega_{r-1}) = Q_1^{r-1} ,$$

and

$$T_1^r = \chi(\omega_r) = Q_1^r .$$

Hence we can conclude that $T_1^1, \ldots, T_1^r$ are in fact the Yangian characters $Q_1^1, \ldots, Q_1^r$.

Remark: Using the Weyl character formula and the recursion relation (3.22), we can conclude that

$$T_i^1(g) = Q_i^1(g) ,$$

for $g = \operatorname{diag}\left(a_1, a_1^{-1}, \ldots, a_r, a_r^{-1}\right)$. *This is done in the following theorem.*

Theorem 4 $T_i^1(g) = Q_i^1(g)$ *for all* i*, where* $g = \operatorname{diag}\left(a_1, a_1^{-1}, \ldots, a_r, a_r^{-1}\right)$.

Proof

Since Q_i^1 is irreducible for all i, we can write

$$Q_i^1 = \mathrm{ch}\left(V(i\omega_1)\right) = \frac{\sum_{w \in W(D_r)} \mathrm{sgn}(w) w \left(a_1^{r-1+i} a_2^{r-2} \dots a_{r-1}\right)}{Q_0},$$

by the Weyl character formula. Then clearly $Q_i^1 = 1$ for $i = 0$ and $-2r+2$, and $Q_i^1 = 0$ for $i = -1, -2, \dots, -2r+3$.

We already know that $T_{i+1}^1 = \mathcal{M} T_i^1$, where $\mathcal{M}$ is a $2r \times 2r$ matrix. Hence, if $T_i^1 = \mathrm{ch}\left(V(i\omega_1)\right)$ is true for any $2r$ values of i, then $T_i^1 = \mathrm{ch}\left(V(i\omega_1)\right)$ must be true for all i.

Notice that the recursion relation (3.22) is true for $n = 1, 0, -1, \dots, -2r+2$ if one puts $T_0^1 = T_{-2r+2}^1 = 1$, $R_0 = R_{-1} = 0$, and $T_{-j}^1 = 0$ for $j = 1, \dots, 2r-3$.

Then clearly for the $2r-1$ values $i = 0, \dots, -2r+2$ we have $Q_i^1(g) = T_i^1(g)$. We also know from the previous theorem that $T_1^1 = Q_1^1$. Hence $Q_i^1(g) = T_i^1(g)$ for all i. This proves the theorem.

3.4 Solving the Equations of the (D_m, A_n) Model

In this section we study the integrable model described by the pair of Dynkin diagrams (D_m, A_n). Using the representation theory of Yangians we solve the equations of the model.

3.4.1 Equations of the model

The equations of the model (D_m, A_n) are $AU = V$, where $A = C(D_m)^{-1} \otimes C(A_n)$, $U = \log(x)$, $V = \log(1-x)$, and $x = (x_{11}, \ldots, x_{mn})$. By exponentiation and a change of variables they can be rewritten as

$$z^{2-C(D_m)} + z^{2-C(A_n)} = z^2 \ . \tag{3.24}$$

We seek a matrix $g \in SO(2m)$ whose Yangian characters satisfy $Q^i_{n+1}(g) = 1$ for $i = 1, 2, \ldots, m$. This leads to the relation

$$z_{ij} = Q^i_j(g) \ ,$$

when $z_{1j} = Q^1_j(g)$ as above. Here z_{ij} are the components of the solutions of (3.24). Write

$$g = \mathrm{diag}(a_1, a_1^{-1}, \ldots, a_m, a_m^{-1}) \in SO(2m) \ .$$

Theorem 5 *Suppose $g \in SO(2m)$ is a matrix that satisfies*

$$Q^i_{n+1}(g) = 1 \ , \tag{3.25}$$

for $i = 1, 2, \ldots, m$, and moreover $Q^i_j(g) \neq 0$ for $j = 1, \ldots, n$. Then g also satisfies *the set of equations*

$$\begin{aligned} Q^i_{n+2}(g) &= 0 \ , \\ Q^i_{n+3}(g) &= 0 \ , \\ &\vdots \\ Q^i_{n+2m-2}(g) &= 0 \ , \\ Q^i_{n+2m-1}(g) &= \pm 1 \ , \end{aligned} \tag{3.26}$$

for $i = 1, 2, \ldots, m$. *In particular*

$$Q^i_{n+2m-1} = \begin{cases} +1 & \text{for } i = 1, 2, \ldots, m-2 \ , \\ +1 & \text{for } i = m-1, m \ , \text{ if } m \equiv 0 \text{ or } 1 \pmod 4 \ , \\ -1 & \text{for } i = m-1, m \ , \text{ if } m \equiv 2 \text{ or } 3 \pmod 4 \ . \end{cases}$$

Proof

This has been proved by Nahm. The idea of the proof is the following. First one considers the variety of generic solutions of equations (3.10) with initial data given by (3.11) and its closure. This excludes solutions with an unwanted pattern of vanishing Q^i_j. Interchanging i and j in equations (3.10) allows to write down explicit algebraic relations between Q^1_j and Q^i_{n+2}. These yield $Q^1_{n+j} = 0$ for $j = 2, \ldots, m-1$. Then one considers an analogous algebraic variety of solutions of equations (3.10) without imposing initial data at $j = 0$. The character formula for Q^1_j is easily generalised to this larger variety. The algebraic equations remain valid and yield $dQ^1_{n+m} = 0$. When one uses the character formula for Q^1_{n+m} this implies $Q^1_{n+m+j} = Q^1_{n+m-j}$.

Lemma 1 *Any matrix $g \in SO(2m)$ that satisfies the equations*

$$\begin{aligned} Q^1_{n+1}(g) &= 1\,, \\ Q^1_{n+2}(g) &= 0\,, \\ Q^1_{n+3}(g) &= 0\,, \\ &\vdots \\ Q^1_{n+2m-2}(g) &= 0\,, \\ Q^1_{n+2m-1}(g) &= 1\,, \end{aligned} \tag{3.27}$$

and the two equations

$$Q^m_{n+1} = 1\,, \tag{3.28}$$

$$Q^m_{n+2m-1} = \pm 1\,, \tag{3.29}$$

also satisfies the set of equations (3.26).

Proof

This follows immediately from substitution of (3.27, 3.28, 3.29) in the Kirillov-Reshetikhin equations (3.10) for D_m.

One can conclude that the problem of finding a matrix g whose characters satisfy (3.25) is equivalent to the problem of finding a matrix g whose characters satisfy (3.27, 3.28, 3.29). We choose to work with the second set of conditions as these involve only those irreducible Yangian representations that remain irreducible as representations of the corresponding Lie algebra. This simplifies matters greatly. To find a matrix $g \in SO(2m)$ whose Yangian characters satisfy the equations (3.27, 3.28, 3.29), we first find a solution of the set of equations (3.27), and show that it is unique. We then show that the same solution satisfies (3.28) and (3.29) when square roots are chosen appropriately.

3.4.2 Some useful facts

Notation

Let $W(D_{m-1})[k]$ denote the Weyl group of D_{m-1} acting on the elements

$$\pm 1, \pm 2, \ldots, \pm(k-1), \pm(k+1), \ldots, \pm m\ .$$

Any element of $W(D_m)$ can be factorised uniquely as $\sigma_k \omega,\ k = 1, \ldots, m$, where $\sigma_1 = 1$, $\sigma_k = (1k)$ for $k \neq 1$, and $\omega \in W(D_{m-1})[k]$. Define the quantity A_k by

$$A_k = \sum_{w \in W(D_{m-1})[k]} \mathrm{sgn}(1k).\mathrm{sgn}(w).w\left((1k)(a_2^{m-2} a_3^{m-3} \ldots a_{m-2}^2 a_{m-1})\right)\ .$$

Lemma 2

$$A_k = (-1)^{k-1} \prod_{t=1}^{m} a_t^{2-m} \prod_{i<j} (a_i a_j - 1)(a_i - a_j)\ ,$$

where the first sum is taken over $t \neq k$, and the second is taken over $i, j \in \{1, \ldots, m\}$ with $i, j \neq k$.

Proof

For $k \neq 1$ we have

$$\begin{aligned}
A_k &= -\sum_{w \in W(D_{m-1})[k]} \operatorname{sgn}(w).w\left((1k)(a_2^{m-2}a_3^{m-3}\dots a_{m-2}^2 a_{m-1})\right) \\
&= -\sum_{w \in W(D_{m-1})[k]} \operatorname{sgn}(w).w\left((a_2^{m-2}a_3^{m-3}\dots a_{k-1}^{m-k+1}a_1^{m-k}a_{k+1}^{m-k-1}\dots a_{m-2}^2 a_{m-1})\right) \\
&= -(-1)^{k-2}\sum_{w \in W(D_{m-1})[k]} \operatorname{sgn}(w).w\left((a_1^{m-2}a_2^{m-3}\dots a_{k-1}^{m-k}a_{k+1}^{m-k-1}\dots a_{m-2}^2 a_{m-1})\right) \ .
\end{aligned}$$

This final expression in fact holds for all k (including $k = 1$). Up to a factor of $(-1)^{k-1}$ this is the Weyl denominator for $D_{m-1}[k]$. Therefore, by the multiplicative formula for the Weyl denominator we can write

$$\begin{aligned}
A_k &= (-1)^{k-1}(a_1^{2-m}a_2^{3-m}\dots a_{k-1}^{k-m}a_{k+1}^{1+k-m}\dots a_m^{-1})\prod_{i<j}(a_ia_j-1)(a_ia_j^{-1}-1) \\
&= (-1)^{k-1}\prod_{t=1}^{m} a_t^{2-m}\prod_{i<j}(a_ia_j-1)(a_i-a_j)\ ,
\end{aligned}$$

where the first sum is taken over $t \neq k$, and the second is taken over $i, j \in \{1, 2, \dots, m\}$ with $i, j \neq k$.

Lemma 3

$$A_1A_2\dots A_m = \pm\frac{\prod_{i>j}(a_i-a_j)^{m-2}(a_ia_j-1)^{m-2}}{\prod_{i=1}^{m} a_i^{(m-1)(m-2)}}\ .$$

Proof

$$\begin{aligned}
A_1A_2\dots A_m &= \left(\prod_{k=1}^{m}(-1)^{k-1}\right)\left(\prod_{i=1}^{m}a_i^{2-m}\right)^{m-1}\prod_{i<j}(a_ia_j-1)^{m-2}(a_i-a_j)^{m-2}\\
&= \left(\prod_{k=1}^{m}(-1)^{k-1}\right)\left(\frac{\prod_{i<j}(a_ia_j-1)^{m-2}(a_i-a_j)^{m-2}}{\prod_{i=1}^{m}a_i^{(m-1)(m-2)}}\right)\\
&= (-1)^{\frac{(m-1)m}{2}}\left(\frac{\prod_{i<j}(a_ia_j-1)^{m-2}(a_i-a_j)^{m-2}}{\prod_{i=1}^{m}a_i^{(m-1)(m-2)}}\right)\\
&= (-1)^{\frac{(m-1)m}{2}}\left(\frac{\prod_{i>j}(a_ia_j-1)^{m-2}(a_i-a_j)^{m-2}(-1)^{m-2}}{\prod_{i=1}^{m}a_i^{(m-1)(m-2)}}\right)\\
&= (-1)^{\frac{(m-1)m}{2}}\left((-1)^{m-2}\right)^{\frac{m(m-1)}{2}}\left(\frac{\prod_{i>j}(a_ia_j-1)^{m-2}(a_i-a_j)^{m-2}}{\prod_{i=1}^{m}a_i^{(m-1)(m-2)}}\right)\\
&= (-1)^{\frac{m(m-1)^2}{2}}\left(\frac{\prod_{i>j}(a_ia_j-1)^{m-2}(a_i-a_j)^{m-2}}{\prod_{i=1}^{m}a_i^{(m-1)(m-2)}}\right)\\
&= \pm\frac{\prod_{i>j}(a_ia_j-1)^{m-2}(a_i-a_j)^{m-2}}{\prod_{i=1}^{m}a_i^{(m-1)(m-2)}}\ .
\end{aligned}$$

Lemma 4

$$\prod_{i=1}^{m}a_i^{m-1}\begin{vmatrix}1 & \dots & \dots & 1\\ (a_1+a_1^{-1}) & \dots & \dots & (a_m+a_m^{-1})\\ \vdots & \vdots & \vdots & \vdots\\ (a_1^{m-1}+a_1^{1-m}) & \dots & \dots & (a_m^{m-1}+a_m^{1-m})\end{vmatrix} = \prod_{i>j}(a_i-a_j)(a_ia_j-1)\ .$$

Proof

Clearly both sides of this equation are polynomials. To prove that they are equivalent, we must show that their zeros coincide. This implies equality up to a constant, which we show to be 1. The proof is in three steps.

