Calculating the Superconformal Index and Seiberg Duality

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Abstract

We develop techniques to calculate an index for four dimensional superconformal field theories. This superconformal index is counting BPS operators which preserve only one supercharge. To calculate the superconformal index we quantize the field theory on $S^3 \times \mathbb{R}$ and show that the twisted theory has an appropriate mass gap. This allows for the interactions to be switched off continuously without the superconformal index being changed. We test those techniques for theories which go through a non-trivial RG flow and for Seiberg dual theories. This leads to the conjecture of some group/number theoretical identities.

1   Introduction

Counting states in string and gauge theories has many applications:

- Knowing the spectrum of a theory is the first step towards understanding it and counting the states is a first step towards understanding the spectrum.
- It allows to do thermodynamics and calculate something like (black hole) entropies.
- It can be used as a check for dualities.

In general, however, it is very difficult to count the full spectrum of a given theory, but for supersymmetric theories it is often possible to count BPS states, which preserve a part of the supersymmetries. This still addresses the problems listed above.

Recently there has been a lot of progress in counting BPS states which preserve two supersymmetries in supersymmetric field theories [**1**, **2**, **3**, **4**, **5**, **6**, **7**]. Those BPS states are chiral operators and are related to holomorphic sections of the moduli space of vacua.

There also has been progress in understanding BPS states which just preserve one supersymmetry [**8**, **9**, **10**, **11**, **12**, **13**]. Those gauge theory states are of great interest in string theory, because they could be microstates of AdS black holes which preserve just one supersymmetry [**14**, **9**].

Most of the progress has been through an index which is a variation of the usual Witten index [**15**] on $S^3 \times \mathbb{R}$ instead of T^4. This superconformal index has several chemical potentials for symmetry generators. Especially, there is one chemical potential μ for an energy operator Ξ that resolves the degeneracy between an infinite number of BPS states.

The Lagrangian for $\mathcal{N} = 1$ supersymmetric field theories on $S^3 \times \mathbb{R}$ [**8**] can be quantized. Because the twisted theory on $S^3 \times \mathbb{R}$ has a mass gap and the superconformal index is a topological quantity the interactions can be switched off continuously in order to calculate the superconformal index. The superconformal index only gets contributions from low lying states in the remaining infinite dimensional harmonic oscillator. This reduces calculating the superconformal index to a combinatoric and group theoretical problem.

This result can be used to calculate the first terms in a $e^{-\mu}$ expansion of the superconformal index for $SU(2)$ SQCD with three flavors and its Seiberg dual. Or other more complicated examples.

In section 2 we review the definition of the superconformal index. In section 3 we demonstrate the arguments for calculating the index on a quantum mechanics theory, which is a consistent truncation of the field theory. Later, in section 4 we generalize the arguments to field theories on $S^3 \times \mathbb{R}$ and propose a general formula (80) and (81) for the superconformal index of a general gauge theory that flows to a conformal fixed point. Finally, in section 5 we calculate a couple of examples involving RG flows and Seiberg duality. This leads to the conjecture of some group/number theoretical identities (83) and (86).

2   From the Witten index to the superconformal index

In most supersymmetric theories, one can define a Witten index. This is done by picking a supercharge Q and its Hermitean conjugate $Q^\dagger$, they satisfy

$$\text{(1)} \qquad\qquad \{Q, Q\} = 0,$$

$$\text{(2)} \qquad\qquad \{Q, Q^\dagger\} = \Delta \qquad \text{and}$$

$$\text{(3)} \qquad\qquad \{Q, (-1)^F\} = 0,$$

where in many cases Δ is the Hamiltonian. Equation (2) implies that the spectrum of Δ is bounded from below by zero. If Δ damps as well as the Hamiltonian H of the theory one can define the Witten index

$$\text{(4)} \qquad\qquad \text{ind} = \text{tr}(-1)^F e^{-\beta \Delta}.$$

This index is an integer, which is independent of β. However, if Δ has infinitely many eigenstates with zero eigenvalue, then the above expression is not well defined and one needs to find a better regulator Ξ, which commutes with Q

$$\text{(5)} \qquad\qquad [\Xi, Q] = 0 \qquad \text{and} \qquad \Xi \geq H$$

to define the modified index

$$\text{(6)} \qquad\qquad \text{ind}(\mu) = \text{tr}(-1)^F e^{-\mu \Xi}.$$

This modified index is actually a generating function for many indices. The coefficients of different powers of $t = e^{-\mu}$ are integers and topological invariants. This is similar to the elliptic genus or Gromov-Witten invariants.

Let us now turn to superconformal field theories. A superconformal field theory is naturally quantized on $S^3 \times \mathbb{R}$, this is called radial quantization. The isometries of $S^3 \times \mathbb{R}$ are $SU(2)_L \times SU(2)_R \times \mathbb{R}$ and the eight supercharges split up into a doublet Q_α under $SU(2)_L$ and a doublet $S_{\dot\alpha}$ under $SU(2)_R$.

To define the superconformal index let us pick the supercharges Q_1 and $Q_1^\dagger$. They satisfy the anticommutation relation

$$(7) \qquad \{Q_1, Q_1^\dagger\} = H - \frac{3}{2}R - 2J_3 = \Delta,$$

where H is the Hamiltonian in radial quantization, R is the $U(1)$ R-charge and J_3 is a generator of $SU(2)_L$. It turns out that in a free theory Δ has infinitely many ground states, i.e. it does not damp well enough. However, there is an operator

$$(8) \qquad \Xi = H - \frac{1}{2}R \geq \frac{2}{3}H$$

which damps as well as the Hamiltonian[1] . Furthermore, the $SU(2)_R$ and possibly an internal symmetry group[2] H commute with Q_α

$$(9) \qquad\qquad [\gamma, Q_\alpha] = 0, \qquad \text{for} \qquad \gamma \in SU(2)_R,$$
$$(10) \qquad\qquad [h, Q_\alpha] = 0, \qquad \text{for} \qquad h \in H.$$

The full superconformal index can be defined as

$$(11) \qquad \text{ind}(\mu, \gamma, h) = \text{tr}(-1)^F e^{-\mu\Xi}\gamma h = \text{tr}(-1)^F e^{-\mu(\Delta + R + 2J^3)}\gamma h$$

and only gets contributions from ground states of Δ.

Note that the superconformal index is a group character of $\mathbb{R}_\Xi \times SU(2)_R \times H$. This allows to split the superconformal index into representations of that group. In the examples involving Seiberg duality we will do this.

The superconformal index is topological and is invariant under continuous deformations of the theory the following sense: A deformation of Q_1 and Δ, that preserves $R + 2J^3$, $\tilde{J}^i$ and the symmetry transformations $h \in H$ as well as the $SU(1|1) \times \mathbb{R}_\Xi \times SU(2)_R \times H$ (super) symmetry group leaves the superconformal index invariant.

This also allows for non-conformal deformations. For this reason the superconformal index can be calculated in a weakly coupled UV theory that flows to a nontrivial conformal fixed point. In the next sections we will develop techniques for calculating the superconformal index for a given theory.

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[1]The inequality is due to a BPS bound implied by the anti-commutation relation $\{Q_\alpha, Q_\beta^\dagger\} = \delta_\alpha^\beta \left(H - \frac{3}{2}R\right) - 4(\sigma^i)^\beta{}_\alpha J_i$.
[2]Those internal symmetries can be continuous or discrete.

3 The superconformal index of the Wess-Zumino quantum mechanics

As we will see in section 4 there is an infinite number of oscillator modes contributing to the superconformal index. This makes calculations fairly cumbersome. For this reason we will illustrate the main arguments of how to calculate the superconformal index in a related quantum mechanical theory, which is a consistent truncation of a Wess-Zumino model on $S^3 \times \mathbb{R}$ [**8**]. In this quantum mechanical theory we will also be able to check those arguments by explicitely constructing the ground states.

3.1 Canonical quatization

The Lagrangian for a single chiral multiplet of R-charge q is given by

(12)
$$L = \left(\partial_t - i\frac{3q-2}{2} \right) \phi^\dagger \left(\partial_t + i\frac{3q-2}{2} \right) \phi - \phi^\dagger \phi - i\psi^\dagger \left(\partial_t + i\frac{3q-5}{2} \right) \psi$$
$$- |W'|^2 - \frac{1}{2}W''\tilde{\psi}\psi - h.c.,$$

where the superpotential is homogenous of R-charge 2, i.e.

