

# Three order parameters in quantum XZ spin-oscillator models with Gibbsian ground states

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## Abstract

Quantum models on the hyper-cubic  $d$ -dimensional lattice of spin- $\frac{1}{2}$  particles interacting with linear oscillators are shown to have three ferromagnetic ground state order parameters. Two order parameters coincide with the magnetization in the first and third directions and the third one is a magnetization in a continuous oscillator variable. The proofs use a generalized Peierls argument and two Griffiths inequalities. The class of spin-oscillator Hamiltonians considered manifest maximal ordering in their ground states. The models have relevance for hydrogen-bond ferroelectrics. The simplest of these is proven to have a unique Gibbsian ground state.

## 1 Introduction

In this paper we consider quantum lattice models of oscillators interacting with spins whose variables are indexed by the sites of the hyper-cubic lattice  $\mathbb{Z}^d$ . For a finite subset  $\Lambda \subset \mathbb{Z}^d$ , the corresponding Hamiltonian  $H_\Lambda$  is expressed in terms of the oscillator variables  $q_\Lambda = (q_x, x \in \Lambda) \in \mathbf{R}^{|\Lambda|}$  and spin- $\frac{1}{2}$  operators  $S_\Lambda^l = (S_x^l, x \in \Lambda, l = 1, 3)$ , defined in the tensor product of the  $2^{|\Lambda|}$ -dimensional Euclidean space and the space of square integrable functions  $\mathbf{L}_\Lambda^2 = (\otimes \mathbf{E}^2)^{|\Lambda|} \otimes L^2(\mathbf{R}^{|\Lambda|})$ ,

$$H_\Lambda = \sum_{x \in \Lambda} [-\partial_x^2 + \mu^2(q_x + \eta \phi_x(S_\Lambda^3))^2 - \mu] + \sum_{A \subset \Lambda} J_A S_{[A]}^1 + V_\Lambda, \quad \mu \geq 0, \eta \in \mathbf{R}. \quad (1.1)$$

Here  $\partial_x$  is the partial derivative with respect to  $q_x$ , and  $J_A$  and  $V_\Lambda$  are real-valued measurable functions, the first of which depends only on  $q_\Lambda$ , the second possibly on  $S_\Lambda^3$  and  $q_\Lambda$ . The potentials  $\phi_x$  are of the form

$$\phi_x(s_\Lambda) = - \sum_{A \subset \Lambda} J'_{x,A} s_{[A]}, \quad (1.2)$$

where  $J'_{x,\emptyset} = 0$  and all the parameters  $J'_{x,A} \geq 0$ . We will also use the notation  $B_{[A]} = \prod_{x \in A} B_x$ .

The scalar product in  $(\otimes \mathbf{E}^2)^{|A|}$  and  $\mathbf{L}_\Lambda^2$  will be denoted by  $(\cdot, \cdot)_0$  and  $(\cdot, \cdot)$ , respectively. We will employ the orthonormal basis  $\psi_\Lambda^0(s_\Lambda)$  of the Euclidean space  $(\otimes \mathbf{E}^2)^{|A|}$ , diagonalizing  $S_\Lambda^3$ , which is chosen in the following standard way:

$$\begin{aligned} \psi_\Lambda^0(s_\Lambda) &= \otimes_{x \in \Lambda} \psi^0(s_x), \quad s_x = \pm \frac{1}{2}, \quad \psi^0(\frac{1}{2}) = (1, 0), \quad \psi^0(-\frac{1}{2}) = (0, 1), \\ S^1 \psi^0(s) &= \frac{1}{2} \psi^0(-s), \quad S^3 \psi^0(s) = s \psi^0(s). \end{aligned}$$

Thus if we write for  $F \in \mathbf{L}_\Lambda^2$ ,

$$F = \sum_{s_\Lambda} F(q_\Lambda, s_\Lambda) \psi_\Lambda^0(s_\Lambda),$$

then

$$(F_1(q_\Lambda), F_2(q_\Lambda))_0 = \sum_{s_\Lambda} F_1(q_\Lambda, s_\Lambda) F_2(q_\Lambda, s_\Lambda),$$

and

$$(F_1, F_2) = \sum_{s_\Lambda} \int F_1(q_\Lambda, s_\Lambda) F_2(q_\Lambda, s_\Lambda) dq_\Lambda,$$

where the integration is performed over  $\mathbf{R}^{|A|}$ .