1. Show that every zero of the RHS is also a zero of the LHS,
2. Show that both sides of the equation have the same order,
3. Show that the constant is equal to 1.

The RHS has a zero when $a_i = a_j$ or when $a_i = a_j^{-1}$ for $i \neq j$. Both of these cases lead to rows i and j being equal in the determinant on the LHS, which implies the whole LHS is zero. Hence every zero of the RHS is also a zero of the LHS.

The left hand side can be rewritten as

$$LHS = \begin{vmatrix} a_1^{m-1} & \dots & \dots & a_m^{m-1} \\ a_1^m + a_1^{m-2} & \dots & \dots & a_m^m + a_m^{m-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_1^{2m-2}+1 & \dots & \dots & a_m^{2m-2}+1 \end{vmatrix} .$$

The term of highest order on the LHS is $a_1^{m-1}a_2^m \dots a_m^{2m-2}$. Hence the LHS has order

$$\sum_{k=m-1}^{2m-2} k = \sum_{k=1}^{2m-2} k - \sum_{k=1}^{m-2} k = \frac{3m(m-1)}{2} = 3\binom{m}{2} .$$

Now for the order of the RHS. There are $\binom{m}{2}$ terms of the form $(a_i - a_j)(a_i a_j - 1)$, and each has order three. Hence the RHS has order $3\binom{m}{2}$. Thus the LHS and the RHS have the same order. This proves that the two sides of the equation are equivalent up to a constant. To calculate this constant compare the coefficient of the term $a_1^{m-1}a_2^m a_3^{m+1}\dots a_{m-1}^{2m-3}a_m^{2m-2}$ on both sides of the equation. On the LHS this term occurs exactly once, coming from a product along the main diagonal of the matrix. On the RHS it arises from the product

$$\prod_{i>j}(a_i a_j)(a_i) = \left(\prod_{i=1}^{m} a_i^{m-1}\right)\left(a_2 a_3^2 \dots a_{m-1}^{m-2}a_m^{m-1}\right) = a_1^{m-1}a_2^{m}\dots a_m^{2m-2}\,.$$

Hence it occurs exactly once on both sides. If the coefficients on both sides agree for one term they must agree for every term, hence the constant is 1. This proves the theorem.

Lemma 5 *Let χ denote some character of a Lie algebra $\mathfrak{g}$, and let g be an element of some maximal torus. Suppose $w_1 \in W(\mathfrak{g})$ satisfies $sgn(w_1) = -1$ such that $\chi(w_1(g)) = \chi(g)$. Then $\sum_{w\in W} sgn(w)\chi(w(g)) = 0$.*

Proof

$$\begin{aligned}\sum_{w\in W}\mathrm{sgn}(w)\chi(w(g)) &= \sum_{w\in W}\mathrm{sgn}(ww_1)\chi(ww_1(g)) \\ &= -\sum_{w\in W}\mathrm{sgn}(w)\chi(w(g)) \\ &= 0\,.\end{aligned}$$

Lemma 6

$$\sum_{k=1}^{m}\left(a_k^{m-j}+a_k^{j-m}\right)A_k = 0$$

for $j = 2,\dots,m$.

Proof

For notational simplicity set $\xi_j^{m-i} = (a_j^{m-i} + a_j^{-m+i})$. Then A_k can be written as

$$A_k = \operatorname{sgn}(1k) \sum_{S_{m-1}} \operatorname{sgn}(w)\, w\left((1k)\xi_2^{m-2}\xi_3^{m-3}\ldots\xi_{m-1}^{1}\right) .$$

Then for $j = 2, \ldots, m$ we get the following:

$$\begin{aligned}
& \sum_{k=1}^{m} \left(a_k^{m-j} + a_k^{-m+j}\right) A_k \\
= & \sum_{k=1}^{m} \xi_k^{m-j} A_k \\
= & \ \xi_1^{m-j} A_1 + \sum_{k=2}^{m} \xi_k^{m-j} A_k \\
= & \sum_{S_{m-1}} \operatorname{sgn}(w)\xi_1^{m-j} w\left(\xi_2^{m-2}\xi_3^{m-3}\ldots\xi_{m-1}^{1}\right) \\
- & \sum_{k=2}^{m} \left(\sum_{S_{m-1}} \operatorname{sgn}(w)\xi_k^{m-j} w\left(\xi_2^{m-2}\ldots\xi_{k-1}^{m-k+1}\xi_1^{m-k}\xi_{k+1}^{m-k-1}\ldots\xi_{m-1}^{1}\right) \right) \\
= & \sum_{S_m} \operatorname{sgn}(w) w\left(\xi_1^{m-j}\xi_2^{m-2}\ldots\xi_j^{m-j}\ldots\xi_k^{m-k}\ldots\right) \\
= & \ 0 .
\end{aligned}$$

3.4.3 Solution of the equations

By theorem 5 we need a matrix $g = \mathrm{diag}(a_1, a_1^{-1}, \ldots, a_m, a_m^{-1}) \in SO(2m)$ whose characters satisfy

$$
\begin{aligned}
Q^1_{n+1}(g) &= 1\ , \\
Q^1_{n+j}(g) &= 0\ ,\ \text{for } j = 2, \ldots, m-1\ , \\
Q^1_{n+m}(g) &= 0\ , \\
Q^1_{n+2m-j}(g) &= 0\ ,\ \text{for } j = 2, \ldots, m-1\ , \\
Q^1_{n+2m-1}(g) &= 1\ .
\end{aligned}
$$

Using the Weyl character formula these can be rewritten as follows

$$\begin{aligned}
& Q^1_{n+1}(g) = 1 \\
\Leftrightarrow \quad & \sum_{i=1}^{m} (a_i^{n+m} + a_i^{-n-m}) A_i = \sum_{i=1}^{m} (a_i^{m-1} + a_i^{1-m}) A_i \\
\Leftrightarrow \quad & \sum_{i=1}^{m} \left((a_i^{n+2m-1} - 1) a_i^{1-m} + (a_i^{-n-2m+1} - 1) a_i^{m-1} \right) A_i = 0 \; ,
\end{aligned}$$

$$\begin{aligned}
& Q^1_{n+j}(g) = 0 \qquad j = 2, \ldots, m-1 \\
\Leftrightarrow \quad & \sum_{i=1}^{m} (a_i^{n+m+j-1} + a_i^{-n-m-j+1}) A_i = 0 \\
\Leftrightarrow \quad & \sum_{i=1}^{m} \left((a_i^{n+2m-1} - 1) a_i^{j-m} + (a_i^{-n-2m+1} - 1) a_i^{m-j} \right) A_i = 0 \; ,
\end{aligned}$$

$$\begin{aligned}
& Q^1_{n+m}(g) = 0 \\
\Leftrightarrow \quad & \sum_{i=1}^{m} (a_i^{n+2m-1} + a_i^{-n-2m+1}) A_i = 0 \\
\Leftrightarrow \quad & \sum_{i=1}^{m} (a_i^{n+2m-1} - 2 + a_i^{-n-2m+1}) A_i = 0 \; ,
\end{aligned}$$

$$\begin{aligned}
& Q^1_{n+2m-j}(g) = 0 \qquad j = 2, \ldots, m-1 \\
\Leftrightarrow \quad & \sum_{i=1}^{m} (a_i^{n+3m-j-1} + a_i^{-n-3m+j+1}) A_i = 0 \\
\Leftrightarrow \quad & \sum_{i=1}^{m} \left((a_i^{n+2m-1} - 1) a_i^{m-j} + (a_i^{-n-2m+1} - 1) a_i^{j-m} \right) A_i = 0 \; ,
\end{aligned}$$

$$\begin{aligned}
& Q^1_{n+2m-1}(g) = 1 \\
\Leftrightarrow \quad & \sum_{i=1}^{m} (a_i^{n+3m-2} + a_i^{-n-3m+2}) A_i = \sum_{i=1}^{m} (a_i^{m-1} + a_i^{1-m}) A_i \\
\Leftrightarrow \quad & \sum_{i=1}^{m} \left((a_i^{n+2m-1} - 1) a_i^{m-1} + (a_i^{-n-2m+1} - 1) a_i^{1-m} \right) A_i = 0 \; .
\end{aligned}$$

Note: In obtaining the above equations we used Lemma 6.

We now have a system of $2m-1$ equations in $2m$ variables. The variables are a_i and a_i^{-1} with $i=1,\ldots,m$. By summing the equations for Q^1_{n+j} and Q^1_{n+2m-j} for $j=1,\ldots,m-1$, we get a system of m equations in m variables. This time the variables are $a_i^{n+2m-1}+a_i^{-n-2m+1}$ for $i=1,\ldots,m$. These equations can be solved exactly. They can be written in matrix form as

$$\begin{pmatrix} A_1 & \ldots & \ldots & A_m \\ (a_1+a_1^{-1})A_1 & \ldots & \ldots & (a_m+a_m^{-1})A_m \\ \vdots & \vdots & \vdots & \vdots \\ (a_1^{m-1}+a_1^{1-m})A_1 & \ldots & \ldots & (a_m^{m-1}+a_m^{1-m})A_m \end{pmatrix} \begin{pmatrix} a_1^{n+2m-1}-2+a_1^{-n-2m+1} \\ a_2^{n+2m-1}-2+a_2^{-n-2m+1} \\ \vdots \\ a_m^{n+2m-1}-2+a_m^{-n-2m+1} \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \tag{3.30}$$

For $i=1,\ldots,m$ this implies

$$\begin{aligned} a_i^{n+2m-1}-2+a_i^{-n-2m+1} &= 0 \\ a_i^{-n-2m+1}\left(a_i^{n+2m-1}-1\right)^2 &= 0 \\ a_i^{n+2m-1} &= 1 \end{aligned}$$

since $a_i \neq 0$.