(13)
$$W = \lambda\phi^n,$$

with $nq = 2$ and λ a complex coupling constant.

Let us proceed in the canonical formalism. The canonical momenta are

(14) $\Pi = \dot{\phi}^\dagger - i\frac{3q-2}{2}\phi^\dagger, \qquad \Pi^\dagger = \dot{\phi} + i\frac{3q-2}{2}\phi, \qquad$ and $\qquad \pi^\alpha = -i\psi_\alpha^\dagger$

and satisfy the commutation relations

(15) $\qquad [\phi, \Pi] = -i \qquad [\phi^\dagger, \Pi^\dagger] = -i \qquad$ and $\qquad \{\psi_\alpha, \pi^\beta\} = -i\delta_\alpha^\beta.$

The Hamiltonian is

(16) $\quad H = |\Pi|^2 - i\frac{3q-2}{2}\Pi\phi + i\frac{3q-2}{2}\phi^\dagger\Pi^\dagger + |\phi|^2 - \frac{3q-5}{2}\psi^\dagger\psi$
$$+ |W'|^2 + W''\psi_1\psi_2 - (W'')^\dagger\psi_1^\dagger\psi_2^\dagger.$$

Since W has R-charge 2, the theory is invariant under the $U(1)$ R-symmetry generated by

(17)
$$R = -iq(\Pi\phi - \phi^\dagger\Pi^\dagger) - (q-1)\psi^\dagger\psi,$$

an $SU(2)_L$ symmetry that acts on the spinors, which is generated by

(18)
$$J^i = -\psi_\alpha^\dagger\sigma^i{}_\alpha{}^\beta\psi_\beta$$

and the supersymmetries generated by

(19) $$Q_\alpha = (\Pi - i\phi^\dagger)\psi_\alpha + i(W')^\dagger \epsilon_{\alpha\beta}\psi_\beta^\dagger.$$

This symmetry group is $SU(2|1)$
(20)

$$\{Q_\alpha, Q_\beta\} = 0, \quad \{Q_\alpha^\dagger, Q_\beta^\dagger\} = 0 \quad \text{and} \quad \{Q_\alpha, Q_\beta^\dagger\} = \delta_\alpha^\beta\,(H - \frac{3}{2}R) - 4\sigma^i{}_\alpha{}^\beta J^i.$$

We now pick the supercharge Q_1, which satisfies the relations

(21) $$\{Q_1, Q_1^\dagger\} = \Delta \equiv H - \frac{3}{2}R - 2J^3 \qquad \text{and}$$

(22) $$[\Xi, Q_1] = 0, \qquad \text{where} \qquad \Xi = H - \frac{1}{2}R$$

and define the superconformal index

(23) $$\mathrm{ind}(\mu) = \mathrm{tr}(-1)^F e^{-\mu\Xi} = \mathrm{tr}(-1)^F t^{\Delta + R + 2J^3},$$

where $t = e^{-\mu}$.

3.2 The free theory

In the next subsection we will show that the superconformal index of the interacting theory can be reduced to the superconformal index of the free theory. For this reason let us start by calculating the superconformal index of the free theory. Let us formulate the theory in terms of the annihilation and creation operators

(24) $$a_1 = \frac{1}{\sqrt{2}}(\Pi^\dagger + i\phi), \qquad\qquad a_1^\dagger = \frac{1}{\sqrt{2}}(\Pi - i\phi^\dagger),$$

(25) $$a_2 = \frac{1}{\sqrt{2}}(\Pi + i\phi^\dagger), \qquad\qquad a_2^\dagger = \frac{1}{\sqrt{2}}(\Pi^\dagger - i\phi),$$

(26) $$b_1 = \psi_1, \qquad\qquad b_1^\dagger = \psi_1^\dagger,$$

(27) $$b_2 = \psi_2, \qquad\qquad b_2^\dagger = \psi_2^\dagger.$$

The twisted Hamiltonian of the free theory is then

(28) $$H - \frac{3}{2}R - 2J^3 = 2a_1^\dagger a_1 + 2b_1^\dagger b_1.$$

It's ground states are

(29) $$|0, m, 0, \eta\rangle.$$

and the commuting $U(1)$ is

(30) $$R + 2J^3 = -qa_1^\dagger a_1 + qa_2^\dagger a_2 - qb_1^\dagger b_1 - (q-2)b_2^\dagger b_2.$$

This implies that the superconformal index is

(31) $$\mathrm{ind}(\mu) = \frac{1 - t^{2-q}}{1 - t^q} = \sum_{m=0}^{n-2} t^{qm},$$

where $qn = 2$. This finite result comes from a cancellation between fermionic and bosonic states with $R + 2J^3 \geq (n-1)q$. To avoid this cancellation in the free theory, one can introduce a chemical potential for an additional internal $U(1)$ symmetry generated by

(32) $$T = -a_1^\dagger a_1 + a_2^\dagger a_2 - b_1^\dagger b_1 - b_2^\dagger b_2.$$

The internal $U(1)$ symmetry commutes with the supercharges and an appropriate chemical potential can be inserted into the superconformal index

(33) $$\mathrm{ind}(\mu, u) = \mathrm{tr}(-1)^F t^{\Delta + R + 2J^3} u^T = \frac{1 - t^{(n-1)q} u^{-1}}{1 - t^q u}.$$

This resolves all the degeneracies in the superconformal index for the free theory.

3.3 The interacting theory

In the interacting theory the extra $U(1)$ symmetry is broken by the superpotential and one expects that the canceling states in (31) are actually lifted by the interaction. We now use two independent arguments to support this claim:

The superconformal index

(34) $$\mathrm{ind}(\mu) = \mathrm{tr}(-1)^F t^\Xi = \mathrm{tr}(-1)^F t^{\Delta + R + 2J^3}$$

is topological and does not change under continuous deformations of Q_1 and Δ as long as $R + 2J^3$, $\tilde{J}^i$ and h remain invariant and the (super) symmetry algebra is preserved. For this reason we can switch off the superpotential as long as we can argue that the eigenstates of $\Xi = \Delta + R + 2J^3$ remain sufficiently localized around the origin in field space. Indeed Ξ has a mass gap and localizes its eigenstates, even though Δ has the zero modes a_2 and b_2.

For this reason the superpotential can be switched off continuously and and the superconformal index can be calculated by counting excitations of free bosonic and fermionic oscillators. The only manifestation of the superpotential is due to the fact that the extra $U(1)$ symmetry is broken

and the cancellations of the previous section can happen, leading to

$$\text{(35)} \qquad \text{ind}(\mu) = \sum_{m=0}^{n-2} t^{qm}.$$

Using a 'zig-zag' argument similar to [**16**, **17**] we can actually verify this result for the superconformal index and show, that in the interacting theory all the canceling states of the free theory calculation are actually lifted. The ground states of the twisted Hamiltonian Δ are in one to one correspondence with the cohomology of the supercharge $Q_1^\dagger$.

The supercharge $Q_1^\dagger$ is the sum

$$\text{(36)} \qquad Q_1^\dagger = \sqrt{2} a_1 b_1^\dagger - i\lambda \left(\frac{a_1 - a_2^\dagger}{\sqrt{2}i} \right)^{n-1} b_2 = Q_1^{(0)\dagger} + \lambda Q_1^{(1)\dagger}$$

of the supercharge of the free theory and an interaction term. We now follow a perturbative procedure and expand a ground state as

$$\text{(37)} \qquad |\Omega\rangle = |\Omega^{(0)}\rangle + \lambda|\Omega^{(1)}\rangle + \lambda^2|\Omega^{(2)}\rangle + \cdots,$$

where all the terms $|\Omega^{(l)}\rangle$ have the same $R + 2J^3$ charge. Then the action of $Q_1^\dagger$ on $|\Omega\rangle$ can be expanded as follows:

$$\text{(38)} \qquad \begin{aligned} Q_1^\dagger |\Omega\rangle = {}& Q_1^{(0)\dagger}|\Omega^{(0)}\rangle + \lambda \left(Q_1^{(0)\dagger}|\Omega^{(1)}\rangle + Q_1^{(1)\dagger}|\Omega^{(0)}\rangle \right) \\ & + \lambda^2 \left(Q_1^{(0)\dagger}|\Omega^{(2)}\rangle + Q_1^{(1)\dagger}|\Omega^{(1)}\rangle \right) + \cdots \end{aligned}$$

For a state to be closed, it needs to satisfy

$$\text{(39)} \qquad Q_1^{(0)\dagger}|\Omega^{(0)}\rangle = 0 \quad \text{and} \quad Q_1^{(0)\dagger}|\Omega^{(l+1)}\rangle + Q_1^{(1)\dagger}|\Omega^{(l)}\rangle = 0$$

and for a state to be exact it needs to satisfy

$$\text{(40)} \qquad |\Omega^{(l)}\rangle = Q_1^{(0)\dagger}|\Lambda^{(l)}\rangle + Q_1^{(1)\dagger}|\Lambda^{(l-1)}\rangle.$$

In the example at hand this expansion actually terminates after the first step.