We require that the Hamiltonian is well defined and bounded from below on  $(\otimes \mathbf{E}^2)^{|A|} \otimes C_0^\infty(\mathbf{R}^{|A|})$ , i.e. the tensor product of the  $2^{|A|}$ -dimensional Euclidean space and the space of infinitely differentiable functions with compact supports. In Theorem 1.2 we formulate conditions for this operator to be essentially self-adjoint.

Our aim is to establish the existence of ferromagnetic long-range order in the oscillator and spin variables in the ground state of these Hamiltonians, implying the existence of the corresponding order parameters. This means that we have to prove that the ground state averages  $\langle \hat{q}_x \hat{q}_y \rangle_\Lambda$  and  $\langle S_x^j S_y^j \rangle_\Lambda$ ,  $j = 1, 3$  are bounded from below uniformly in  $\Lambda$  by positive numbers (here  $\hat{q}_x$  is the operator of multiplication by  $q_x$ ). Occurrence of ferromagnetic long-range order implies the existence of spin order parameters (magnetization in the first and third directions)  $M_\Lambda^l = |\Lambda|^{-1} \sum_{x \in \Lambda} S_x^l$ , ( $l = 1, 3$ ), and an oscillator order parameter  $Q_\Lambda = |\Lambda|^{-1} \sum_{x \in \Lambda} q_x$  in the thermodynamic limit ( $\Lambda \rightarrow \mathbf{Z}^d$ ) since the ground state averages of their squares are bounded from below uniformly in  $\Lambda$  by a positive constant.

Our study is inspired by hydrogen-bond ferroelectric crystal models, considered in [VS, K, KP], whose Hamiltonians may be identified with the Hamiltonian (1.1) with a special choice of  $J_A, V_\Lambda$ . Such Hamiltonians describe the interaction between heavy ions (oscillators with constant frequency) and protons (spins). The second term with  $J_A = 0$  for  $|A| \geq 2$ , corresponds to the energy of protons, tunnelling along hydrogen bonds from one well to another, and  $J_x$  is associated with the tunnelling frequency. The last term in the expression for the Hamiltonian  $V_\Lambda = - \sum_{A \subseteq \Lambda} J_A'' S_{[A]}^3$  describes many-body interaction between protons ( $J_A''$  is the intensity of the  $|A|$ -body interaction).

The simplest interaction between ions and protons is described by the Hamiltonian

$$H_\Lambda = \sum_{x \in \Lambda} [-\partial_x^2 + \mu^2 q_x^2 + \eta' q_x \phi_x(S_\Lambda^3)] + \sum_{x \in \Lambda} J_x S_x^1, \quad (1.3)$$

where

$$\phi_x(S_\Lambda^3) = \sum_{y \in \Lambda} J(x-y) S_y^3, \quad J(x) = J(-x). \quad (1.4)$$

The Hamiltonian (1.3) reduces to the one given by (1.1) if  $\eta' = 2\eta\mu^2$ ,  $J_A = 0$ ,  $|A| \geq 2$  and

$$V_\Lambda + \mu|\Lambda| = -(\mu\eta)^2 \sum_{x \in \Lambda} \phi_x^2(S_\Lambda^3) = -(\mu\eta)^2 \sum_{x, y \in \Lambda} J_\Lambda^{(2)}(x, y) S_x^3 S_y^3, \quad (1.5)$$

where  $J_\Lambda^{(2)}(x, y) = \sum_{x' \in \Lambda} J(x - x') J(y - x')$ .