Theorem 6 *Any matrix* $g = \mathrm{diag}(a_1, a_1^{-1}, \ldots, a_m, a_m^{-1}) \in SO(2m)$*, whose entries satisfy*

$$a_i^{n+2m-1} = 1 \;, \tag{3.31}$$

for $i = 1, \ldots, m$*, is a solution of the equations (3.27).*

Proof

$$\begin{aligned}
a_i^{n+2m-1} = 1 \quad &\Rightarrow \quad a_i^{n+m} = a_i^{1-m} \\
&\Rightarrow \quad \textstyle\sum_{w \in W(D_m)} \mathrm{sgn}(w).w(a_1^{n+m} a_2^{m-2} \ldots a_{m-1}) \\
&= \quad \textstyle\sum_{w \in W(D_m)} \mathrm{sgn}(w).w(a_1^{1-m} a_2^{m-2} \ldots a_{m-1}) \\
&= \quad \textstyle\sum_{w \in W(D_m)} \mathrm{sgn}(w).w(w_1(a_1^{1-m} a_2^{m-2} \ldots a_{m-1})) \\
& \text{where } w_1 \text{ is the identity with signs } (- + \ldots + -) \\
&= \quad \textstyle\sum_{w \in W(D_m)} \mathrm{sgn}(w).w(a_1^{m-1} a_2^{m-2} \ldots a_{m-1}) \\
&\Rightarrow \quad Q^1_{n+1}(g) = 1 \;.
\end{aligned}$$

$$\begin{aligned}
a_i^{n+2m-1} = 1 \quad &\Rightarrow \quad a_i^{n+3m-2} = a_i^{m-1} \\
&\Rightarrow \quad \textstyle\sum_{w \in W(D_m)} \mathrm{sgn}(w).w(a_1^{n+3m-2} a_2^{m-2} \ldots a_{m-1}) \\
&= \quad \textstyle\sum_{w \in W(D_m)} \mathrm{sgn}(w).w(a_1^{m-1} a_2^{m-2} \ldots a_{m-1}) \\
&\Rightarrow \quad Q^1_{n+2m-1}(g) = 1 \;.
\end{aligned}$$

$$
\begin{aligned}
a_i^{n+2m-1} = 1 \;\; &\Rightarrow \;\; a_i^{n+m+j-1} = a_i^{j-m} \\
&\Rightarrow \;\; \textstyle\sum_{w \in W(D_m)} \mathrm{sgn}(w).w(a_1^{n+m+j-1} a_2^{m-2} \ldots a_{m-1}) \\
&= \;\; \textstyle\sum_{w \in W(D_m)} \mathrm{sgn}(w).w(a_1^{-m+j} a_2^{m-2} \ldots a_{m-1}) \\
&= \;\; 0\ , \text{ by Lemma 5 using } w_1 = (1j) \\
&\Rightarrow \;\; Q^1_{n+j}(g) = 0 \quad \text{for} \quad j = 2, \ldots, m-1\ .
\end{aligned}
$$

$$
\begin{aligned}
a_i^{n+2m-1} = 1 \;\; &\Rightarrow \;\; a_i^{n+3m-j-1} = a_i^{m-j} \\
&\Rightarrow \;\; \textstyle\sum_{w \in W(D_m)} \mathrm{sgn}(w).w(a_1^{n+3m-j-1} a_2^{m-2} \ldots a_{m-1}) \\
&= \;\; \textstyle\sum_{w \in W(D_m)} \mathrm{sgn}(w).w(a_1^{m-j} a_2^{m-2} \ldots a_{m-1}) \\
&= \;\; 0\ , \text{ by Lemma 5 using } w_1 = (1j) \\
&\Rightarrow \;\; Q^1_{n+2m-j}(g) = 0 \quad \text{for} \quad j = 2, \ldots, m-1\ .
\end{aligned}
$$

$$
\begin{aligned}
a_i^{n+2m-1} = 1 \;\; &\Rightarrow \;\; \textstyle\sum_{w \in W(D_m)} \mathrm{sgn}(w).w(a_1^{n+2m-1} a_2^{m-2} \ldots a_{m-1}) \\
&= \;\; \textstyle\sum_{w \in W(D_m)} \mathrm{sgn}(w).w(a_2^{m-2} \ldots a_{m-1}) \\
&= \;\; 0\ , \text{ by Lemma 5 using } w_1 = (1m) \\
&\Rightarrow \;\; Q^1_{n+m}(g) = 0\ .
\end{aligned}
$$

Theorem 7 *Condition (3.31) is necessary for solutions of the* (D_m, A_n) *equations, provided* $a_i \neq a_j^{\pm 1}$ *for* $i, j = 1, 2, \ldots, m$ *and* $i \neq j$.

Proof

The solution is unique if and only if the determinant of the matrix in equation (3.30) is non-zero.

$$
\begin{aligned}
\det &= \begin{vmatrix} A_1 & \dots & \dots & A_m \\ (a_1 + a_1^{-1})A_1 & \dots & \dots & (a_m + a_m^{-1})A_m \\ \vdots & \vdots & \vdots & \vdots \\ (a_1^{m-1} + a_1^{1-m})A_1 & \dots & \dots & (a_m^{m-1} + a_m^{1-m})A_m \end{vmatrix} \\
&= \prod_{i=1}^{m} A_i \begin{vmatrix} 1 & \dots & \dots & 1 \\ (a_1 + a_1^{-1}) & \dots & \dots & (a_m + a_m^{-1}) \\ \vdots & \vdots & \vdots & \vdots \\ (a_1^{m-1} + a_1^{1-m}) & \dots & \dots & (a_m^{m-1} + a_m^{1-m}) \end{vmatrix} \\
&= \prod_{i=1}^{m} A_i \left(\frac{\prod_{i>j}(a_i - a_j)(a_i a_j - 1)}{\prod_{i=1}^{m} a_i^{m-1}} \right) \\
&= \left(\pm \frac{\prod_{i>j}(a_i - a_j)^{m-2}(a_i a_j - 1)^{m-2}}{\prod_{i=1}^{m} a_i^{(m-1)(m-2)}} \right) \left(\frac{\prod_{i>j}(a_i - a_j)(a_i a_j - 1)}{\prod_{i=1}^{m} a_i^{m-1}} \right)
\end{aligned}
$$

This is non-zero provided $a_i \neq a_j$ and $a_i \neq a_j^{-1}$ for $i \neq j$.

Remark: Nahm's proof of theorem 5 also implies that $a_i \neq a_j^{\pm 1}$ *is necessary for* $i \neq j$.

We now show that this particular choice of the matrix g also satisfies the equations (3.28, 3.29).

Theorem 8 *A matrix* $g = \mathrm{diag}(a_1, a_1^{-1}, \dots, a_m, a_m^{-1}) \in SO(2m)$*, whose entries satisfy* $a_i^{n+2m-1} = 1$ *for* $i = 1, 2, \dots, m$*, is a solution of the equation* $Q_{n+1}^{m}(g) = 1$*,* *provided square roots of the a_i are chosen to satisfy*

$$\prod_{i=1}^{m} a_i^{\frac{n+2m-1}{2}} = \begin{cases} +1 & \textit{if } m \equiv 0 \pmod 4 \ \textit{ or } \ m \equiv 1 \pmod 4 \,, \\ -1 & \textit{if } m \equiv 2 \pmod 4 \ \textit{ or } \ m \equiv 3 \pmod 4 \,. \end{cases} \tag{3.32}$$

Proof

$$Q^m_{n+1}(g) = 1$$

$$\begin{aligned}
\Leftrightarrow \quad & \sum_{w \in W(D_m)} \operatorname{sgn}(w).w\left(a_1^{\frac{n+2m-1}{2}} a_2^{\frac{n+2m-1}{2}-1} \ldots a_{m-1}^{\frac{n+2m-1}{2}-(m-2)} a_m^{\frac{n+2m-1}{2}-(m-1)}\right) \\
= \quad & \sum_{w \in W(D_m)} \operatorname{sgn}(w).w\left(a_1^{m-1} a_2^{m-2} \ldots a_{m-2}^{2} a_{m-1}\right)
\end{aligned}$$

$$\begin{aligned}
\Leftrightarrow \quad & \sum_{w \in W(D_m)} \operatorname{sgn}(w).w\left(a_1^{\frac{n+2m-1}{2}} \ldots a_m^{\frac{n+2m-1}{2}}\right).w\left(a_2^{-1} a_3^{-2} \ldots a_{m-1}^{-m+2} a_m^{-m+1}\right) \\
= \quad & \sum_{w \in W(D_m)} \operatorname{sgn}(w).w\left(a_1^{m-1} a_2^{m-2} \ldots a_{m-2}^{2} a_{m-1}\right)
\end{aligned}$$

$$\begin{aligned}
\Leftrightarrow \quad & a_1^{\frac{n+2m-1}{2}} \ldots a_m^{\frac{n+2m-1}{2}} \sum_{w \in W(D_m)} \operatorname{sgn}(w).w\left(a_2^{-1} a_3^{-2} \ldots a_{m-1}^{-m+2} a_m^{-m+1}\right) \\
= \quad & \sum_{w \in W(D_m)} \operatorname{sgn}(w).w\left(a_1^{m-1} a_2^{m-2} \ldots a_{m-2}^{2} a_{m-1}\right)
\end{aligned}$$

$$\begin{aligned}
\Leftrightarrow \quad & a_1^{\frac{n+2m-1}{2}} \ldots a_m^{\frac{n+2m-1}{2}} \sum_{w \in W(D_m)} \operatorname{sgn}(w).w\left(a_2^{-1} a_3^{-2} \ldots a_{m-1}^{-m+2} a_m^{-m+1}\right) \\
= \quad & \sum_{w \in W(D_m)} \operatorname{sgn}(w).\operatorname{sgn}(\hat{w}).w\left(a_m^{m-1} a_{m-1}^{m-2} \ldots a_3^{2} a_2\right) \ ,
\end{aligned}$$

where

$$\hat{w} = \begin{cases} (1m)(2,m-1)\dots(\frac{m}{2},\frac{m}{2}+1) & \text{for } m \text{ even }, \\ (1m)(2,m-1)\dots(\frac{m-1}{2},\frac{m+1}{2}) & \text{for } m \text{ odd }. \end{cases} \tag{3.33}$$

Clearly for the equation $Q^m_{n+1}(g) = 1$ to be satisfied, we must choose the signs $\pm\sqrt{a_i}$ such that the equation

$$\prod_{i=1}^{m} a_i^{\frac{n+2m-1}{2}} = sgn(\hat{w})$$

is satisfied. This amounts to choosing square root signs that satisfy

$$\prod_{i=1}^{m} a_i^{\frac{n+2m-1}{2}} = \begin{cases} +1 & \text{if } m \equiv 0 \pmod 4 \text{ or } m \equiv 1 \pmod 4 , \\ -1 & \text{if } m \equiv 2 \pmod 4 \text{ or } m \equiv 3 \pmod 4 . \end{cases} \tag{3.34}$$

Theorem 9 *Let $g = \mathrm{diag}(a_1, a_1^{-1}, \dots, a_m, a_m^{-1})$ be a matrix whose entries satisfy the conditions (3.31) and (3.34). Then g satisfies the equation $Q^m_{n+2m-1} = \pm 1$.*

Proof

The character $Q^m_{n+2m-1}(g)$ has numerator

$$\begin{aligned} & \sum_{w \in W(D_m)} \mathrm{sgn}(w).w\left(\prod_{i=1}^{m} a_i^{\frac{n+2m-1}{2}}\right).w\left(a_1^{m-1}a_2^{m-2}\dots a_{m-2}^2 a_{m-1}\right) \\ = \ & \pm \sum_{w \in W(D_m)} \mathrm{sgn}(w).w\left(a_1^{m-1}a_2^{m-2}\dots a_{m-2}^2 a_{m-1}\right) \\ = \ & \pm\, Q_0 \ , \end{aligned}$$

where Q_0 denotes the Weyl denominator. Hence by (3.34)

$$Q^m_{n+2m-1}(g) = \begin{cases} +1 & \text{if } m \equiv 0 \,(\mathrm{mod}\, 4) \text{ or } m \equiv 1 \,(\mathrm{mod}\, 4) \;, \\ -1 & \text{if } m \equiv 2 \,(\mathrm{mod}\, 4) \text{ or } m \equiv 3 \,(\mathrm{mod}\, 4) \;. \end{cases}$$

3.5 Effective Central Charge Calculations

Given a model described by a pair of Dynkin diagrams, we can calculate the effective central charge of the corresponding conformal field theory using the dilogarithm formulae. In this section we carry out a detailed example of this calculation for the (D_3, A_2) case, and we summarise the results of such calculations for numerous other cases.

3.5.1 Detailed example

To show how these calculations work we look at one particular example in more detail. Consider the model described by the pair of Dynkin diagrams (D_3, A_2). We solve the equations $AU = V$ and use the solutions to calculate the effective central charge (and other values of $c - 24h$) of the corresponding conformal field theory.