At leading order, we have to find the cohomology of $Q_1^{(0)\dagger}$. As we have seen before, the cohomology of $Q_1^{(0)\dagger}$ can be represented by $|0, m, 0, \eta\rangle$, the ground states of the free Hamiltonian. The $Q_1^{(0)\dagger}$-exact states have the form $|m_1, m_2, 1, \eta\rangle$.

Next, equation (39) implies that $|\Omega^{(0)}\rangle$ needs to be $Q_1^{(1)\dagger}$-closed inside the $Q_1^{(0)\dagger}$-cohomology

$$\text{(41)} \qquad Q_1^{(1)\dagger}|\Omega^{(0)}\rangle = -Q_1^{(0)\dagger}|\Omega^{(1)}\rangle$$

which only has a solution for $\eta = 0$, where it reduces to

$$(42) \qquad\qquad Q_1^{(0)\dagger}|\Omega^{(1)}\rangle = 0,$$

i.e. $|\Omega^{(1)}\rangle$ has to be again in the $Q_1^{(0)\dagger}$-cohomology. Furthermore, the relation (40) implies that states of the form

$$(43) \qquad\qquad |\Omega^{(0)}\rangle \sim |0, m+n-1, 0, 0\rangle$$

are exact. One can now iterate the procedure for $|\Omega^{(1)}\rangle$ and get the same result. For this reason one can just choose

$$(44) \qquad\qquad |\Omega^{(l)}\rangle = 0 \qquad \text{for} \qquad l \geq 1$$

and the expansion terminates after the first step.This shows that the cohomology of $Q_1^\dagger$ can be represented by the states

$$(45) \qquad\qquad |0, m, 0, 0\rangle, \qquad m = 0, \cdots, n-2$$

proving the claim before.

The Wess-Zumino model actually also has a discrete $\mathbb{Z}_n$ symmetry left of the $U(1)$ in the free theory and one could define the superconformal index

$$(46) \qquad\qquad \text{ind}(\mu, \omega) = \text{tr}(-1)^F t^\Xi g = \sum_{m=0}^{n-2} t^{qm} \omega^m,$$

where $g \in \mathbb{Z}_n$ and $\omega^n = 1$. However this does not lead to any new information.

4   The superconformal index for field theories

In this section we will generalize the quantum mechanical result of the previous section to field theory. In [8] the Lagrangian for $\mathcal{N} = 1$ field theory on $S^3 \times \mathbb{R}$ was derived. For a Wess-Zumino model of a chiral multiplet with R-charge q the Lagrangian is

$$(47) \qquad \mathcal{L}_0 = \left(\partial_t - i\frac{3q-2}{2}\right)\phi^* \left(\partial_t + i\frac{3iq-2}{2}\right)\phi - 4\sigma_i^{(L)}\phi^*\sigma_i^{(L)}\phi - \phi^*\phi$$

$$+ i\bar\psi\gamma^0 \left(\partial_t + i\frac{3q-2}{2}\right)\psi - 2i\bar\psi\gamma^i \left(\sigma_i^{(L)} + \frac{1}{8}\epsilon_{ijk}\gamma^{jk}\right)\psi + F^*F,$$

$$(48) \qquad \mathcal{L}_W = W'(\phi)F - \frac{1}{2}W''(\phi)\tilde\psi\psi + \text{h.c.}$$

with

$$(49) \qquad\qquad W'(\phi) = c\phi^{-\frac{q-2}{q}},$$

where $\sigma_i^{(L)}$ are the derivatives with respect to the Killing vector fields generating $SU(2)_L$ and the spinors are represented in a vielbein basis given by the right invariant 1-forms. The Lagrangian for a gauge theory is given in appendix B. In general the superconformal R-charges q have to be fixed by a-maximization [**18**].

As we did for the quantum mechanical theory, let us first understand the free field theory.

4.1 The free chiral multipet

The Lagrangian for the boson in a free chiral multiplet of R-charge q is

(50) $$\mathcal{L}_\phi = (\partial_t - i\frac{3q-2}{2})\phi^\dagger(\partial_t + i\frac{3q-2}{2})\phi - 4\sigma_i^{(L)}\phi^\dagger\sigma_i^{(L)}\phi - \phi^\dagger\phi.$$

When the scalar field is decomposed into spherical harmonics

(51) $$\phi = \sum_{j,j_3,\tilde{j}_3} \phi_{j,j_3,\tilde{j}_3} Y_{j,j_3,\tilde{j}_3}.$$

the Lagrangian has the form

(52) $L_\phi =$
$$\sum_{j,j_3,\tilde{j}_3} \left((\partial_t - i\frac{3q-2}{2})\phi^\dagger_{j,j_3,\tilde{j}_3}(\partial_t + i\frac{3q-2}{2})\phi_{j,j_3,\tilde{j}_3} - (2j+1)^2\phi^\dagger_{j,j_3,\tilde{j}_3}\phi_{j,j_3,\tilde{j}_3} \right).$$

The canonical momenta are

(53) $\Pi_{j,j_3,\tilde{j}_3} = (\partial_t - i\frac{3q-2}{2})\phi^\dagger_{j,j_3,\tilde{j}_3}$ and $\Pi^\dagger_{j,j_3,\tilde{j}_3} = (\partial_t + i\frac{3q-2}{2})\phi_{j,j_3,\tilde{j}_3},$

the Hamiltonian is

(54)
$$H_\phi = \sum_{j,j_3,\tilde{j}_3} \left(\Pi_{j,j_3,\tilde{j}_3}\Pi^\dagger_{j,j_3,\tilde{j}_3} - i\frac{3q-2}{2}\Pi_{j,j_3,\tilde{j}_3}\phi_{j,j_3,\tilde{j}_3} + i\frac{3q-2}{2}\phi^\dagger_{j,j_3,\tilde{j}_3}\Pi^\dagger_{j,j_3,\tilde{j}_3} + \right.$$
$$\left. (2j+1)^2\phi^\dagger_{j,j_3,\tilde{j}_3}\phi_{j,j_3,\tilde{j}_3} \right),$$

the R-charge is

(55) $$R_\phi = -iq \sum_{j,j_3,\tilde{j}_3} \left(\Pi_{j,j_3,\tilde{j}_3}\phi_{j,j_3,\tilde{j}_3} - \phi^\dagger_{j,j_3,\tilde{j}_3}\Pi^\dagger_{j,j_3,\tilde{j}_3} \right),$$

the J^3 charge is

(56) $$J^3_\phi = -i \sum_{j,j_3,\tilde{j}_3} j_3 \left(\Pi_{j,j_3,\tilde{j}_3}\phi_{j,j_3,\tilde{j}_3} - \phi^\dagger_{j,j_3,\tilde{j}_3}\Pi^\dagger_{j,j_3,\tilde{j}_3} \right)$$

and the $\tilde{J}^3$ charge is

(57) $$\tilde{J}_\phi^3 = -i \sum_{j,j_3,\tilde{j}_3} \tilde{j}_3 \left(\Pi_{j,j_3,\tilde{j}_3} \phi_{j,j_3,\tilde{j}_3} - \phi_{j,j_3,\tilde{j}_3}^\dagger \Pi_{j,j_3,\tilde{j}_3}^\dagger \right).$$

The twisted Hamiltonian is then

(58) $\Delta_\phi = H_\phi - \dfrac{3}{2} R_\phi - 2 J_\phi^3 =$
$$2 \sum_{j,j_3,\tilde{j}_3} \left((j + j_3 + 1) a_{1,j,j_3,\tilde{j}_3}^\dagger a_{1,j,j_3,\tilde{j}_3} + (j - j_3) a_{2,j,j_3,\tilde{j}_3}^\dagger a_{2,j,j_3,\tilde{j}_3} \right),$$

with the ladder (annihilation) operators

(59) $$a_{1,j,j_3,\tilde{j}_3} = \frac{1}{\sqrt{4j+2}} \left(\Pi_{j,j_3,\tilde{j}_3}^\dagger + i(2j+1)\phi \right) \qquad \text{and}$$