In Section 2 we shall show that if  $J_A$  and  $V_\Lambda$  do not depend on the oscillator variables, the oscillator and spin subsystems can be decoupled with the help of a translation of the oscillator variables on each one-dimensional subspace of  $(\otimes \mathbf{C}^2)^{|\Lambda|}$ . In the case  $V_\Lambda = 0$  the spin part of the decoupled Hamiltonian can be mapped by a duality transformation to a quantum Ising model. From a mathematical point of view the case  $V_\Lambda \neq 0$  has more relevance to interesting physical phenomena. The following theorem will be proved in Section 2.

**Theorem 1.1.** *Let the Hamiltonian be given by (1.3) and (1.4) with finite-range  $J(x)$  or by (1.1) with  $J_A = 0$ ,  $|A| \geq 2$  and a translation invariant finite-range potential*

$$V_\Lambda = - \sum_{A \subseteq \Lambda} J_A'' S_{[A]}^3$$

which is independent of  $q_\Lambda$ . Then, for sufficiently small  $J_A'' > 0$ , there exists a function  $U_*(s_\Lambda)$  such that the vector-valued function

$$\Psi_\Lambda(q_\Lambda) = \sum_{s_\Lambda} e^{-\frac{1}{2}U_*(s_\Lambda)} \psi_\Lambda^0(s_\Lambda) \psi_{0\Lambda}(q_\Lambda + \eta\phi_\Lambda(s_\Lambda)) \quad (1.6)$$

is a ground state of the respective Hamiltonians, and it is unique if  $J_x < 0$  for all  $x \in \Lambda$ . Here the summation is performed over the  $|\Lambda|$ -fold Cartesian product:  $s_\Lambda \in \{-\frac{1}{2}, \frac{1}{2}\}^{|\Lambda|}$ , and  $\psi_{0,\Lambda}$  is the free oscillator ground state:

$$\psi_{0\Lambda}(q_\Lambda) = \prod_{x \in \Lambda} \psi_0(q_x) \text{ where } \psi_0(q) = (\mu\pi^{-1})^{\frac{1}{4}} \exp\{-\frac{\mu}{2}q^2\}.$$

Moreover, the following equality holds for  $x \neq y$ :

$$\langle \hat{q}_x \hat{q}_y \rangle_\Lambda = \eta^2 \langle \phi_x(S_\Lambda^3) \phi_y(S_\Lambda^3) \rangle_\Lambda, \quad (1.7)$$

where the averages  $\langle \cdot \rangle_\Lambda$  and  $\langle \cdot \rangle_{*\Lambda}$  correspond to the ground state (1.6) and the Ising model with potential energy  $U_*$ , respectively.

The vector in (1.6) is Gibbsian since it can be represented in the following form

$$\Psi_\Lambda(q_\Lambda) = \sum_{s_\Lambda} e^{-\frac{1}{2}U(s_\Lambda; q_\Lambda)} \prod_{x \in \Lambda} \psi^0(s_x) \psi_0(q_x). \quad (1.8)$$

For the cases considered in Theorem 1.1 we have  $U(s_\Lambda; q_\Lambda) = U_*(s_\Lambda) + \mu \sum_{x \in \Lambda} [(q_x + \eta\phi_x(s_\Lambda))^2 - q_x^2]$ .

In the general case  $\psi_0$  need not be Gaussian.

Gibbsian ground states were introduced by Kirkwood and Thomas in [KT] for the simplest XZ spin- $\frac{1}{2}$  models with Hamiltonians coinciding with the sum of the second and third terms in the right-hand side in (1.1). Matsui in [?, ?] proposed a large class of spin- $\frac{1}{2}$  XZ-type models in which Gibbsian ground states exist. The method was further extended by Datta and Kennedy [DK]. Theorem 1.1 is in fact proven easily applying the results of [KT]. Indeed, if one drops  $\psi_{0\Lambda}$  in the right-hand side of (1.6) then the resulting state will coincide with the ground state

of the spin part of (1.1). Using results of [KT] one can prove an analogue of (1.6) when  $J''_{x,y}$  is sufficiently large and  $J''_A = 0$  for  $|A| \neq 2$  implying occurrence of spin long-range order (lro) in the third direction.