The equations of this model are $AU = V$ where $A = C(D_3)^{-1} \otimes C(A_2)$, $U = \log(x)$, $V = \log(1-x)$, and $x = (x_{11}, x_{12}, x_{21}, x_{22}, x_{31}, x_{32})$. Exponentiation leads to a set of algebraic equations

$$x^A = 1 - x \; . \tag{3.35}$$

By the change of variable

$$x = z^{-C(D_3)\otimes I_2} \ , \tag{3.36}$$

the equations (3.35) can be written as

$$z^{(2-C(D_3))\otimes I_2} + z^{I_3\otimes(2-C(A_2))} = z^2 \ ,$$

or more explicitly as

$$(z_{ij})^2 = z_{ij^\star} + \begin{cases} z_{2j}z_{3j} & \text{if } i = 1 \ , \\ z_{1j} & \text{if } i = 2, 3 \ , \end{cases} \tag{3.37}$$

where $z = (z_{11}, z_{12}, z_{21}, z_{22}, z_{31}, z_{32})$. Here $\star$ denotes interchanging the indices 1 and 2 (i.e. $j^\star = 3 - j$). The first step is to solve these equations. The z_{ij} are the characters $Q^i_j(g)$ of representations of the Yangian $Y(D_3)$, for some specially chosen matrix $g \in SO(6)$.

Solution

Choose the matrix $g = \text{diag}(a_1, a_1^{-1}, a_2, a_2^{-1}, a_3, a_3^{-1}) \in SO(6)$ so that the entries satisfy

$$a_i^7 = 1, \ \text{with } a_i \neq a_j^{\pm 1} \text{ for } i \neq j \ .$$

As discussed earlier the square root signs $\pm\sqrt{a_i}$ must be carefully chosen in order to satisfy the equations $a_1^{\frac{7}{2}} a_2^{\frac{7}{2}} a_3^{\frac{7}{2}} = -1$. We rule out any choice of g that does not satisfy $Q^i_j(g) \neq 0$ for $j = 1, \ldots, n$ (these are the non-admissable solutions mentioned earlier). This results in four choices for the matrix g (up to permutation).

$$(a_1, a_2, a_3) = \begin{cases} \left(1, e^{\frac{2\pi i}{7}}, e^{\frac{4\pi i}{7}}\right) \ , \\ \left(1, e^{\frac{4\pi i}{7}}, e^{\frac{6\pi i}{7}}\right) \ , \\ \left(1, e^{\frac{2\pi i}{7}}, e^{\frac{6\pi i}{7}}\right) \ , \\ \left(e^{\frac{2\pi i}{7}}, e^{\frac{4\pi i}{7}}, e^{\frac{6\pi i}{7}}\right) \ . \end{cases}$$

Characters

The three fundamental Yangian characters are written in terms of entries of g as

$$\begin{aligned} Q_1^1(g) &= a_1 + a_2 + a_3 + a_1^{-1} + a_2^{-1} + a_3^{-1} \ , \\ Q_1^2(g) &= a_1^{-1/2} a_2^{-1/2} a_3^{-1/2} + a_1^{-1/2} a_2^{1/2} a_3^{1/2} + a_1^{1/2} a_2^{-1/2} a_3^{1/2} + a_1^{1/2} a_2^{1/2} a_3^{-1/2} \ , \\ Q_1^3(g) &= a_1^{1/2} a_2^{1/2} a_3^{1/2} + a_1^{-1/2} a_2^{-1/2} a_3^{1/2} + a_1^{1/2} a_2^{-1/2} a_3^{-1/2} + a_1^{-1/2} a_2^{1/2} a_3^{-1/2} \ . \end{aligned}$$

We use the Weyl character formula to compute these characters. The corresponding values of $Q_2^1(g)$, $Q_2^2(g)$ and $Q_2^3(g)$ can then be calculated using the Kirillov-Reshetikhin equations. The KR equations for D_3 are

$$\begin{aligned} (Q_j^1)^2 - Q_{j-1}^1 Q_{j+1}^1 &= Q_j^2 Q_j^3 \ , \\ (Q_j^2)^2 - Q_{j-1}^2 Q_{j+1}^2 &= Q_j^1 \ , \\ (Q_j^3)^2 - Q_{j-1}^3 Q_{j+1}^3 &= Q_j^1 \ . \end{aligned}$$

The identification $z_{ij} = Q_j^i(g)$ means that we have now found the solutions $z = (z_{11}, z_{12}, z_{21}, z_{22}, z_{31}, z_{32})$ of the equations $z^{(2-C(D_3))\otimes I_2} + z^{I_3 \otimes (2-C(A_2))} = z^2$.

We compute the values of x_{ij} using the relation

$$x = z^{-C(D_3)\otimes I_2} \quad \Rightarrow \quad \begin{cases} x_{11} &= z_{11}^{-2} z_{21} z_{31} \ , \\ x_{12} &= z_{12}^{-2} z_{22} z_{32} \ , \\ x_{21} &= z_{11} z_{21}^{-2} \ , \\ x_{22} &= z_{12} z_{22}^{-2} \ , \\ x_{31} &= z_{11} z_{31}^{-2} \ , \\ x_{32} &= z_{12} z_{32}^{-2} \ . \end{cases}$$

The logarithms of these solutions must be chosen so as to satisfy

$$\log(x) = (-C(D_3) \otimes I_2) \log(z) \ ,$$

which is equivalent to the equations

$$\begin{cases} u_{11} = \log(x_{11}) &= -2\log(z_{11}) + \log(z_{21}) + \log(z_{31}) \ , \\ u_{12} = \log(x_{12}) &= -2\log(z_{12}) + \log(z_{22}) + \log(z_{32}) \ , \\ u_{21} = \log(x_{21}) &= \log(z_{11}) - 2\log(z_{21}) \ , \\ u_{22} = \log(x_{22}) &= \log(z_{12}) - 2\log(z_{22}) \ , \\ u_{31} = \log(x_{31}) &= \log(z_{11}) - 2\log(z_{31}) \ , \\ u_{32} = \log(x_{32}) &= \log(z_{12}) - 2\log(z_{32}) \ , \end{cases}$$

and

$$\log(1 - x) = (-I_3 \otimes C(A_2)) \log(z) \ ,$$

which is equivalent to

$$\begin{cases} v_{11} = \log(1-x_{11}) &= -2\log(z_{11}) + \log(z_{12})\ , \\ v_{12} = \log(1-x_{12}) &= \log(z_{11}) - 2\log(z_{12})\ , \\ v_{21} = \log(1-x_{21}) &= -2\log(z_{21}) + \log(z_{22})\ , \\ v_{22} = \log(1-x_{22}) &= \log(z_{21}) - 2\log(z_{22})\ , \\ v_{31} = \log(1-x_{31}) &= -2\log(z_{31}) + \log(z_{32})\ , \\ v_{32} = \log(1-x_{32}) &= \log(z_{31}) - 2\log(z_{32})\ . \end{cases}$$

Then the pairs $(u_{jk}, v_{jk}) \equiv (\log(x_{jk}), \log(1-x_{jk}))$ are the solutions of the equations $AU = V$.

In this particular example there are four solutions of the equations $x^A = 1 - x$. We label these $x^i = (x^i_{11}, \ldots, x^i_{32})$ for $i = 0, 1, 2, 3$. We calculate the values of $c - 24h_i$ using the formula

$$c - 24h_i = \frac{6}{\pi^2} \sum_{jk=11,\ldots,32} L(u^i_{jk}, v^i_{jk})\ .$$

For any given model, the effective central charge, c_{eff}, is the value of $c - 24h_i$ arising from the unique solution x^0, whose components $x^0_{jk} \in \mathbb{R}$ all satisfy $0 < x^0_{jk} < 1$.

For the case (D_3, A_2) the four choices of the matrix g give rise to four different values of $c - 24h_i$. These are

$$\begin{aligned} g &= \left(1, e^{\frac{2\pi i}{7}}, e^{\frac{4\pi i}{7}}\right) &&\Rightarrow\ c_{\text{eff}} = \tfrac{24}{7}\ , \\ g &= \left(1, e^{\frac{4\pi i}{7}}, e^{\frac{6\pi i}{7}}\right) &&\Rightarrow\ c - 24h_1 = -\tfrac{72}{7} \mod 24\mathbb{Z}\ , \\ g &= \left(1, e^{\frac{2\pi i}{7}}, e^{\frac{6\pi i}{7}}\right) &&\Rightarrow\ c - 24h_2 = -\tfrac{120}{7} \mod 24\mathbb{Z}\ , \\ g &= \left(e^{\frac{2\pi i}{7}}, e^{\frac{4\pi i}{7}}, e^{\frac{6\pi i}{7}}\right) &&\Rightarrow\ c - 24h_3 = 0 \mod 24\mathbb{Z}\ . \end{aligned}$$

3.5.2 Effective central charge for other models

These calculations have been carried out for many different models. The results are summarised in the following table:

Pair	c_{eff}	$c - 24h_1$	$c - 24h_2$	$c - 24h_3$
(A_1, A_1)	$1/2$	-	-	-
(A_1, A_2)	$4/5$	$-44/5$	-	-
(A_1, A_3)	1	-	-	-
(A_1, A_4)	$8/7$	$32/7$	$-40/7$	-
(A_2, A_1)	$6/5$	$-54/5$	-	-
(A_2, A_2)	2	-	-	-
(A_2, A_3)	$18/7$	6	$162/7$	-
(A_2, A_4)	3	9	27	-9
(A_3, A_1)	2	-	-	-
(A_3, A_2)	$24/7$	0	$-72/7$	$-120/7$
(A_3, A_3)	$9/2$	-	-	-
(D_3, A_1)	2	-	-	-
(D_3, A_3)	$9/2$	-	-	-
(D_4, A_1)	3	-	-	-
(D_4, A_2)	$16/3$	$-32/3$	-16	$-307/3$
(D_4, A_3)	$36/5$	$24/5$	-	-

3.6 (D_m, A_n) as coset models

Consider the model described by the pair of Dynkin diagrams (X, Y). Its effective central charge is known or conjectured to be

$$c_{\text{eff}}(X,Y) = \frac{r(X)r(Y)h(X)}{h(X)+h(Y)} \ ,$$

where r denotes the rank, and h the dual Coxeter number of a Lie algebra. Then for the case (D_m, A_n) we expect the effective central charge to be

$$c_{\text{eff}}(D_m, A_n) = \frac{(2m-2)mn}{2m+n-1} \ .$$

Now consider the coset model

$$\frac{(D_m)_{n+1}}{u(1)^m} \ . \tag{3.38}$$

This model has central charge

$$\frac{(n+1)(2m^2-m)}{n+1+2m-2} - m = \frac{(2m-2)mn}{2m+n-1} \ .$$

Clearly this coincides with the value of $c_{\text{eff}}(D_m, A_n)$.

This is extremely good evidence to suggest that the model described by the Dynkin diagrams (D_m, A_n) is a unitary model described by the coset

$$\frac{(D_m)_{n+1}}{u(1)^m} \ .$$

In all cases checked so far, the h-values calculated from the dilog formulae (3.6, 3.7) for the model (D_m, A_n) also arise as h-values of the coset model (3.38).

Chapter 4

Nahm's Conjecture

In the previous chapter we studied a family of matrices related to pairs of Dynkin diagrams. Each matrix A had the special property that solutions of $A \log x = \log(1 - x)$ gave torsion elements of the Bloch group. This lead to a nice relationship with conformal field theory. In this chapter we consider a number of 2×2 matrices that don't fall into this Cartan matrix pattern but nevertheless yield torsion elements of the Bloch group, at least for the special 'minimal' solution. We show that these matrices are again related to conformal field theory.

4.1 Overview

A **q-hypergeometric series** is a series of the form $\sum_{n=0}^{\infty} A_n(q)$, where $A_0(q)$ is a rational function, and $A_n(q) = R(q, q^n)A_{n-1}(q)$ for all $n \geq 1$ for some rational function $R(x, y)$ with $\lim_{x\to 0} \lim_{y\to 0} R(x, y) = 0$.