(60) $$a_{2,j,j_3,\tilde{j}_3} = \frac{1}{\sqrt{4j+2}} \left(\Pi_{j,j_3,\tilde{j}_3} + i(2j+1)\phi^\dagger \right).$$

The zero modes are given by

(61) $$a_{2,j,j,\tilde{j}_3} \qquad \text{and} \qquad a_{2,j,j,\tilde{j}_3}^\dagger$$

and the $R + 2J^3$ charge is

(62) $$R_\phi + 2J_\phi^3 = - \sum_{j,j_3,\tilde{j}_3} (q + 2j_3) \left(a_{1,j,j_3,\tilde{j}_3}^\dagger a_{1,j,j_3,\tilde{j}_3} - a_{2,j,j_3,\tilde{j}_3}^\dagger a_{2,j,j_3,\tilde{j}_3} \right).$$

As in the quantum mechanical theory, the theory has an extra $U(1)$ symmetry generated by

(63) $$T_\phi = - \sum_{j,j_3,\tilde{j}_3} \left(a_{1,j,j_3,\tilde{j}_3}^\dagger a_{1,j,j_3,\tilde{j}_3} - a_{2,j,j_3,\tilde{j}_3}^\dagger a_{2,j,j_3,\tilde{j}_3} \right).$$

The generating function for all the oscillator modes contributing to the superconformal index is

(64) $$f_\phi(t, y, u) = \frac{t^q u}{(1 - ty)(1 - \frac{t}{y})},$$

where t is the fugacity for the $R + 2J^3$-charge, u is the fugacity for the T-charge and y is the fugacity for the $\tilde{J}^3$-charge.

Similarly, the Lagrangian for the fermion in a free chiral multiplet of R-charge q is

(65) $$\mathcal{L}_\psi = i\bar{\psi}\gamma^0 \left(\partial_0 + i\frac{3q - 2}{2} \right) \psi - 2i\bar{\psi}\gamma^i \left(\sigma_i^{(L)} + \frac{1}{8}\epsilon_{ijk}\gamma^{jk} \right) \psi.$$

A similar expansion in spherical harmonics leads to a generating function

$$(66) \qquad f_\psi(t, y, u) = -\frac{t^{2-q}u^{-1}}{(1-ty)(1-\frac{t}{y})}$$

for the fermionic oscillator modes contributing to the superconformal index leading to the generating function

$$(67) \qquad f_\Phi(t, y, u) = \frac{t^q u - t^{2-q}u^{-1}}{(1-ty)(1-\frac{t}{y})}$$

for the oscillator modes of a free chiral multiplet contributing to the superconformal index.

To get the superconformal index one has to count all the oscillator excitations, which is done by the Plethystic exponential [**19**, **9**, **1**, **2**]
$$(68)$$
$$\operatorname{ind}(\mu, y, u) = g_\Phi(t, y, u) = \operatorname{Pexp}(f_\Phi(t, y, u)) = \exp\left(\sum_{l=1}^\infty \frac{1}{l} f_\Phi(t^l, y^l, u^l)\right),$$

which in this case is

$$(69) \qquad \operatorname{ind}(\mu, y, u) = \prod_{m,n} \frac{1 - t^{2-q+m+n}y^{m-n}u^{-1}}{1 - t^{q+m+n}y^{m-n}u}.$$

4.2   Wess-Zumino models

Let us now consider an ungauged theory of chiral multiplets with interactions given by a homogenous superpotential. Similar as in the quantum mechanical theory in section 3.3, there is a mass gap due to the R-charge dependent 'curvature couplings' in the Lagrangian. For this reason, the superpotential can actually be switched off for the purpose of calculating the superconformal index.

However, the superpotential breaks the group G of internal symmetries of the free theory to a subgroup H and only this subgroup can be used to resolve degeneracies in the superconformal index. If the interacting theory has chiral multiplets Φ_i of R-charge q_i transforming in the representation R_i of the global symmetry group H, then the generating function for the oscillator modes contributing to the superconformal index is

$$(70) \qquad f_\Phi(t, \gamma, h) = \sum_i \frac{t^{q_i}\chi_{R_i}(h) - t^{2-q_i}\chi_{\bar{R}_i}(h)}{1 - t\chi_f(\gamma) + t^2}$$

and the superconformal index is the Plethystic exponential thereof

$$(71) \qquad \text{ind}(\mu, \gamma, h) = \exp\left(\sum_{l=1}^{\infty} \frac{1}{l} f_\Phi(t^l, \gamma^l, h^l)\right).$$

Note that the superconformal index can again be viewed as a character of the $\mathbb{R}_\Xi \times SU(2)_R \times H$ group. This allows to extract contributions transforming in given irreducible representations.

For a Wess-Zumino model of a single chiral multiplet with a superpotential

$$(72) \qquad W(\phi) = \phi^n,$$

the global $U(1)$-symmetry is broken to $\mathbb{Z}_n$ and the superconformal index is generated by

$$(73) \qquad f_\Phi(t, \gamma, \omega) = \frac{t^q \omega - t^{2-q}\omega^{-1}}{1 - t\chi_f(\gamma) + t^2} = \frac{t^q \omega - (t^q \omega)^{n-1}}{(1 - ty)(1 - \frac{t}{y})},$$

where $nq = 2$ and $\omega^n = 1$. One can extract the lowest contributions, which are singlets under the $SU(2)_R$

$$(74) \qquad \text{ind}(\mu, \gamma, \omega) = 1 + t^q \omega + \cdots + (t^q \omega)^{n-2} + \cdots$$

Those contributions agree with the superconformal index of the quantum mechanical theory and they are due to the chiral primaries.

If the superpotential includes a mass term for a field, say

$$(75) \qquad W(\phi) = \phi^2$$

then this superfield should be integrated out in the IR, i.e. just disappear from the low energy theory. For this reason it should not contribute to the superconformal index. Indeed, such a superfield has R-charge 1, the global symmetry is broken to $\mathbb{Z}_2$ or nothing and the superfield does not contribute to the superconformal index

$$(76) \qquad f_\Phi(t, \gamma, \omega) = \frac{t\omega - t\omega^{-1}}{1 - t\chi_f(\gamma) + t^2} = 0.$$

4.3   Gauge theories

A free gauge theory can be treated in a similar way as the free chiral multiplet except for a few modifications related to gauge fixing and the Gauss law constraint. The most convenient gauge in the case at hand is the temporal gauge $A_0 = 0$. In the free limit, the dynamics of the gauge theory is described by free abelian gauge fields together with an overall Gauss law constraint imposing the condition, that all states are gauge singlets.

The generating function for oscillator modes of a free abelian vector multiplet contributing to the superconformal index is

$$(77) \qquad f_V(\mu, \gamma) = \frac{2t^2 - t\chi_f(\gamma)}{1 - t\chi_f(\gamma) + t^2}.$$

For a nonabelian gauge theory with gauge group G one has to substitute the group character in the adjoint representation

$$(78) \qquad f_V(\mu, \gamma, g) = \frac{2t^2 - t\chi_f(\gamma)}{1 - t\chi_f(\gamma) + t^2} \chi_{adj}(g).$$

The matter fields in representations r_i of the gauge group G lead to contributions of the form

$$(79) \qquad f_\Phi(t, \gamma, h, g) = \sum_i \frac{t^{q_i}\chi_{R_i}(h)\chi_{r_i}(g) - t^{2-q_i}\chi_{\bar{R}_i}(h)\chi_{\bar{r}_i}(g)}{1 - t\chi_f(\gamma) + t^2}$$

so that the total generating function for the oscillator modes is

$$(80) \quad f(t, \gamma, h, g) =$$
$$\frac{2t^2 - t\chi_f(\gamma)}{1 - t\chi_f(\gamma) + t^2} \chi_{adj}(g) + \sum_i \frac{t^{q_i}\chi_{R_i}(h)\chi_{r_i}(g) - t^{2-q_i}\chi_{\bar{R}_i}(h)\chi_{\bar{r}_i}(g)}{1 - t\chi_f(\gamma) + t^2}.$$

The superconformal index is then given by the gauge invariant terms in the Plethystic exponential

$$(81) \qquad \mathrm{ind}(\mu, \gamma, h) = \int_G [dg] \, \exp\left(\sum_{l=1}^\infty \frac{1}{l} f(t^l, \gamma^l, h^l, g^l)\right).$$

As for ungauged theories the gauge theories have a mass gap and the arguments of section 3.3 go through. For this reason the superconformal index can be calculated in the free theory. The interactions manifest themselves in the preserved global symmetry group H. This agrees with the fact that the superconformal index is a topological quantity and should only depend on the universality class of the theory. Here the universality class of the theory is just given by the matter content and the symmetry group of the theory. All interactions that are consistent with the symmetry group are allowed.