A rigorous analysis of a mean-field version of the Hamiltonian (1.1) with  $\phi_x$  given by (1.4), where  $J_x \neq 0$ ,  $J''_{x,y}$  are the same at all lattice sites and proportional to  $|\Lambda|^{-1}$ , and  $J_A = 0$  for  $|A| > 2$ , was carried out in [KP] in the frame-work of the Bogoliubov approximating Hamiltonian method. Here, the occurrence of spin and oscillator ordering (the corresponding order parameters are non-zero) for two-body interaction between protons at non-zero temperatures was proved.

In [DS] we showed how to associate a Hamiltonian with a prescribed Gibbsian ground state and we established the existence of long-range order in  $S^1$  and  $S^3$  for a wide class of spin- $\frac{1}{2}$  models (see the Remark at the end of the paper). In Theorem 1.3 we extend this method to models of the type (1.1).

If  $V_\Lambda$  admits a simple dependence on  $q_\Lambda$  then one can prove that the existence of a Gibbsian ground state implies maximal ordering in the system:

**Theorem 1.2.** *Let  $\phi_x$  be defined in (1.4) and*

$$V_\Lambda = -\frac{1}{2} \sum_{A \subseteq \Lambda} J_A v_{[A]}, \quad v_x = \cosh(\frac{1}{2}u_x) + 2S_x^3 \sinh(\frac{1}{2}u_x), \quad u_x = 2\eta\mu\phi_x(q_\Lambda).$$

(Note that  $J_A$  is the same as in the second term of (1.1).) Let  $\|J\|_1 = \sum_x |J(x)| < \infty$ , where the summation is performed over  $\mathbf{Z}^d$ , and assume that the functions  $J_A$ ,  $A \subset \mathbf{Z}^d$ , are negative and bounded. Then

I.  $H_\Lambda$  is positive definite and essentially self-adjoint on the set  $(\otimes \mathbf{C}^2)^{|\Lambda|} \otimes C_0^\infty(\mathbf{R}^{|\Lambda|})$ . Moreover,  $\Psi_\Lambda$  given by (1.4) with  $U_*(s_\Lambda) = -\eta^2\mu \sum_{x \in \Lambda} \phi_x^2(s_\Lambda)$ , is its ground state with zero eigenvalue and it is unique if for all  $x \in \Lambda$  the inequality  $J_x \leq J_- < 0$  holds, where  $J_-$  does not depend on  $q_\Lambda$ ;

II. (1.7) is true and the ground state magnetization in the first direction  $M_\Lambda^1$  is non-zero in the thermodynamic limit in an arbitrary dimension  $d$ . If  $J(x) \geq 0$  and  $J(0) = 1, J(1) > 0$  then for sufficiently large  $\eta^2\mu J(1)$  the ground state magnetization in the third direction  $M_\Lambda^3$  and the oscillator order parameter  $Q_\Lambda$  are non-zero in the thermodynamic limit in dimensions  $d \geq 2$ .

Statement I of Theorem 1.2 follows from the following theorem which gives a general prescription of how to construct  $V_\Lambda$  yielding the Hamiltonian with Gibbsian ground state (1.8) (Cf. [DS]). The function  $V_\Lambda$  in Theorem 1.2 coincides with that of Theorem 1.3 if we put  $\alpha = 0$ .

The existence of order parameters in Statement II of Theorem 1.2 follows from Theorems 4.2 and 4.3 and its corollary. Note that the condition of large  $\beta$  in Theorem 4.2 follows from the corresponding condition in Theorem 1.2 putting  $\alpha = 0$ .