The question of when a q-hypergeometric series is also modular is an interesting open question in mathematics. There are a handful of known examples of such series, the most famous ones being given by the Rogers-Ramanujan identities

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \prod_{n \equiv \pm 1 \bmod 5} \frac{1}{1-q^n} \qquad (|q|<1) ,$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_n} = \prod_{n \equiv \pm 2 \bmod 5} \frac{1}{1-q^n} \qquad (|q|<1) .$$

These are modular functions up to factors $q^{-1/60}$ and $q^{11/60}$ respectively.

In general, the problem of understanding the overlap between q-hypergeometric series and modular functions is still completely unsolved. Nahm's conjecture takes a first step towards tackling this problem by considering a special case involving r-fold hypergeometric series. (These are defined as above but with n running over $(\mathbb{Z}_{\geq 0})^r$ rather than $\mathbb{Z}_{\geq 0}$).

Let A be a positive definite symmetric $r \times r$ matrix, B a vector of length r, and C a scalar, all three with rational coefficients. Define a function $f_{A,B,C}$ by the r-fold q-hypergeometric series

$$f_{A,B,C}(z) = \sum_{n=(n_1,\ldots,n_r) \in (\mathbb{Z}_{\geq 0})^r} \frac{q^{\frac{1}{2}nAn^t+Bn+C}}{(q)_{n_1}\ldots(q)_{n_r}} ,$$

where $(q)_n = (1-q)(1-q^2)\ldots(1-q^n)$. We can ask the question, when is $f_{A,B,C}$ a modular function? Nahm's conjucture doesn't fully answer this question, but predicts which matrices A can occur.

Given any positive definite symmetric $r \times r$ matrix $A = (A_{ij})$ with real coefficients, we can consider the system of equations

$$x_i = \prod_{j=1}^{r} (1 - x_j)^{A_{ij}} \ . \tag{4.1}$$

This system has a finite number of solutions $x = (x_1, \ldots, x_r)$. Again the unique solution whose components are all real and between 0 and 1 is denoted by $x^0 = (x_1^0, \ldots, x_r^0)$.

Note: The related set of equations $1 - x_i = \prod_{j=1}^{r} {x_j}^{A_{ij}}$ corresponds to the set of equations (4.1) with A replaced by A^{-1}. There is a duality between these two cases. In particular the effective central charges are related by

$$c_{\text{eff}}(A) + c_{\text{eff}}(A^{-1}) = r \ .$$

Let F denote the number field $\mathbb{Q}(x_1, \ldots, x_r)$. Given any solution $x = (x_1, \ldots, x_r)$ of (4.1), define the element $\xi_x \in \mathbb{Z}(F)$ by $\xi_x = [x_1] + \ldots + [x_r]$. ξ_x in an element of the Bloch group $\mathcal{B}(F)$.

Conjecture 1 **(Nahm's Conjecture)** *Let A be a positive definite symmetric $r \times r$ matrix with rational coefficients. Then the following are equivalent:*

1. *The element ξ_x is a torsion element of $\mathcal{B}(F)$ for all solutions $x = (x_1, \ldots, x_r)$.*

2. *There exist* $B \in \mathbb{Q}^r$ *and* $C \in \mathbb{Q}$ *such that* $f_{A,B,C}(z)$ *is a modular function.*

For the case $r = 1$ this conjecture is proved in [27]. It is expected that modular functions $f_{A,B,C}$ that arise in this way are characters of certain rational conformal field theories. We show later that this is certainly true in at least one case.

4.2 Terhoeven's Matrices

In his PhD thesis [39], Michael Terhoeven looked for rational 2×2 matrices for which ξ_{x^0} was a torsion element of the Bloch group. To do this he carried out a systematic search of all 2×2 matrices of the form

$$A = \frac{1}{m} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in M_2(\mathbb{Q}) ,$$

with $a, b, c, m \leq 11$. Due to time constraints we study only nine of Terhoeven's matrices:

$$A = \begin{pmatrix} 11 & 9 \\ 9 & 8 \end{pmatrix} \quad A = \begin{pmatrix} 8 & 5 \\ 5 & 4 \end{pmatrix} \quad A = \begin{pmatrix} 4 & 3 \\ 3 & 3 \end{pmatrix}$$

$$A = \begin{pmatrix} 8 & 3 \\ 3 & 2 \end{pmatrix} \quad A = \tfrac{1}{2}\begin{pmatrix} 5 & 4 \\ 4 & 4 \end{pmatrix} \quad A = \tfrac{1}{3}\begin{pmatrix} 8 & 1 \\ 1 & 2 \end{pmatrix}$$

$$A = \tfrac{1}{9}\begin{pmatrix} 8 & 3 \\ 3 & 0 \end{pmatrix} \quad A = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix} \quad A = \tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

We chose to look at this particular subset of Terhoeven's matrices since less is known about them than about the remaining matrices. In particular, for his other matrices it was already known either that all solutions gave rise to torsion elements or that all solutions gave rise to non-torsion elements. These nine matrices are particularly interesting in that they give rise to a mixture of torsion and non-torsion elements.

A similar search carried out by Don Zagier, with $a, b, c, m \leq 100$, resulted in one further matrix, namely

$$A = \begin{pmatrix} 24 & 19 \\ 19 & 16 \end{pmatrix} .$$

We don't repeat the calculations in this case as they are more lengthy and it is already known [27] that this matrix doesn't satisfy the stronger condition of having all solutions of (4.1) being torsion.

Notice that the matrices

$$A = \frac{1}{9}\begin{pmatrix} 8 & 3 \\ 3 & 0 \end{pmatrix} \quad \text{and} \quad A = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} ,$$

are not positive definite. We have included the calculations carried out in these cases although it is not yet clear exactly how they fit with Nahm's conjecture.

For each of these matrices we solve the equations $x = (1-x)^A$, and in cases where ξ_x is a torsion element of the Bloch group for all solutions x, we calculate the values of B and C for which $f_{A,B,C}$ is modular.

Note: For simplicity we solve the equations $x^A = 1 - x$ instead of $x = (1-x)^A$ in some cases. To recover the desired solutions we need only replace x by $1 - x$.

4.2.1 Equations

$$\boxed{A = \begin{pmatrix} 11 & 9 \\ 9 & 8 \end{pmatrix}}$$

We get the following set of equations in two variables

$$\begin{aligned} 1 - x_1 &= x_1^{11} x_2^{9} \,, \\ 1 - x_2 &= x_1^{9} x_2^{8} \,. \end{aligned}$$

These reduce to the one-variable equation

$$x^{77}(x_1^8 - x_1^7 + 8x_1^6 - 9x_1^5 - x_1^4 + 2x_1^3 + 3x_1^2 - 3x_1 + 1)(x_1^4 - 8x_1^3 + 3x_1^2 + 2x_1 + 1) = 0\ .$$

ξ_x is not a torsion element of $\mathcal{B}(F)$ for all solutions x. Let x_1 be any solution of the equation

$$x_1^8 - x_1^7 + 8x_1^6 - 9x_1^5 - x_1^4 + 2x_1^3 + 3x_1^2 - 3x_1 + 1 = 0\ .$$

Then for all the corresponding pair (x_1, x_2) we have checked that $D(x_1) + D(x_2) \neq 0$ in each case. Hence ξ_x is not torsion for these particular solutions. The solution $x_1 = 0$ is again torsion, although somewhat trivial since it just leads to the pair $(x_1, x_2) = (0, 1)$ which gives $L(\xi_x) = L(1)$. Although we don't always refer to it specifically, this trivial solution appears in many of the cases below.

In the case where x_1 is a solutions of $x_1^4 - 8x_1^3 + 3x_1^2 + 2x_1 + 1 = 0$, we have checked numerically that the corresponding pair (x_1, x_2) satisfy $D(x_1) + D(x_2) = 0$. This suggests that ξ_x is torsion for these four solutions. This second factor has Galois group D_4.

$$\boxed{A = \begin{pmatrix} 8 & 5 \\ 5 & 4 \end{pmatrix}}$$

We get the following set of equations in two variables

$$\begin{aligned} 1 - x_1 &= x_1^8 x_2^5 \; , \\ 1 - x_2 &= x_1^5 x_2^4 \; . \end{aligned}$$

These reduce to the one-variable equation

$$x^{24}(x_1^4 - x_1^3 + 3x_1^2 - 3x_1 + 1)(x_1^4 + x_1^3 + 3x_1^2 - 3x_1 - 1) = 0 \; .$$

ξ_x is not a torsion element of $\mathcal{B}(F)$ for all of the solutions. In particular we showed that ξ_x is not torsion for solutions that annihilate the first factor (i.e. $D(\xi_x) \neq 0$). We checked numerically that solutions annihilating the second factor give rise to ξ_x that are torsion ($D(\xi_x) = 0$ numerically). This second factor has Galois group D_4.

$$\boxed{A = \begin{pmatrix} 4 & 3 \\ 3 & 3 \end{pmatrix}}$$

We get the following set of equations in two variables

$$\begin{aligned} 1 - x_1 &= x_1^4 x_2^3 \; , \\ 1 - x_2 &= x_1^3 x_2^3 \; . \end{aligned}$$

These reduce to the one-variable equation

$$x^8(4x_1^2 - 2x_1 - 1)(2x_1^2 - 2x_1 + 1) = 0 \; .$$

We checked numerically that ξ_x is a torsion element of $\mathcal{B}(F)$ for the two solutions that annihilate the first factor, and proved that ξ_x is not torsion for the remaining solutions. The first factor has Galois group S_2.

$$\boxed{A = \begin{pmatrix} 8 & 3 \\ 3 & 2 \end{pmatrix}}$$

We get the following set of equations in two variables

$$\begin{aligned} 1 - x_1 &= x_1^8 x_2^3 \ , \\ 1 - x_2 &= x_1^3 x_2^2 \ . \end{aligned}$$

These reduce to the one-variable equation

$$x^8(x_1^2 - x_1 + 1)^2(x_1^4 + 2x_1^3 + x_1^2 - 2x_1 - 1) = 0 \ .$$

We checked numerically that ξ_x is a torsion element of $\mathcal{B}(F)$ for the four solutions that annihilate the second factor. For the first factor, if x_1 is a root of $x_1^2 - x_1 + 1$, then $(x_1, x_2) = (x, x)$ and $(x, 1/x)$ solve the set of two-variable equations. The element (x, x) is not torsion since $2D(x) \neq 0$, but the element $(x, 1/x)$ seems to be torsion. The second factor has Galois group D_4.

$$\boxed{A = 1/2 \begin{pmatrix} 5 & 4 \\ 4 & 4 \end{pmatrix}}$$

We get the following set of equations in two variables

$$\begin{aligned} 1 - x_1 &= x_1^{5/2} x_2^2 \ , \\ 1 - x_2 &= x_1^2 x_2^2 \ . \end{aligned}$$

These reduce to the one-variable equation

$$x^{5/2}(1 - x_1 + 2x_1^2 - 2x_1^3 - x_1^{3/2} + x_1^{5/2} - x_1^{7/2}) = 0 \ .$$

Since solving the equations in this case requires a careful choice of square roots, it is perhaps more appropriate to consider this equation as a polynomial of degree 7 in the variable $y = x_1^{1/2}$. The resulting equation is

$$y^5(y^3 - y + 1)(y^4 + 2y^3 - y - 1) = 0 \ .$$

Again ξ_x is not torsion for all solutions. In particular ξ_x is not torsion for those solutions that annihilate the first factor, but (numerically) seems to be torsion for solutions that annihilate the second factor. The Galois group of the second factor is D_4.