5   Examples

Let us now calculate the superconformal index for a couple examples.

5.1 Leigh-Strassler flows

First let us look at the renormalization group flow of $\mathcal{N} = 4$ super Yang-Mills theory that is deformed by a mass term for one of the adjoint chiral multiplets. This flow was discussed by Leigh and Strassler [20] and in the context of holographic renormalization group flows by Freedman, Gubser, Pilch and Warner [21, 22]. In this theory the superpotential for the three adjoint chiral multiplets X, Y and Z is deformed by a mass term for Z

$$(82) \qquad W = \text{tr}[X,Y]Z \mapsto W = \text{tr}[X,Y]Z + \frac{m}{2}\,\text{tr}\,Z^2.$$

This breaks the $SO(6)$ R-symmetry to a global $SU(2)$ symmetry and a $U(1)_R$ symmetry. The charges of the chiral multiplets are

Field	$SU(2)$	$U(1)_R$
(X,Y)	2	$\frac{1}{2}$
Z	1	1

As already mentioned in section 4.2 the scalar Z has R-charge 1 and does not contribute to the superconformal index. This obviously agrees with the theory in the infrared, where Z has been integrated out. The same arguments apply for the Klebanov-Witten flow [23, 24, 25] and related flows.

5.2 Seiberg duality for a $SU(2)$ super Yang-Mills theory with three flavors

Let us now turn to a more interesting example and compare the superconformal index of two theories related by a strong coupling duality, Seiberg duality of SQCD in the conformal window $\frac{3}{2}N_c \leq N_f \leq 3N_c$. Seiberg duality [26] is the duality in the IR between an 'electric' $SU(N_c)$ gauge theory with N_f flavors and a 'magnetic' $SU(N_f - N_c)$ gauge theory with N_f flavors.

For simplicity let us first look at $SU(2)$ SQCD with 3 flavors. This theory has actually 6 quark fields transforming under a global $SU(6)$ flavor symmetry group. The oscillators contributing to the superconformal index are summarized in the following table

Theory	Field	$SU(2)_c$	$SU(6)$	$SU(2)_R$	3Ξ
electric	$\phi^{(l)}$	2	6	$l+1$	$1+3l$
	$\psi^{(l)}$	2	$\bar{6}$	$l+1$	$5+3l$
	$\lambda^{(l)}$	3	1	$l \oplus (l+2)$	$3+3l$
	$F^{(l)}$	3	1	$(l+1)\oplus(l+1)$	$6+3l$
magnetic	$\phi_M^{(l)}$	N/A	15	$l+1$	$2+3l$
	$\psi_M^{(l)}$	N/A	$\overline{15}$	$l+1$	$4+3l$

where l labels the appropriate harmonic on S^3.

The first terms in the t-expansion of the superconformal index of the electric theory are summarized in the following table

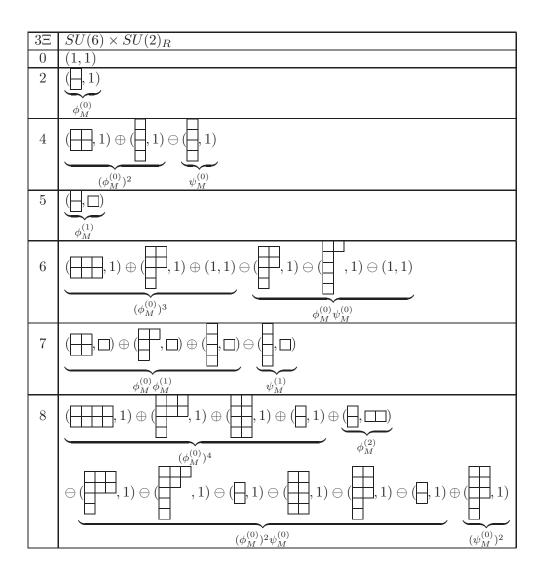| 3Ξ	$SU(6) \times SU(2)_R$
0	$(1,1)$
2	$\underbrace{(\ydiagram{2},1)}_{(\phi^{(0)})^2}$
4	$\underbrace{(\ydiagram{2,2},1)}_{(\phi^{(0)})^4}$
5	$\underbrace{(\ydiagram{2},\ydiagram{1}) \oplus (\ydiagram{1,1},\ydiagram{1})}_{\phi^{(0)}\phi^{(1)}} \ominus \underbrace{(\ydiagram{2},\ydiagram{1})}_{(\phi^{(0)})^2\lambda^{(0)}}$
6	$\underbrace{(\ydiagram{3,3},1)}_{(\phi^{(0)})^6} \oplus \underbrace{(1,1)}_{(\lambda^{(0)})^2} \ominus (1,1) \ominus \underbrace{(\ydiagram{2,1,1,1},1)}_{\phi^{(0)}\psi^{(0)}}$
7	$\underbrace{(\ydiagram{3,1},\ydiagram{1}) \oplus (\ydiagram{2,2},\ydiagram{1}) \oplus (\ydiagram{2,1,1},\ydiagram{1})}_{(\phi^{(0)})^3\phi^{(1)}} \ominus \underbrace{(\ydiagram{3,1},\ydiagram{1})}_{(\phi^{(0)})^4\lambda^{(0)}}$
8	$\underbrace{(\ydiagram{2},1) \oplus (\ydiagram{2},1) \oplus (\ydiagram{2},\ydiagram{2}) \oplus (\ydiagram{1,1},1)}_{F^{(0)}(\phi^{(0)})^2 \quad (\lambda^{(0)})^2(\phi^{(0)})^2} \ominus \underbrace{(\ydiagram{2},1) \ominus (\ydiagram{2},\ydiagram{2})}_{\lambda^{(1)}(\phi^{(0)})^2}$ $\oplus \underbrace{(\ydiagram{4,2},1)}_{(\phi^{(0)})^8} \ominus \underbrace{(\ydiagram{2},\ydiagram{2}) \ominus (\ydiagram{2},1) \ominus (\ydiagram{1,1},\ydiagram{2}) \ominus (\ydiagram{1,1},1)}_{\phi^{(0)}\phi^{(1)}\lambda^{(0)}} \oplus \underbrace{(\ydiagram{1,1},\ydiagram{2}) \oplus (\ydiagram{2},1)}_{(\phi^{(1)})^2}$ $\oplus \underbrace{(\ydiagram{2},\ydiagram{2}) \oplus (\ydiagram{1,1},\ydiagram{2})}_{\phi^{(0)}\phi^{(2)}} \ominus \underbrace{(\ydiagram{2,1,1,1},1) \ominus (\ydiagram{2},1) \ominus (\ydiagram{1,1},1)}_{(\phi^{(0)})^3\psi^{(0)}}$

A similar table can be made for the magnetic theory

3Ξ	$SU(6) \times SU(2)_R$
0	$(1,1)$
2	$(\,\yng(2)\,,1)$ $\underbrace{}_{\phi_M^{(0)}}$
4	$(\,\yng(2,2)\,,1) \oplus (\,\yng(1,1),1) \ominus (\,\yng(1,1),1)$ $\underbrace{}_{(\phi_M^{(0)})^2} \underbrace{}_{\psi_M^{(0)}}$
5	$(\,\yng(1,1),\square)$ $\underbrace{}_{\phi_M^{(1)}}$
6	$(\,\yng(3,3),1) \oplus (\,\yng(2,1),1) \oplus (1,1) \ominus (\,\yng(2,1),1) \ominus (\,\yng(1,1,1),1) \ominus (1,1)$ $\underbrace{}_{(\phi_M^{(0)})^3} \underbrace{}_{\phi_M^{(0)}\psi_M^{(0)}}$
7	$(\,\yng(2,1),\square) \oplus (\,\yng(2,1),\square) \oplus (\,\yng(1,1),\square) \ominus (\,\yng(1,1),\square)$ $\underbrace{}_{\phi_M^{(0)}\phi_M^{(1)}} \underbrace{}_{\psi_M^{(1)}}$
8	$(\,\yng(4,4),1) \oplus (\,\yng(3,2),1) \oplus (\,\yng(2,2),1) \oplus (\,\yng(1,1),1) \oplus (\,\yng(1),\yng(2))$ $\underbrace{}_{(\phi_M^{(0)})^4} \underbrace{}_{\phi_M^{(2)}}$ $\ominus (\,\yng(3,2),1) \ominus (\,\yng(3,2),1) \ominus (\,\yng(1,1),1) \ominus (\,\yng(2,2),1) \ominus (\,\yng(2,1),1) \ominus (\,\yng(1,1),1) \oplus (\,\yng(2,1),1)$ $\underbrace{}_{(\phi_M^{(0)})^2\psi_M^{(0)}} \underbrace{}_{(\psi_M^{(0)})^2}$