**Theorem 1.3.** *Let  $S_\Lambda^{3A} = (-S_A^3, S_{\Lambda \setminus A}^3)$  and*

$$V_\Lambda = -\frac{1}{2} \sum_{A \subseteq \Lambda} J_A e^{-\frac{1}{2}W_A(S_\Lambda^3)}, \quad W_A(S_\Lambda^3) = U(S_\Lambda^{3A}; q_\Lambda) - U(S_\Lambda^3; q_\Lambda), \quad (1.9)$$

where

$$U(s_\Lambda; q_\Lambda) = 2\mu\eta \sum_{x \in \Lambda} q_x \phi_x(s_\Lambda) + \alpha U_0(s_\Lambda). \quad (1.10)$$

Then

I. the Gibbsian vector  $\Psi_\Lambda$ , given by (1.8), is an eigenvector of  $H_\Lambda$  with zero eigenvalue. Besides, equalities (1.6) and (1.7) hold with

$$U_*(s_\Lambda) = -\eta^2 \mu \sum_{x \in \Lambda} \phi_x^2(s_\Lambda) + \alpha U_0(s_\Lambda). \quad (1.11)$$

Moreover, if the functions  $J_A, A \subset \mathbf{Z}^d$ , are negative and bounded, then

II.

$$H'_\Lambda = - \sum_{A \subseteq \Lambda} J_A P_{[A]} \quad (1.12)$$

is positive definite on  $(\otimes \mathbf{C}^2)^{|\Lambda|} \otimes C_0^\infty(\mathbf{R}^{|\Lambda|})$ , where

$$P_{[A]} = \frac{1}{2} e^{-\frac{1}{2} W_A(S_\Lambda^3)} - S_{[A]}^1; \quad (1.13)$$

III.  $H_\Lambda$  is essentially self-adjoint on a set containing  $(\otimes \mathbf{C}^2)^{|\Lambda|} \otimes C_0^\infty(\mathbf{R}^{|\Lambda|})$ ;

VI.  $H_\Lambda$  is positive definite and  $\Psi_\Lambda$  is its ground state;

V.  $\Psi_\Lambda$  is the unique ground state if for all  $x \in \Lambda$  the inequality  $J_x \leq J_- < 0$  holds, where  $J_-$  does not depend on  $q_\Lambda$ .

**Remark 1.1** If the functions  $J_A$  are only negative but not bounded then Statement II still holds and  $\Psi_\Lambda$  is the ground state of the self-adjoint extension of the Hamiltonian preserving positive definiteness.

The expression for  $V_\Lambda$  in Theorem 1.2 is derived from (1.9) for the case  $U_0 = 0$  with the help of the equalities

$$W_A(S_\Lambda^3) = -2 \sum_{x \in A} u_x S_x^3, \quad e^{aS^3} = \cosh(\frac{1}{2}a) + 2S^3 \sinh(\frac{1}{2}a), \quad (S^3)^2 = \frac{1}{4}I,$$

implying that

$$J''_A = \frac{1}{2} \prod_{x \in A} 2 \sinh(\frac{1}{2}u_x) \sum_{A' \subseteq \Lambda \setminus A} J_{A \cup A'} \prod_{y \in A'} \cosh(\frac{1}{2}u_y).$$

The expression for  $V_\Lambda$  can be calculated for certain  $U_*$  [DS]. We will prove an analog of Statement II in Theorem 1.2 for a general ferromagnetic  $U_0$ . It turns out that lro in  $S^1$  occurs even in one-dimensional systems and  $U_0$  need not be ferromagnetic if it satisfies some additional conditions (see Theorem 4.3). For anti-ferromagnetic  $U_0$ , specifically,

$$U_*(s_\Lambda) = -\eta^2 \mu \sum_{x \in \Lambda} \phi_x^2(s_\Lambda) + \alpha \sum_{\langle x, y \rangle} s_x s_y$$

( $\alpha > 0$ ), we can prove that the spin long-range order in the third direction will be anti-ferromagnetic, generating a staggered magnetization. In order to prove the occurrence of spin lro at a non-zero temperature for our spin-oscillator Hamiltonians one would have to generalize the technique developed in [TY] for spin  $XZ$ -type Hamiltonians.