$$\boxed{A = 1/3 \begin{pmatrix} 8 & 1 \\ 1 & 2 \end{pmatrix}}$$

We get the following set of equations in two variables

$$\begin{aligned} 1 - x_1 &= x_1^{8/3} x_2^{1/3} \ , \\ 1 - x_2 &= x_1^{1/3} x_2^{2/3} \ . \end{aligned}$$

These reduce to the one-variable equation

$$(x_1^2 - x_1 + 1)(x_1^6 + x_1^5 - 2x_1^3 + 2x_1 - 1) = 0 \ .$$

ξ_x is (numerically) a torsion element for six of the eight solutions, namely the six that annihilate the second factor. The Galois group of the second factor is $A_4 \times C_2$.

$$\boxed{A = 1/9 \begin{pmatrix} 8 & 3 \\ 3 & 0 \end{pmatrix}}$$

We get the following set of equations in two variables

$$\begin{aligned} 1 - x_1 &= x_1^{8/9} x_2^{1/3} \ , \\ 1 - x_2 &= x_1^{1/3} \ . \end{aligned}$$

These reduce to the one-variable equation

$$x_2(x_2^3 - 2x_2^2 + x_2 + 1)(x_2^4 - 6x_2^3 + 12x_2^2 - 9x_2 + 1) = 0 \ .$$

ξ_x is (numerically) a torsion element for four of the seven non-zero solutions, the four annihilating the second factor. This factor has Galois group D_4.

$$\boxed{A = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}}$$

We get the following set of equations in two variables

$$\begin{aligned} 1 - x_1 &= x_1^4 x_2 \,, \\ 1 - x_2 &= x_1 x_2 \,. \end{aligned}$$

These reduce to the one-variable equation

$$x_1^4 + x_1^2 - 1 = 0 \,,$$

whose Galois group is again D_4. We checked numerically that ξ_x is a torsion element of $\mathcal{B}(F)$ for all solutions.

$$\boxed{A = 1/2 \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}$$

We get the following set of equations in two variables

$$\begin{aligned} 1 - x_1 &= x_1^{1/2} x_2^{1/2} \,, \\ 1 - x_2 &= x_1^{1/2} \,. \end{aligned}$$

These reduce to the one-variable equation

$$x_2(x_2^3 - 5x_2^2 + 6x_2 - 1) = 0 \,.$$

ξ_x is a (numerically) torsion element of $\mathcal{B}(F)$ for all solutions. Ignoring the trivial factor (whose Galois group is S_1), this equation has Galois group A_3.

Note: All Galois group calculations were carried out using the programme Pari. This can be downloaded free from http://pari.math.u-bordeaux.fr/.

4.2.2 Torsion elements of the Bloch group

In the context of Nahm's conjecture, the two cases of greatest interest are $A = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}$ and $A = 1/2 \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. In these cases **all** solutions of the equations $x^A = 1-x$ yield torsion elements of the Bloch group, whereas for the other matrices A considered above, only some of the solutions yield torsion elements. Hence the conjecture suggests that, for these two values of A, there exist values $B \in \mathbb{Q}^2$ and $C \in \mathbb{Q}$ such that the function $f_{A,B,C}(z)$ is modular. We now proceed to calculate these B and C values.

Notice that the matrix $A = 1/2 \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ is not in fact positive definite. For this reason we are not entirely sure how it fits in with Nahm's conjecture. However, since all solutions yield torsion elements of the Bloch group, this case is clearly worth examining in more detail.

4.2.3 B and C values

Suppose A is an $r \times r$ matrix. Then in the individual terms defining $f_{A,B,C}$, the denominators involve finite sub-products of $\eta(z)^r$ where, (for $q = e^{2\pi i z}$),

$$\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \ .$$

Hence, for a 2×2 matrix A, we can expect $\eta(z)^2 f_{A,B,0}(z)$ to be a holomorphic modular form. For values of B giving rise to modular $f_{A,B,C}$, the series $\eta(z)^2 f_{A,B,0}(z)$ should have small coefficients, whereas for other B the coefficients will be much bigger. By considering the coefficients in this series for different values of B, it is possible to identify those B-values that give rise to modular $f_{A,B,C}$.

For example, consider the matrix $A = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}$. $B = (0, 1/2)$ leads to modular $f_{A,B,C}$, while $B = (0, 0)$ does not lead to modular $f_{A,B,C}$. In each case let us look at the coefficients of q^{25}, q^{50}, q^{75}, q^{100} in the series $\eta(z)^2 f_{A,B,0}(z)$ (divided by $q^{1/12}$ for simplicity). $B = (0, 1/2)$ gives $1q^{25}$, $1q^{50}$, $3q^{75}$, and $5q^{100}$, while $B = (0, 0)$ gives $-3q^{25}$, $-20q^{50}$, $171q^{75}$, and $110q^{100}$. Clearly the coefficients in the non-modular case grow significantly more quickly than those in the modular case.

Furthermore, as a useful check, we expect that any B-value that leads to modular $f_{A,B,0}$, should give rise to a rational value of C.

Given the vector $B = (B_1, B_2)$, C can then be calculated [39] using the equation:

$$\begin{aligned}
C \;=\; & \frac{\phi_2(B_i)}{2}\left(\frac{1-x_i}{x_i}\right) - \frac{1}{2}\phi_1(B_i)\left(\frac{1-x_i}{x_i}\right)T^{-1}_{ij}\phi_1(B_j)\left(\frac{1-x_j}{x_j}\right) \\
& + \frac{1}{2}\phi_1(B_i)\left(\frac{1-x_i}{(x_i)^2}\right)T^{-1}_{ij} - \frac{1}{2}\left(\frac{1-x_i}{(x_i)^2}\right)T^{-1}_{ii}T^{-1}_{ij}\phi_1(B_j)\left(\frac{1-x_j}{x_j}\right) \\
& + \frac{1}{8}\left(\frac{(1-x_i)(2-x_i)}{(x_i)^3}\right)(T_{ii})^{-1} \\
& - \frac{1}{12}\left(\frac{1-x_i}{(x_i)^2}\right)(T^{-1}_{ij})^3\left(\frac{1-x_j}{(x_j)^2}\right) \\
& - \frac{1}{8}\left(\frac{1-x_i}{(x_i)^2}\right)T^{-1}_{ii}T^{-1}_{ij}T^{-1}_{jj}\left(\frac{1-x_j}{(x_j)^2}\right) \; .
\end{aligned}$$

Here ϕ_i denotes the i^{th} Bernoulli polynomial, and the matrix $T = (T_{ij})$ is given by

$$T_{ij} = (A)^{-1}_{ij} + \delta_{ij}\left(\frac{1-x_i}{x_i}\right) \; .$$

For the matrix $A = (4, 1, 1, 1)$, we find that the values of B and C for which $f_{A,B,C}(z)$ is modular are

$$B = (0, 1/2) \; , \quad C = 1/120 \; ,$$
$$B = (2, 1/2) \; , \quad C = 49/120 \; .$$

These values agree with the calculations of Don Zagier [27].

For the matrix $A = 1/2(1, 1, 1, 0)$, the corresponding B and C values are

$$B = (0, 1/2) \; , \qquad C = -1/84 \; ,$$
$$B = (1/2, 1/2) \; , \quad C = 5/84 \; ,$$
$$B = (0, 1) \; , \qquad C = -1/21 \; .$$

4.2.4 $f_{A,B,C}$ as characters of rational CFTs

For the case $A = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}$ we calculate the functions $f_{A,B,C}$ explicitly. Up to order 6 these are

$$\begin{aligned} f_{A,\left(2,\frac{1}{2}\right),\frac{49}{120}} &= q^{\frac{49}{120}}\left(1+q+q^2+2q^3+3q^4+4q^5+6q^6+\ldots\right) , \\ f_{A,\left(0,\frac{1}{2}\right),\frac{1}{120}} &= q^{\frac{1}{120}}\left(1+q+2q^2+3q^3+4q^4+6q^5+8q^6+\ldots\right) . \end{aligned}$$

These expansions seem to be the same as the characters of the $(5,4)$-minimal model given on page 243 of [18]. This leads us to expect that the matrix $A = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}$ describes some integrable perturbation of the $(5,4)$-minimal model.

Recall that a perturbation of a CFT gives rise to a theory that, in general, is no longer conformally invariant. However, there are some cases in which the new theory turns out to be integrable, meaning that it has an infinite number of conservation laws, and hence a much greater chance of being solved exactly. So, what we assume above is that the matrix $A = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}$ describes some theory that arises as a perturbation of the $(5,4)$-minimal model. As mentioned earlier, the scattering matrix of any quantum field theory provides important insight into the theory itself. We expect that, using the matrix $A = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}$, it should be possible to construct the scattering matrix of the corresponding integrable theory. Having done this, we could then hope to be able to identify the particular integrable perturbation. It would be interesting to explore this connection in more detail at a later stage.

For the case $A = 1/2(1,1,1,0)$ the functions $f_{A,B,C}$ are more complicated. They are

$$\begin{aligned}
f_{A,\left(0,\frac{1}{2}\right),-\frac{1}{84}} &= q^{-\frac{1}{84}}\Big((1+2q+3q^2+6q^3+10q^4+16q^5+\ldots) \\
&\quad +q^{\frac{1}{4}}(1+2q+5q^2+8q^3+15q^4+24q^5+\ldots) \\
&\quad +q^{\frac{1}{2}}(1+2q+4q^2+7q^3+12q^4+19q^5+\ldots)\Big)\ , \\
f_{A,\left(\frac{1}{2},\frac{1}{2}\right),\frac{5}{84}} &= q^{\frac{5}{84}}\Big((1+q+3q^2+5q^3+8q^4+13q^5+\ldots) \\
&\quad +q^{\frac{1}{2}}(1+2q+3q^2+6q^3+10q^4+16q^5+\ldots) \\
&\quad +q^{\frac{3}{4}}(1+2q+4q^2+8q^3+13q^4+22q^5+\ldots)\Big)\ , \\
f_{A,(0,1),-\frac{1}{21}} &= q^{-\frac{1}{21}}\Big((1+2q+3q^2+6q^3+10q^4+16q^5+\ldots) \\
&\quad +q^{\frac{1}{4}}(1+q+2q^2+3q^3+5q^4+8q^5+\ldots) \\
&\quad +q^{\frac{3}{4}}(q+2q^2+3q^3+6q^4+9q^5+\ldots)\Big)\ .
\end{aligned}$$

These expansions seem to have some modular properties, and indeed the fact that the B-values gave rise to rational C-values is good evidence in favour of this theory. As yet we have not managed to find a conformal field theory with these characters, nor even managed to identify the expansions themselves as modular forms. However, we are optimistic that with more work these expansions can be identified as the characters of some conformal field theory.

4.2.5 Effective central charge calculations

We have already seen that, given a solution $x = (x_1, \ldots, x_r)$ of (4.1) for which ξ_x is a torsion element of the Bloch group, we can apply the mapping

$$\frac{6}{\pi^2} \sum_{i=1}^{r} L(x_i) = c - 24h \mod 24 .$$

This gives some information about the corresponding conformal field theory.

Since ξ_{x^0} is a torsion element of the Bloch group for each of Terhoeven's matrices (and their inverses), we can calculate the corresponding value of the effective central charge in each case. Furthermore we can calculate $c-24h$ (mod 24) for any solution x of (4.1) for which ξ_x is a torsion element. The results of some such calculations are summarised below.

A^{-1}	c_{eff}	$c - 24h_1$	$c - 24h_2$
$(4,1,1,1)^{-1}$	$7/10$	$127/10$	$343/10$
$1/2(1,1,1,0)$	$6/7$	$150/7$	$54/7$
$(11,9,9,8)^{-1}$	$3/10$	$27/10$	-
$(8,5,5,4)$	$8/5$	$32/5$	-
$(4,3,3,3)$	$3/2$	-	-
$(8,3,3,2)$	$3/2$	$27/2$	-
$1/2(5,4,4,4)$	$7/5$	$103/5$	-
$1/3(8,1,1,2)$	$8/7$	$128/7$	$116/7$
$1/9(8,3,3,0)$	$4/5$	$76/5$	-

4.2.6 Significance of the Galois group

The **Kronecker-Weber theorem** states that every finite abelian extension F of $\mathbb{Q}$ (i.e. every algebraic number field F whose Galois group over $\mathbb{Q}$ is abelian) is a subfield of a cyclotomic field.