After taking into account cancellations between fermionic and bosonic states, those two expansions agree. From the physics of Seiberg duality it is expected, that the indices calculated on the electric and on the magnetic side actually agree to all orders in the expansion. This leads to the conjecture of a fairly involved group/number theory identity:

$$(83) \qquad \int_{SU(2)} [dg] \exp\left(\sum_{l=1}^{\infty} \frac{1}{l} f_e(t^l, \gamma^l, h^l, g^l)\right) = \exp\left(\sum_{l=1}^{\infty} \frac{1}{l} f_m(t^l, \gamma^l, h^l)\right),$$

where

$$(84) \qquad f_e(t,\gamma,h,g) = \frac{(2t^2 - t\chi_f(\gamma))\chi_3(g) + (t^{\frac{1}{3}}\chi_6(h) - t^{\frac{5}{3}}\chi_{\bar{6}}(h))\chi_2(g)}{1 - t\chi_f(\gamma) + t^2},$$

$$(85) \qquad f_m(t,\gamma,h) = \frac{t^{\frac{2}{3}}\chi_{15}(h) - t^{\frac{4}{3}}\chi_{\overline{15}}(h)}{1 - t\chi_f(\gamma) + t^2}.$$

5.3 More general Seiberg duality for SQCD in the conformal window

Calculating the superconformal index for more general $SU(N_c)$ SQCD with N_f flavors and its Seiberg dual is quite cumbersome, but we will check a few interesting terms in the t expansion. The oscillator modes contributing to the superconformal index are

Field	Ξ	G	H	$SU(2)_R$
$\phi_Q^{(l)}$	$1 - \frac{N_c}{N_f} + l$	f	$(N_f,1)_1$	$l+1$
$\phi_{\bar{Q}}^{(l)}$	$1 - \frac{N_c}{N_f} + l$	$\bar{f}$	$(1,\overline{N_f})_{-1}$	$l+1$
$\psi_Q^{(l)}$	$1 + \frac{N_c}{N_f} + l$	$\bar{f}$	$(\overline{N_f},1)_{-1}$	$l+1$
$\psi_{\bar{Q}}^{(l)}$	$1 + \frac{N_c}{N_f} + l$	f	$(1,N_f)_1$	$l+1$
$\lambda^{(l)}$	$1 + l$	Adj	$(1,1)_0$	$l \oplus (l+2)$
$F^{(l)}$	$2 + l$	Adj	$(1,1)_0$	$(l+1) \oplus (l+1)$

for the electric theory, where $G = SU(N_c)$ and H is the $SU(N_f) \times SU(N_f) \times U(1)_B$ flavor symmetry group, and

Field	Ξ	G'	H	$SU(2)_R$
$\phi_q^{(l)}$	$\frac{N_c}{N_f} + l$	f	$(\overline{N_f},1)_{\frac{N_c}{N_f-N_c}}$	$l+1$
$\phi_{\bar{q}}^{(l)}$	$\frac{N_c}{N_f} + l$	$\bar{f}$	$(1,N_f)_{-\frac{N_c}{N_f-N_c}}$	$l+1$
$\phi_M^{(l)}$	$2 - 2\frac{N_c}{N_f} + l$	1	$(N_f,\overline{N_f})_0$	$l+1$
$\psi_q^{(l)}$	$2 - \frac{N_c}{N_f} + l$	$\bar{f}$	$(N_f,1)_{-\frac{N_c}{N_f-N_c}}$	$l+1$
$\psi_{\bar{q}}^{(l)}$	$2 - \frac{N_c}{N_f} + l$	f	$(1,\overline{N_f})_{\frac{N_c}{N_f-N_c}}$	$l+1$
$\psi_M^{(l)}$	$2\frac{N_c}{N_f} + l$	1	$(\overline{N_f},N_f)_0$	$l+1$
$\lambda^{(l)}$	$1 + l$	Adj	$(1,1)_0$	$l \oplus (l+2)$
$F^{(l)}$	$2 + l$	Adj	$(1,1)_0$	$(l+1) \oplus (l+1)$

for the magnetic theory, with $G' = SU(N_f - N_c)$. We will now assume that $\frac{1}{3} \leq \frac{N_c}{N_f} \leq \frac{2}{3}$ is generic and N_c is large.

In the electric theory there are the following contributions to the superconformal index

Ξ	$H \times SU(2)_R$
0	$(1,1,1)_0$
$2\frac{N_c}{N_f}$	$-$
$1 + 2\frac{N_c}{N_f}$	$-$
$2 - 2\frac{N_c}{N_f}$	$\underbrace{(f,\bar{f},1)_0}_{\phi_Q^{(0)}\phi_{\bar{Q}}^{(0)}}$
2	$\ominus \underbrace{(1,1,1)_0 \ominus (Adj,1,1)_0}_{\phi_Q^{(0)}\psi_Q^{(0)}} \ominus \underbrace{(1,1,1)_0 \ominus (1,Adj,1)_0}_{\phi_{\bar{Q}}^{(0)}\psi_{\bar{Q}}^{(0)}}$ $\oplus \underbrace{(1,1,1)_0}_{(\lambda^{(0)})^2}$
$2 + 2\frac{N_c}{N_f}$	$\underbrace{(\bar{f},f,1)_0}_{\psi_Q^{(0)}\psi_{\bar{Q}}^{(0)}}$