The interesting and important property of the models considered in Theorem 1.3, is that their Hamiltonians are simply related to generators of stationary Markovian processes. We believe that it is possible to apply the same mathematical technique for proving existence of

an order and phase transitions in equilibrium quantum systems and nonequilibrium stochastic systems (see [SII]).

Our paper is organized as follows. In the next sections Theorem 1.1 and Theorem 1.3 are proven. In Section 4, a proof of the existence of lro and associated order parameters for our general Hamiltonian is presented in Theorems 4.2 and 4.3. Theorem 4.2 is a consequence of the Basic Theorem 4.1 guaranteeing existence of lro and a spin order parameter in a general ferromagnetic Ising model. A proof of Statement II of Theorem 1.2 is a direct consequence of Theorems 4.2 and 4.3.

## 2 Proof of Theorem 1.1

We define the translation operators  $T_x(\phi)$  and  $T_\Lambda = \prod_{x \in \Lambda} T_x(\phi)$  as follows:

$$(T_x(\phi)F)(q_\Lambda) = \sum_{s_\Lambda} F(q_x + \eta\phi_x(s_\Lambda), q_{\Lambda \setminus x}; s_\Lambda) \psi_\Lambda^0(s_\Lambda), \quad (2.1)$$

or

$$(T_x(\phi)F)(q_\Lambda; s_\Lambda) = F(q_x + \eta\phi_x(s_\Lambda), q_{\Lambda \setminus x}; s_\Lambda).$$

Let

$$h_x = -\partial_x^2 + \mu^2(q_x + \eta\phi_x(s_\Lambda^3))^2 - \mu \quad (2.2)$$

and

$$h_x^0 = -\partial_x^2 + \mu^2 q_x^2 - \mu = (-\partial_x + \mu q_x)(\partial_x + \mu q_x) \quad (2.3)$$

and put

$$h_\Lambda = \sum_{x \in \Lambda} h_x, \quad h_\Lambda^0 = \sum_{x \in \Lambda} h_x^0.$$

Then

$$h_x = T_x(\phi)h_x^0 T_x^{-1}(\phi), \quad h_\Lambda = T_\Lambda h_\Lambda^0 T_\Lambda^{-1}, \quad (2.4)$$

where we took into account that differentiation commutes with  $T_\Lambda$ . This means that the Hamiltonian  $\tilde{H}_\Lambda = T_\Lambda^{-1} H_\Lambda T_\Lambda = h_\Lambda^0 + \tilde{V}_\Lambda$  is decomposed into the sum of a free oscillator Hamiltonian and a pure spin  $XZ$ -type Hamiltonian  $\tilde{V}_\Lambda$  if  $J_A$  and  $V_\Lambda$  do not depend on oscillator variables. Then, by Theorem 2.1 of [KT], this pure spin Hamiltonian possesses a ground state of the form

$$\Psi_\Lambda^0 = \sum_{s_\Lambda} e^{-\frac{1}{2}U_*(s_\Lambda)} \psi_\Lambda^0(s_\Lambda)$$

and hence  $\tilde{H}_\Lambda$  has a ground state  $\tilde{\Psi}_\Lambda$  equal to

$$\tilde{\Psi}_\Lambda(q_\Lambda) = \sum_{s_\Lambda} e^{-\frac{1}{2}U_*(s_\Lambda)} \psi_\Lambda^0(s_\Lambda) \psi_{0\Lambda}(q_\Lambda).$$

Thus  $\Psi_\Lambda(q_\Lambda) = T_\Lambda \tilde{\Psi}_\Lambda(q_\Lambda)$  is a ground state of  $H_\Lambda$ . Uniqueness of the ground state follows from the arguments given below in the proof of Statement V of Theorem 1.3.