For a given abelian extension F of $\mathbb{Q}$ there is in fact a minimal cyclotomic field that contains F. The **conductor** of F is defined to be the smallest integer n such that F lies inside the field generated by the n^{th} roots of unity. Hence we can say that $\mathrm{Gal}(F)$ is abelian if and only if F is generated by n^{th} roots of unity. We can conclude that if an equation has abelian Galois group, its solutions can be expressed in terms of roots of unity.

Now let G be a group and $\mathrm{com}(G)$ be its commutator subgroup. $G/\mathrm{com}(G)$ is abelian, so G is an extension of an abelian group by $\mathrm{com}(G)$. Of the cases considered so far, the pairs of Cartan matrices give rise to equations with abelian Galois groups, while the equations arising from Terhoeven's matrices have Galois groups that are abelian or have $\mathrm{com}(G) = \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. The pattern is not yet understood completely.

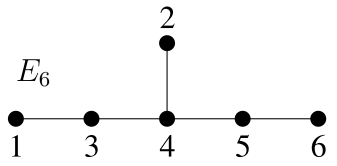
Chapter 5

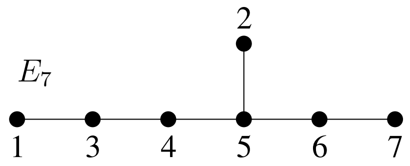
Integrable Models Described by Exceptional Lie Algebras

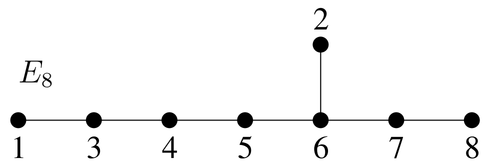
In this chapter we consider the integrable models described by pairs of Dynkin diagrams (E_m, T_1). There are three distinct models of this type, since m can take values 6, 7, or 8. For each model the equations are of the form $U = AV$, where the matrix A is given by $A = C(E_m) \otimes C(T_1)^{-1}$, $U = \log(x)$, $V = \log(1-x)$, and $x = (x_1, \ldots, x_m)$. In each case we solve the equations and use these solutions to compute the effective central charge of the corresponding conformal field theory. Since the matrices we deal with in this chapter are bigger than the 2×2 matrics considered in the previous chapter, the calculations are more difficult. We do not go as far as to calculate the explicit modular forms associated to each (E_m, T_1) theory, neither do we try to identify the CFTs themselves. Both of these would be interesting future projects.

5.1 Exceptional Lie Algebras

The E family of exceptional Lie algebras has three members, namely E_6, E_7, and E_8. Their Dynkin diagrams are







The corresponding Cartan matrices are

$$C(E_6) = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$C(E_7) = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$C(E_8) = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

Their Coxeter numbers are

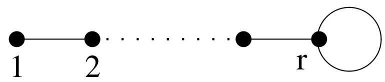
$$\begin{aligned} h(E_6) &= 12\,, \\ h(E_7) &= 18\,, \\ h(E_8) &= 30\,. \end{aligned}$$

5.2 Dynkin Diagrams T_r

The 'Tadpole' diagram, T_r, is got by folding the diagram A_{2r} in the middle, to get a pairwise identification of the vertices

$$T_r = A_{2r}/\mathbb{Z}_2 \ .$$

This gives the Dynkin diagram



The Cartan matrix of T_r is identical to that of A_r except for the entry $C(T_r)_{rr} = 1$. In particular for T_1 we get the 1×1 Cartan matrix

$$C(T_1) = 1 \ .$$

The Coxeter number of T_r is $h(T_r) = 2r + 1$.

5.3 Pairs of Dynkin Diagrams (E_m, T_1)

5.3.1 Solving the equations of the model

Case 1: $\mathbf{A} = \mathbf{A}(\mathbf{E_6}, \mathbf{T_1})$

Here we consider the pair of Dynkin diagrams $X = E_6$ and $Y = T_1$. The matrix $A = A(X, Y)$ is given by

$$A(E_6, T_1) = C(E_6) \otimes C(T_1)^{-1} = C(E_6)\ .$$

Exponentiating the equations $U = AV$ leads to the following set of algebraic equations:

$$\begin{aligned}
x_1 &= \frac{(1-x_1)^2}{1-x_3}\ , \\
x_2 &= \frac{(1-x_2)^2}{1-x_4}\ , \\
x_3 &= \frac{(1-x_3)^2}{(1-x_1)(1-x_4)}\ , \\
x_4 &= \frac{(1-x_4)^2}{(1-x_2)(1-x_3)(1-x_5)}\ , \\
x_5 &= \frac{(1-x_5)^2}{(1-x_4)(1-x_6)}\ , \\
x_6 &= \frac{(1-x_6)^2}{1-x_5}\ .
\end{aligned}$$

To solve these equations we write them in terms of the two variables x_6 and x_1. This leads to two possible solutions; either $x_6 = x_1$, and x_6 satisfies the equation

$$(5x_6^2 - 5x_6 + 1)(x_6 - 1)^3 = 0 \ , \tag{5.1}$$

or $x_6 \neq x_1$, and x_6 satisfies the following polynomial of degree 7

$$x_6(x_6^2 - 3x_6 + 1)(x_6 - 1)^4 = 0 \ . \tag{5.2}$$

We exclude any 'solutions' that imply $x_i = \infty$ for any $i = 1, \ldots, 6$ (this excludes all solutions arising from (5.2) and some of those arising from (5.1)). This leaves two distinct solutions given by

$$x_1 = \frac{1}{2} \pm \frac{\sqrt{5}}{10} \ . \tag{5.3}$$

The remaining x_i are given by

$$\begin{aligned} x_2 &= \frac{5}{2}x_1 - 1 \ , \\ x_3 &= 4x_1 - 2 \ , \\ x_4 &= 10 - \frac{25}{2}x_1 \ , \\ x_5 &= 4x_1 - 2 \ , \\ x_6 &= x_1 \ . \end{aligned}$$

Clearly these solutions reflect the symmetry of the E_6 Dynkin diagram.

More explicitly we have

$$
\begin{aligned}
x_1 = x_6 = \frac{1}{2} + \frac{\sqrt{5}}{10} \quad &\Rightarrow \quad x_2 = \frac{1+\sqrt{5}}{4} \\
&\Rightarrow \quad x_3 = \frac{2}{\sqrt{5}} = x_5 \\
&\Rightarrow \quad x_4 = \frac{15-5\sqrt{5}}{4} \ ,
\end{aligned}
$$

where $\sqrt{5}$ denotes either square root of 5.

Case 2: $\mathbf{A} = \mathbf{A}(\mathbf{E_7}, \mathbf{T_1})$

In this case $(X, Y) = (E_7, T_1)$. The matrix $A(X, Y)$ is given by

$$
A(E_7, T_1) = C(E_7) \otimes C(T_1)^{-1} = C(E_7) \ .
$$

We want to solve to following set of algebraic equations arising from the model

$$
\begin{aligned}
x_1 &= \frac{(1-x_1)^2}{1-x_3} \ , \\
x_2 &= \frac{(1-x_2)^2}{1-x_4} \ , \\
x_3 &= \frac{(1-x_3)^2}{(1-x_1)(1-x_4)} \ , \\
x_4 &= \frac{(1-x_4)^2}{(1-x_2)(1-x_3)(1-x_5)} \ , \\
x_5 &= \frac{(1-x_5)^2}{(1-x_4)(1-x_6)} \ , \\
x_6 &= \frac{(1-x_6)^2}{(1-x_5)(1-x_7)} \ , \\
x_7 &= \frac{(1-x_7)^2}{(1-x_6)} \ .
\end{aligned}
$$

As before, these equations can be solved analytically. We reduce the equations to a polynomial of degree 14 in x_7. This can be factorised using MAPLE. We then solve the equation

$$(2x_7-1)(3x_7^2-9x_7+5)(x_7^3-x_7^2-2x_7+1)(x_7^2-3x_7+1)(x_7-1)^6=0\ .$$

The solutions are given by

$$\begin{aligned} x_7 &= -\frac{3}{2}+\frac{\sqrt{21}}{2}\ , \\ x_7 &= -6\cos\left(\frac{2\pi}{7}\right)+2\cos\left(\frac{3\pi}{7}\right)+4\cos\left(\frac{\pi}{7}\right)\ , \end{aligned}$$

and their Galois conjugates (given explicitly in later calculations). Again we have excluded any solutions with $x_i=\infty$ for any $i=1,\ldots,7$.

Case 3: $\mathbf{A}=\mathbf{A}(\mathbf{E_8},\mathbf{T_1})$

In this case $(X,Y)=(E_8,T_1)$. The matrix A is given by

$$A(X,Y)=C(E_8)\otimes C(T_1)^{-1}=C(E_8)\ .$$

The algebraic equations obtained from the model are

$$
\begin{aligned}
x_1 &= \frac{(1-x_1)^2}{1-x_3} \ , \\
x_2 &= \frac{(1-x_2)^2}{1-x_4} \ , \\
x_3 &= \frac{(1-x_3)^2}{(1-x_1)(1-x_4)} \ , \\
x_4 &= \frac{(1-x_4)^2}{(1-x_2)(1-x_3)(1-x_5)} \ , \\
x_5 &= \frac{(1-x_5)^2}{(1-x_4)(1-x_6)} \ , \\
x_6 &= \frac{(1-x_6)^2}{(1-x_5)(1-x_7)} \ , \\
x_7 &= \frac{(1-x_7)^2}{(1-x_6)(1-x_8)} \ , \\
x_8 &= \frac{(1-x_8)^2}{(1-x_7)} \ .
\end{aligned}
$$

As in the previous case we get a polynomial of degree 17 in x_8. After factorisation this is

$$(2x_8-1)(x_8^3-2x_8^2-x_8+1)(x_8^5-3x_8^4-3x_8^3+4x_8^2+x_8-1)(x_8-1)^8=0 \ .$$

There relevant solutions are

$$x_8 = 4\cos\left(\frac{2\pi}{11}\right) - 2\cos\left(\frac{4\pi}{11}\right) - 2\cos\left(\frac{5\pi}{11}\right) - 2\cos\left(\frac{\pi}{11}\right) \ , \tag{5.4}$$

and its four algebraic conjugates, got by replacing $\cos(2n\pi/11)$ by $\cos(2an\pi/11)$ with a ranging over the quadratic residues modulo 11.

5.3.2 Effective central charge calculations

Each $x = (x_1, x_2, \ldots, x_m)$ satisfies the set of algebraic equations $x = (1-x)^A$. Logarithms of these solutions must be chosen so that they satisfy the original equations of the model, namely $\log(x) = A\log(1-x)$. To do this we define $\log(x)$ in terms of $\log(z)$, where $z = (z_1, \ldots, z_m)$ is the variable introduced earlier

$$\begin{aligned} 1 - x = z^{-C(Y)} \quad &\Rightarrow \quad v_i = \log(1-x) = -C(Y)\log(z) \ , \\ x = z^{-C(X)} \quad &\Rightarrow \quad u_i = \log(x) = -C(X)\log(z) \ . \end{aligned}$$

The effective central charge is calculated using the formula

$$c_{\text{eff}} = \frac{6}{\pi^2}\sum_{i=1}^{m} L(u_i^0, v_i^0) \ . \tag{5.5}$$

Here (u_i^0, v_i^0) corresponds to the special solution whose components satisfy $x_i \in \mathbb{R}$ and $0 < x_i < 1$ for all i.