In the magnetic theory the contributions look quite different

Ξ	$H \times SU(2)_R$
0	$(1,1,1)_0$
$2\frac{N_c}{N_f}$	$\underbrace{(\bar{f},f,1)_0}_{\phi_q^{(0)}\phi_{\bar{q}}^{(0)}} \ominus \underbrace{(\bar{f},f,1)_0}_{\psi_M^{(0)}}$
$1+2\frac{N_c}{N_f}$	$\ominus \underbrace{(\bar{f},f,2)_0}_{\phi_q^{(0)}\phi_{\bar{q}}^{(0)}\lambda^{(0)}} \oplus \underbrace{(\bar{f},f,2)_0}_{\phi_q^{(0)}\phi_{\bar{q}}^{(1)}} \oplus \underbrace{(\bar{f},f,2)_0}_{\phi_q^{(1)}\phi_{\bar{q}}^{(0)}} \ominus \underbrace{(\bar{f},f,2)_0}_{\psi_M^{(1)}}$
$2-2\frac{N_c}{N_f}$	$\underbrace{(f,\bar{f},1)_0}_{\phi_M^{(0)}}$
2	$\ominus \underbrace{(1,1,1)_0 \ominus (Adj,1,1)_0}_{\phi_q^{(0)}\psi_q^{(0)}} \ominus \underbrace{(1,1,1)_0 \ominus (1,Adj,1)_0}_{\phi_{\bar{q}}^{(0)}\psi_{\bar{q}}^{(0)}}$ $\ominus \underbrace{(1,1,1)_0 \ominus (Adj,1,1)_0 \ominus (1,Adj,1)_0 \ominus (Adj,Adj,1)_0}_{\phi_M^{(0)}\psi_M^{(0)}}$ $\oplus \underbrace{(1,1,1)_0 \oplus (Adj,1,1)_0 \oplus (1,Adj,1)_0 \oplus (Adj,Adj,1)_0}_{\phi_q^{(0)}\phi_{\bar{q}}^{(0)}\psi_M^{(0)}} \oplus \underbrace{(1,1,1)_0}_{(\lambda^{(0)})^2}$
$2+2\frac{N_c}{N_f}$	$\underbrace{(f \otimes \Lambda^2\bar{f}, \bar{f} \otimes S^2 f, 1)_0 \oplus (f \otimes S^2\bar{f}, \bar{f} \otimes \Lambda^2 f, 1)_0}_{\phi_M^{(0)}(\psi_M^{(0)})^2}$ $\oplus \underbrace{(f \otimes \Lambda^2\bar{f}, \bar{f} \otimes \Lambda^2 f, 1)_0 \oplus (f \otimes S^2\bar{f}, \bar{f} \otimes S^2 f, 1)_0}_{\phi_M^{(0)}(\phi_q^{(0)})^2(\phi_{\bar{q}}^{(0)})^2}$ $\ominus \underbrace{(f \otimes \Lambda^2\bar{f}, \bar{f} \otimes \Lambda^2 f, 1)_0 \ominus (f \otimes \Lambda^2\bar{f}, \bar{f} \otimes S^2 f, 1)_0}_{\phi_M^{(0)}\psi_M^{(0)}\phi_q^{(0)}\phi_{\bar{q}}^{(0)}}$ $\ominus \underbrace{(f \otimes S^2\bar{f}, \bar{f} \otimes \Lambda^2 f, 1)_0 \ominus (f \otimes S^2\bar{f}, \bar{f} \otimes S^2 f, 1)_0}_{\phi_M^{(0)}\psi_M^{(0)}\phi_q^{(0)}\phi_{\bar{q}}^{(0)}}$ $\ominus \underbrace{(f \otimes \Lambda^2\bar{f}, f, 1)_0 \ominus (f \otimes S^2\bar{f}, f, 1)_0}_{\psi_q^{(0)}(\phi_q^{(0)})^2\phi_{\bar{q}}^{(0)}} \ominus \underbrace{(\bar{f}, \bar{f} \otimes \Lambda^2 f, 1)_0 \ominus (\bar{f}, \bar{f} \otimes S^2 f, 1)_0}_{\psi_{\bar{q}}^{(0)}(\phi_{\bar{q}}^{(0)})^2\phi_q^{(0)}}$ $\oplus \underbrace{(f \otimes \Lambda^2\bar{f}, f, 1)_0 \oplus (f \otimes S^2\bar{f}, f, 1)_0}_{\psi_q^{(0)}\phi_q^{(0)}\psi_M^{(0)}} \oplus \underbrace{(\bar{f}, \bar{f} \otimes \Lambda^2 f, 1)_0 \oplus (\bar{f}, \bar{f} \otimes S^2 f, 1)_0}_{\psi_{\bar{q}}^{(0)}\phi_{\bar{q}}^{(0)}\psi_M^{(0)}}$ $\oplus \underbrace{(\bar{f},f,1)_0 \oplus (\bar{f},f,1)_0}_{F^{(0)}\phi_q^{(0)}\phi_{\bar{q}}^{(0)}} \ominus \underbrace{(\bar{f},f,1)_0}_{(\lambda^{(0)})^2\psi_M^{(0)}} \ominus \underbrace{(\bar{f},f,1)_0 \ominus (\bar{f},f,3)_0}_{\lambda^{(0)}\phi_q^{(1)}\phi_{\bar{q}}^{(0)}}$ $\ominus \underbrace{(\bar{f},f,1)_0 \ominus (\bar{f},f,3)_0}_{\lambda^{(0)}\phi_q^{(0)}\phi_{\bar{q}}^{(1)}} \ominus \underbrace{(\bar{f},f,1)_0 \ominus (\bar{f},f,3)_0}_{\lambda^{(1)}\phi_q^{(0)}\phi_{\bar{q}}^{(0)}} \ominus \underbrace{(\bar{f},f,3)_0}_{\psi_M^{(2)}} \oplus \underbrace{(\bar{f},f,3)_0}_{\phi_q^{(0)}\phi_{\bar{q}}^{(2)}}$ $\oplus \underbrace{(\bar{f},f,3)_0}_{\phi_q^{(2)}\phi_{\bar{q}}^{(0)}} \oplus \underbrace{(\bar{f},f,1)_0 \oplus \bar{f},f,3)_0}_{\phi_q^{(1)}\phi_{\bar{q}}^{(1)}} \oplus \underbrace{(\bar{f},f,1)_0 \oplus (\bar{f},f,1)_0 \oplus \bar{f},f,3)_0}_{(\lambda^{(0)})^2\phi_q^{(0)}\phi_{\bar{q}}^{(0)}}$

but there are cancellations such that the conformal indices of the electric and the magnetic theories agree at the given levels. Note that we checked only states with vanishing baryon number, as is usual for large N_c. This again leads to the conjecture of a fairly involved group/number theory identity:

$$(86) \quad \int\limits_{SU(N_c)} [dg_1] \exp\left(\sum_{l=1}^{\infty} \frac{1}{l} f_e(t^l, \gamma^l, h_1^l, h_2^l, b^l, g_1^l)\right)$$

$$= \int\limits_{SU(N_f-N_c)} [dg_2] \exp\left(\sum_{l=1}^{\infty} \frac{1}{l} f_m(t^l, \gamma^l, h_1^l, h_2^l, b^l, g_2^l)\right)$$

where f_e and f_m are given by (80) using the matter content of the electric and the magnetic theory respectively and $(h_1, h_2)_b \in SU(N_f) \times SU(N_f) \times U(1)_B$.

At the lowest orders in the t-expansion there seem to be mainly contributions from chiral operators and some 'relations' in the form of fermions. However, at higher orders the structure is much more complicated. Also, the gauginos typically contribute at levels, where a lot of other modes contribute. For this reason it seems hard to separate out the gaugino contributions.

6   Discussion

We have established, that the superconformal index can be calculated by reducing a interacting field theory at the conformal fixed point to a free $\mathcal{N} = 1$ supersymmetric field theory on $S^3 \times \mathbb{R}$. Using this result we checked a few examples of field theory dualities and found agreement at lower orders in the t-expansion of the full superconformal index.

It would be very interesting to prove in general that the superconformal index of dual theories agrees. The simple dependence of equations (80) and (81) for the superconformal index on the field content and the universal form of the denominator of f suggests, that there might be a simple geometrical argument involving the moduli space of vacua and the chiral ring or anomaly matching.

There are also still interesting examples involving String theory on $AdS_5 \times Y^{(p,q)}$ [27, 28] and its CFT duals [29]. In many of those examples the field theory descriptions are related by Seiberg dualities. Those models might be treated using large N matrix model technologies [9, 10].

Also, it is clear from the tables in sections 5.2 and 5.3 that there are net contributions to the superconformal index from fermions and from bosons

coming in different representations of the global symmetry group. This might partly explain the missing states in the black hole entropy counting of [**9**]. Those different states cannot really be seen very well by just looking at the characters without decomposing them into irreps: If the characters are written in terms of Cartan generators, often the coefficients of those have all the same sign, but when decomposed into irreps there are contributions with different signs.

It would be interesting to do an argument similar to the spectral sequence argument in section 3.3 to see whether the superconformal index actually counts all the $\frac{1}{4}$ BPS states in a generic theory or whether there are bigger cancellations inside the superconformal index of an interacting theory. This might lead to a description of those $\frac{1}{4}$ BPS states similar to [**30**] but with much less supersymmetry. This might be compared to gravitational black hole states or D-brane configurations similar to [**31**, **32**, **6**, **33**].