We now prove (1.7). From the orthonormality of the basis it follows that

$$\begin{aligned} \|\Psi_\Lambda\|^2 &= \int (\Psi_\Lambda(q_\Lambda), \Psi_\Lambda(q_\Lambda))_0 dq_\Lambda \\ &= \sum_{s_\Lambda} e^{-U_*(s_\Lambda)} \int \psi_{0\Lambda}^2(q_\Lambda + \eta\phi_\Lambda(s_\Lambda)) dq_\Lambda \\ &= \sum_{s_\Lambda} \exp\{-U_*(s_\Lambda)\} =: Z_\Lambda. \end{aligned}$$

By a similar argument we have

$$\begin{aligned}
Z_\Lambda \langle \hat{q}_x \hat{q}_y \rangle_\Lambda &= \int q_x q_y (\Psi_\Lambda(q_\Lambda), \Psi_\Lambda(q_\Lambda))_0 dq_\Lambda \\
&= \sum_{s_\Lambda} e^{-U_*(s_\Lambda)} \int q_x q_y \psi_{0\Lambda}^2(q_\Lambda + \eta \phi_\Lambda(s_\Lambda)) dq_\Lambda \\
&= \sum_{s_\Lambda} e^{-U_*(s_\Lambda)} \int q_x \psi_0^2(q_x + \eta \phi_x(s_\Lambda)) dq_x \int q_y \psi_0^2(q_y + \eta \phi_y(s_\Lambda)) dq_y \\
&= \eta^2 \sum_{s_\Lambda} e^{-U_*(s_\Lambda)} \phi_x(s_\Lambda) \phi_y(s_\Lambda).
\end{aligned}$$

Here we applied the equality  $\int q \psi_0^2(q) dq = 0$ . We also have

$$\begin{aligned}
Z_\Lambda \langle \phi_x(S_\Lambda^3) \phi_y(S_\Lambda^3) \rangle_\Lambda &= \int (\Psi_\Lambda(q_\Lambda), \phi_x(S_\Lambda^3) \phi_y(S_\Lambda^3) \Psi_\Lambda(q_\Lambda))_0 dq_\Lambda \\
&= \sum_{s_\Lambda} \phi_x(s_\Lambda) \phi_y(s_\Lambda) e^{-U_*(s_\Lambda)} \int \psi_{0\Lambda}^2(q_\Lambda + \eta \phi_\Lambda(s_\Lambda)) dq_\Lambda \\
&= \sum_{s_\Lambda} e^{-U_*(s_\Lambda)} \phi_x(s_\Lambda) \phi_y(s_\Lambda).
\end{aligned}$$

Thus (1.7) holds. Existence of lro follows from Proposition 3.2 of [KT]. Q.E.D.

### 3 Proof of Theorem 1.3

**Proof of Statement I.** Our Hamiltonian can be rewritten as follows

$$H_\Lambda = \sum_{x \in \Lambda} h_x - \sum_{A \subseteq \Lambda} J_A P_{[A]}, \quad (3.1)$$

where

$$P_{[A]} = \frac{1}{2} e^{-\frac{1}{2} W_A(S_\Lambda^3)} - S_{[A]}^1.$$

For simplicity we omit  $q_\Lambda$  in  $U(S_\Lambda^3; q_\Lambda)$ . The remarkable fact is that the symmetric operator  $P_A$  and the harmonic oscillator operators  $h_x$  have a common eigenvector  $\Psi_\Lambda$  with zero eigenvalue. (1.6) follows from (1.8), (1.10) and (1.11). Note that the space of ground states of the operator  $h_x$  (eigenfunctions with the zero eigenvalue) is  $2^{|\Lambda|}$ -fold degenerate since  $S_x^3$  is diagonal and the Laplacian is translation invariant. From (1.6) and the definition of  $T_x(\phi)$  it follows that  $T_x^{-1}(\phi) \Psi_\Lambda$  is equal to  $\psi_0(q_x)$  multiplied by a function independent of  $q_x$ :