Other values of $c - 24h$ can be calculated (modulo $24\mathbb{Z}$) from the remaining solutions by

$$c - 24h = \frac{6}{\pi^2}\sum_{i=1}^{m} L(u_i, v_i) \ . \tag{5.6}$$

Case 1: $A(E_6, T_1)$

The relations

$$1 - x = z^{-C(T_1)\otimes I_6} \qquad \text{and} \qquad x = z^{-C(E_6)} \ ,$$

give rise to the following set of equations for $\log(x_i)$ and $\log(1-x_i)$.

$$
\begin{aligned}
v_i = \log(1-x_i) &= -\log(z_i)\ ,\ \text{for } i = 1,\ldots,6\ , \\
&\text{and} \\
u_1 = \log(x_1) &= -2\log(z_1) + \log(z_3)\ , \\
u_2 = \log(x_2) &= -2\log(z_2) + \log(z_4)\ , \\
u_3 = \log(x_3) &= \log(z_1) - 2\log(z_3) + \log(z_4)\ , \\
u_4 = \log(x_4) &= \log(z_2) + \log(z_3) - 2\log(z_4) + \log(z_5)\ , \\
u_5 = \log(x_5) &= \log(z_4) - 2\log(z_5) + \log(z_6)\ , \\
u_6 = \log(x_6) &= \log(z_5) - 2\log(z_6)\ .
\end{aligned}
$$

This choice of logs must (and does) satisfy the following equations ($U = AV$).

$$
\begin{aligned}
\log(x_1) &= 2\log(1-x_1) - \log(1-x_3)\ , \\
\log(x_2) &= 2\log(1-x_2) - \log(1-x_4)\ , \\
\log(x_3) &= -\log(1-x_1) + 2\log(1-x_3) - \log(1-x_4)\ , \\
\log(x_4) &= -\log(1-x_2) - \log(1-x_3) + 2\log(1-x_4) - \log(1-x_5)\ , \\
\log(x_5) &= -\log(1-x_4) + 2\log(1-x_5) - \log(1-x_6)\ , \\
\log(x_6) &= -\log(1-x_5) + 2\log(1-x_6)\ .
\end{aligned}
$$

Substituting the two solutions (5.3) into the equations (5.5) and (5.6) gives the following results:

$$
\begin{aligned}
x_1 &= \frac{1}{2} - \frac{\sqrt{5}}{10} \quad \Rightarrow \quad c - 24h = -\frac{24}{5} \mod 24\mathbb{Z} \, , \\
x_1 &= \frac{1}{2} + \frac{\sqrt{5}}{10} \quad \Rightarrow \quad c_{\text{eff}} = \frac{24}{5} \, .
\end{aligned}
$$

Case 2: $A(E_7, T_1)$

Using the same method in this case gives the following results:

$$
\begin{aligned}
x_1 = -\frac{3}{2} + \frac{\sqrt{21}}{2} \quad &\Rightarrow \quad c_{\text{eff}} = 6 \, , \\
x_1 = -\frac{3}{2} - \frac{\sqrt{21}}{2} \quad &\Rightarrow \quad c - 24h = -18 \mod 24\mathbb{Z} \, , \\
x_1 = -6\cos\left(\frac{2\pi}{7}\right) + 2\cos\left(\frac{3\pi}{7}\right) + 4\cos\left(\frac{\pi}{7}\right) \quad &\Rightarrow \quad c - 24h = -\frac{30}{7} \mod 24\mathbb{Z} \, , \\
x_1 = -4\cos\left(\frac{2\pi}{7}\right) + 6\cos\left(\frac{3\pi}{7}\right) + 2\cos\left(\frac{\pi}{7}\right) \quad &\Rightarrow \quad c - 24h = -\frac{78}{7} \mod 24\mathbb{Z} \, , \\
x_1 = -2\cos\left(\frac{2\pi}{7}\right) + 4\cos\left(\frac{3\pi}{7}\right) + 6\cos\left(\frac{\pi}{7}\right) \quad &\Rightarrow \quad c - 24h = -\frac{102}{7} \mod 24\mathbb{Z} \, .
\end{aligned}
$$

Case 3: $A(E_8, T_1)$

Again the same calculation gives

$$
\begin{aligned}
& x_1 = 4\cos\left(\frac{2\pi}{11}\right) - 2\cos\left(\frac{4\pi}{11}\right) - 2\cos\left(\frac{5\pi}{11}\right) - 2\cos\left(\frac{\pi}{11}\right) \\
\Rightarrow \quad & c - 24h = -\frac{40}{11} \mod 24\mathbb{Z}\,,
\end{aligned}
$$

$$
\begin{aligned}
& x_1 = 2\cos\left(\frac{2\pi}{11}\right) + 4\cos\left(\frac{4\pi}{11}\right) + 2\cos\left(\frac{3\pi}{11}\right) - 2\cos\left(\frac{\pi}{11}\right) \\
\Rightarrow \quad & c - 24h = -\frac{160}{11} \mod 24\mathbb{Z}\,,
\end{aligned}
$$

$$
\begin{aligned}
& x_1 = -4\cos\left(\frac{5\pi}{11}\right) + 2\cos\left(\frac{4\pi}{11}\right) - 2\cos\left(\frac{3\pi}{11}\right) + 2\cos\left(\frac{\pi}{11}\right) \\
\Rightarrow \quad & c_{\text{eff}} = \frac{80}{11}\,,
\end{aligned}
$$

$$
\begin{aligned}
& x_1 = -4\cos\left(\frac{3\pi}{11}\right) + 2\cos\left(\frac{4\pi}{11}\right) + 2\cos\left(\frac{2\pi}{11}\right) + 2\cos\left(\frac{5\pi}{11}\right) \\
\Rightarrow \quad & c - 24h = -\frac{112}{11} \mod 24\mathbb{Z}\,,
\end{aligned}
$$

$$
\begin{aligned}
& x_1 = -4\cos\left(\frac{\pi}{11}\right) - 2\cos\left(\frac{2\pi}{11}\right) - 2\cos\left(\frac{5\pi}{11}\right) - 2\cos\left(\frac{3\pi}{11}\right) \\
\Rightarrow \quad & c - 24h = -\frac{208}{11} \mod 24\mathbb{Z}\,.
\end{aligned}
$$

Note: Similar calculations have been carried out by Klassen and Melzer for the cases (A_1, E_6), (A_1, E_7), and (A_1, E_8). For more details see [14].

Summary

Certain integrable models are described by pairs (X, Y) of ADET Dynkin diagrams. At high energy these models are expected to have a conformally invariant limit. The S-matrix of the model determines algebraic equations, whose solutions are mapped to the central charge and scaling dimensions of the corresponding conformal field theory. We study the equations of the (D_m, A_n) model and find all solutions explicitly using the representation theory of Lie algebras and related Yangians. These mathematically rigorous results are in agreement with the expectations arising from physics. We also investigate the overlap between certain q-hypergeometric series and modular functions. We study a particular class of 2-fold q-hypergeometric series, denoted $f_{A,B,C}$. Here A is a positive definite, symmetric, 2×2 matrix, B is a vector of length 2, and C is a scalar, all three with rational entries. It turns out that for certain choices of the matrix A, the function $f_{A,B,C}$ can be made modular. We calculate the corresponding values of B and C. It is expected that functions $f_{A,B,C}$ arising in this way are characters of some rational conformal field theory. We show that this is true in at least one case, namely $A = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}$.

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Appendix A

Useful Method of Solving Equations

This appendix describes a useful method of solving equations. In chapter 5 it is used to simplify certain solutions arising from equations of the (E_m, T_1) model.

A.1 General Method

Suppose we wish to find the roots of an irreducible polynomial

$$x^3 + a_1x^2 + a_2x + a_3 = 0 \;, \tag{A.1}$$

with rational a_i. This equation has three roots, say x_1, x_2, and x_3. Suppose that

$$x_1 = \sum_{i=1}^{3} b_i(\omega^i + \omega^{-i}) = b_1(\omega + \omega^{-1}) + b_2(\omega^2 + \omega^{-2}) + b_3(\omega^3 + \omega^{-3}) \;,$$

with rational b_i, and $\omega = exp\left(\frac{2\pi i}{7}\right)$.

Galois group

The Galois group of ω is generated by $\gamma : \omega \mapsto \omega^3$. This action extends to other elements as

$$\gamma : \omega \mapsto \omega^3 \mapsto \omega^2 \mapsto \omega^6 \mapsto \omega^4 \mapsto \omega^5 \mapsto \omega \; .$$

Hence the Galois group is $\mathbb{Z}_6$

$$\gamma : 1 \mapsto 3 \mapsto 2 \mapsto 6 \mapsto 4 \mapsto 5 \mapsto 1 \; .$$

Solutions

Using the action of the Galois group we can find the other two roots by

$$x = \sum_{k=1}^{3} b_k \left(\omega^{\gamma(k)} + \omega^{-\gamma(k)}\right) \; .$$

It follows that the three roots of the equation are

$$\begin{aligned} x_1 &= b_1(\omega + \omega^{-1}) + b_2(\omega^2 + \omega^{-2}) + b_3(\omega^3 + \omega^{-3}) \; , \\ x_2 &= b_1(\omega^3 + \omega^{-3}) + b_2(\omega + \omega^{-1}) + b_3(\omega^2 + \omega^{-2}) \; , \\ x_3 &= b_1(\omega^2 + \omega^{-2}) + b_2(\omega^3 + \omega^{-3}) + b_3(\omega + \omega^{-1}) \; . \end{aligned} \tag{A.2}$$

Coefficients

In terms of the three roots (A.2) the equation (A.1) becomes

$$
\begin{aligned}
& x^3 + a_1 x^2 + a_2 x + a_3 \\
= \;& x^3 + (b_1 + b_2 + b_3)x^2 + \left(-2 \sum b_i^2 + (-2+5) \sum_{i<j} b_i b_j \right) x + \ldots
\end{aligned}
$$

Solving for $\{a_1, a_2, a_3\}$ in terms of $\{b_1, b_2, b_3\}$ gives

$$
\begin{aligned}
a_1 &= b_1 + b_2 + b_3 \ , \\
a_2 &= -2 \sum b_i^2 + 3 \sum_{i<j} b_i b_j = -\frac{7}{2} \sum b_i^2 + \frac{3}{2} \left(\sum b_i \right)^2 \ .
\end{aligned}
$$

This can be rearranged to get

$$
\begin{aligned}
\sum_{i=1}^{3} b_i &= a_1 \ , \\
\sum_{i=1}^{3} b_i^2 &= -\frac{2}{7} \left(a_2 - \frac{3}{2} a_1^2 \right) \ .
\end{aligned}
$$

A.2 Example

We use the method described above to find the roots of the equation

$$
x^3 - 6x^2 + 5x - 1 = 0 \ .
$$

Solution

We can solve for $\{b_1, b_2, b_3\}$ to get

$$\begin{aligned}(a_1, a_2) = (-6, 5) \quad &\Rightarrow \quad b_1 + b_2 + b_3 = -6 \quad \text{and} \quad b_1^2 + b_2^2 + b_3^2 = 14 \\ &\Rightarrow \quad (b_1, b_2, b_3) = (-3, -2, -1)\ .\end{aligned}$$

The 3 roots of the equation turn out to be

$$\begin{aligned} x &= -4\cos\left(\frac{2\pi}{7}\right) + 6\cos\left(\frac{3\pi}{7}\right) + 2\cos\left(\frac{\pi}{7}\right)\ , \\ x &= -2\cos\left(\frac{2\pi}{7}\right) + 4\cos\left(\frac{3\pi}{7}\right) + 6\cos\left(\frac{\pi}{7}\right)\ , \\ x &= -6\cos\left(\frac{2\pi}{7}\right) + 2\cos\left(\frac{3\pi}{7}\right) + 4\cos\left(\frac{\pi}{7}\right)\ . \end{aligned}$$