Appendix A   Some conventions

We follow the Metric conventions $(-+++)$ and the gamma matrices are

$$(87) \qquad \gamma^0 = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \gamma^i = 2 \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix},$$

$$(88) \qquad \gamma^5 = i\gamma^{0123} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix},$$

where the Pauli matrices are

$$(89) \quad \sigma^1_{(P)} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2_{(P)} = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3_{(P)} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The Hermitean conjugation is defined as

$$(90) \qquad (\gamma^\mu)^\dagger = -C\gamma^\mu C^{-1} \quad \text{and} \quad \bar\psi = \psi^\dagger C \quad \text{with} \quad C = \gamma^0,$$

the transpose as

$$(91) \qquad (\gamma^\mu)^t = D\gamma^\mu D^{-1} \quad \text{and} \quad \tilde\psi = \psi^t D \quad \text{with} \quad D = \gamma^{135}$$

and the complex conjugation as

$$(92) \qquad (\gamma^\mu)^* = -B\gamma^\mu B^{-1} \quad \text{and} \quad *\psi = B^{-1}\psi^* \quad \text{with} \quad B = \gamma^2.$$

Appendix B   Lagrangian for a $\mathcal{N} = 1$ gauge theory on $S^3 \times \mathbb{R}$

The Lagrangian for a $\mathcal{N} = 1$ gauge theory on $S^3 \times \mathbb{R}$ [**8**] is written down in a vielbein basis of right invariant 1-forms and the spatial derivatives are

derivatives with respect to the Killing vector fields generated by $SU(2)_L$.

$$(93) \qquad \mathcal{L}_g = \frac{1}{g^2}\Bigg(4\,\mathrm{tr}\,\mathcal{F}_{0i}\mathcal{F}_{0i} - 8\,\mathrm{tr}\,\mathcal{F}_{ij}\mathcal{F}_{ij}$$

$$+ i\,\mathrm{tr}\,\bar{\lambda}\gamma^0\mathcal{D}_0\lambda - 2i\,\mathrm{tr}\,\bar{\lambda}\gamma^i\left(\mathcal{D}_i + \frac{1}{8}\epsilon_{ijk}\gamma^{jk}\right)\lambda - \mathrm{tr}\,D^2\Bigg),$$

$$(94) \qquad \mathcal{L}_\theta = \theta\epsilon_{ijk}\mathcal{F}_{0i}\mathcal{F}_{jk},$$

$$(95) \qquad \mathcal{L}_0 = \left(\mathcal{D}_0 - i\frac{3q-2}{2}\right)\phi^\dagger\left(\mathcal{D}_0 + i\frac{3iq-2}{2}\right)\phi - 4\mathcal{D}_i\phi^\dagger\mathcal{D}_i\phi - \phi^\dagger\phi$$

$$+ i\bar{\psi}\gamma^0\left(\mathcal{D}_0 + i\frac{3q-2}{2}\right)\psi - 2i\bar{\psi}\gamma^i\left(\mathcal{D}_i + \frac{1}{8}\epsilon_{ijk}\gamma^{jk}\right)\psi + F^\dagger F$$

$$+ 2i\phi^\dagger D^{(\rho)}\phi - 2i\phi^\dagger\tilde{\lambda}^{(\rho)}\psi + 2\bar{\psi}(*\lambda)^{(\rho)}\phi,$$

$$(96) \qquad \mathcal{L}_W = W'(\phi)F - \frac{1}{2}W''(\phi)\tilde{\psi}\psi + \mathrm{h.c.}$$

Acknowledgments

References

[1] S. Benvenuti, B. Feng, A. Hanany, and Y.-H. He, "Counting bps operators in gauge theories: Quivers, syzygies and plethystics," `hep-th/0608050`.

[2] B. Feng, A. Hanany, and Y.-H. He, "Counting gauge invariants: The plethystic program," *JHEP* **03** (2007) 090, `hep-th/0701063`.

[3] A. Hanany and C. Römelsberger, "Counting bps operators in the chiral ring of n = 2 supersymmetric gauge theories or n = 2 braine surgery," `hep-th/0611346`.

[4] D. Forcella, A. Hanany, and A. Zaffaroni, "Baryonic generating functions," `hep-th/0701236`.

[5] D. Forcella, "Bps partition functions for quiver gauge theories: Counting fermionic operators," `arXiv:0705.2989 [hep-th]`.

[6] G. Mandal and N. V. Suryanarayana, "Counting 1/8-bps dual-giants," *JHEP* **03** (2007) 031, `hep-th/0606088`.

[7] F. A. Dolan, "Counting bps operators in n=4 sym," `arXiv:0704.1038 [hep-th]`.

[8] C. Römelsberger, "Counting chiral primaries in n = 1, d=4 superconformal field theories," *Nucl. Phys.* **B747** (2006) 329–353, `hep-th/0510060`.

[9] J. Kinney, J. M. Maldacena, S. Minwalla, and S. Raju, "An index for 4 dimensional super conformal theories," `hep-th/0510251`.

[10] Y. Nakayama, "Index for orbifold quiver gauge theories," *Phys. Lett.* **B636** (2006) 132–136, `hep-th/0512280`.

[11] Y. Nakayama, "Index for supergravity on ads(5) x t**(1,1) and conifold gauge theory," *Nucl. Phys.* **B755** (2006) 295–312, `hep-th/0602284`.

[12] Y. Nakayama, "Finite n index and angular momentum bound from gravity," `hep-th/0701208`.

[13] M. Bianchi, F. A. Dolan, P. J. Heslop, and H. Osborn, "N = 4 superconformal characters and partition functions," `hep-th/0609179`.

[14] J. B. Gutowski and H. S. Reall, "Supersymmetric ads(5) black holes," *JHEP* **02** (2004) 006, `hep-th/0401042`.

[15] E. Witten, "Constraints on supersymmetry breaking," *Nucl. Phys.* **B202** (1982) 253.

[16] S. Kachru and E. Witten, "Computing the complete massless spectrum of a landau- ginzburg orbifold," *Nucl. Phys.* **B407** (1993) 637–666, `hep-th/9307038`.

[17] R. Bott and L. Tu, *Differential Forms In Algebraic Topology*. Springer Verlag.

[18] K. Intriligator and B. Wecht, "The exact superconformal r-symmetry maximizes a," *Nucl. Phys.* **B667** (2003) 183–200, `hep-th/0304128`.

[19] O. Aharony, J. Marsano, S. Minwalla, K. Papadodimas, and M. Van Raamsdonk, "The hagedorn / deconfinement phase transition in weakly coupled large n gauge theories," *Adv. Theor. Math. Phys.* **8** (2004) 603–696, `hep-th/0310285`.

[20] R. G. Leigh and M. J. Strassler, "Exactly marginal operators and duality in four-dimensional n=1 supersymmetric gauge theory," *Nucl. Phys.* **B447** (1995) 95–136, `hep-th/9503121`.

[21] D. Z. Freedman, S. S. Gubser, K. Pilch, and N. P. Warner, "Renormalization group flows from holography supersymmetry and a c-theorem," *Adv. Theor. Math. Phys.* **3** (1999) 363–417, `hep-th/9904017`.

[22] K. Pilch and N. P. Warner, "N = 1 supersymmetric renormalization group flows from iib supergravity," *Adv. Theor. Math. Phys.* **4** (2002) 627–677, `hep-th/0006066`.

[23] I. R. Klebanov and E. Witten, "Superconformal field theory on threebranes at a calabi-yau singularity," *Nucl. Phys.* **B536** (1998) 199–218, `hep-th/9807080`.

[24] N. Halmagyi, K. Pilch, C. Römelsberger, and N. P. Warner, "The complex geometry of holographic flows of quiver gauge theories," *JHEP* **09** (2006) 063, `hep-th/0406147`.

[25] N. Halmagyi, K. Pilch, C. Römelsberger, and N. P. Warner, "Holographic duals of a family of n = 1 fixed points," *JHEP* **08** (2006) 083, `hep-th/0506206`.

[26] N. Seiberg, "Electric - magnetic duality in supersymmetric nonabelian gauge theories," *Nucl. Phys.* **B435** (1995) 129–146, `hep-th/9411149`.

[27] J. P. Gauntlett, D. Martelli, J. Sparks, and D. Waldram, "Sasaki-einstein metrics on s(2) x s(3)," *Adv. Theor. Math. Phys.* **8** (2004) 711–734, `hep-th/0403002`.

[28] D. Martelli and J. Sparks, "Toric geometry, sasaki-einstein manifolds and a new infinite class of ads/cft duals," *Commun. Math. Phys.* **262** (2006) 51–89, `hep-th/0411238`.

[29] S. Benvenuti, S. Franco, A. Hanany, D. Martelli, and J. Sparks, "An infinite family of superconformal quiver gauge theories with sasaki-einstein duals," *JHEP* **06** (2005) 064, `hep-th/0411264`.

[30] D. Berenstein, "A toy model for the ads/cft correspondence," *JHEP* **07** (2004) 018, `hep-th/0403110`.

[31] A. Mikhailov, "Giant gravitons from holomorphic surfaces," *JHEP* **11** (2000) 027, `hep-th/0010206`.

[32] C. E. Beasley, "Bps branes from baryons," *JHEP* **11** (2002) 015, `hep-th/0207125`.

[33] I. Biswas, D. Gaiotto, S. Lahiri, and S. Minwalla, "Supersymmetric states of n = 4 yang-mills from giant gravitons," `hep-th/0606087`.