$$T_x^{-1}(\phi) \Psi_\Lambda(q_\Lambda) = \sum_{s_\Lambda} e^{-\frac{1}{2} U_*(s_\Lambda)} \psi_\Lambda^0(s_\Lambda) \psi_{0\Lambda}(q_{\Lambda \setminus x} + \eta \phi_{\Lambda \setminus x}(s_{\Lambda \setminus x})) \psi_0(q_x).$$

Hence  $h_x^0 T_x^{-1}(\phi) \Psi_\Lambda = 0$  and

$$h_x \Psi_\Lambda = T_x(\phi) h_x^0 T_x^{-1}(\phi) \Psi_\Lambda = 0.$$

The proof that  $\Psi_\Lambda$  is an eigenvector of  $P_{[A]}$  with zero eigenvalue is inspired by our previous paper [DS]. Taking into consideration the equalities

$$S_{[A]}^1 \psi_\Lambda^0(s_\Lambda) = \frac{1}{2} \psi_\Lambda^0(s_\Lambda^A) = \frac{1}{2} \psi_\Lambda^0(s_{\Lambda \setminus A}, -s_A), \quad S_x^3 \psi_\Lambda^0(s_\Lambda) = s_x \psi_\Lambda^0(s_\Lambda)$$

we obtain

$$(\psi_{0\Lambda})^{-1}P_{[A]}\Psi_\Lambda = \frac{1}{2} \sum_{s_\Lambda} \left( e^{-\frac{1}{2}W_A(s_\Lambda)}\psi_\Lambda^0(s_\Lambda) - \psi_\Lambda^0(s_{\Lambda\setminus A}, -s_A) \right) e^{-\frac{1}{2}U(s_\Lambda)} \quad (3.2)$$

$$= \frac{1}{2} \sum_{s_\Lambda} \left( \psi_\Lambda^0(s_\Lambda)e^{-\frac{1}{2}U(s_\Lambda^A)} - \psi_\Lambda^0(s_{\Lambda\setminus A}, -s_A)e^{-\frac{1}{2}U(s_\Lambda)} \right) \quad (3.3)$$

$$= \frac{1}{2} \sum_{s_\Lambda} \left( e^{-\frac{1}{2}U(s_\Lambda^A)} - e^{-\frac{1}{2}U(s_\Lambda)} \right) \psi_\Lambda^0(s_\Lambda) = 0. \quad (3.4)$$

Here we changed signs of the spin variables  $s_A$  in the first term in the sum in  $s_\Lambda$ . Q.E.D.

**Proof of Statement II.** The following proposition will do the job.

**Proposition 3.1.** *The operator  $P_A$  is positive definite.*

**Proof.** We have to show that  $(P_A F(q_\Lambda), F(q_\Lambda))_0 \geq 0$ . Let us define the operator

$$P_A^+ = e^{\frac{1}{2}U(S_\Lambda^3)} P_A e^{-\frac{1}{2}U(S_\Lambda^3)}. \quad (3.5)$$

It is not difficult to check in the basis  $\psi_\Lambda^0$  that

$$P_A^+ = e^{-\frac{1}{2}W_A(S_\Lambda^3)} \left( \frac{1}{2}I - S_{[A]}^1 \right), \quad (3.6)$$

where  $I$  is the unit operator.

For the operator  $P_A^+$  we have

$$P_A^+ F = \sum_{s_\Lambda} (P_A^+ F)(s_\Lambda, q_\Lambda) \psi_\Lambda^0(s_\Lambda)$$

and

$$(P_A^+ F)(s_\Lambda, q_\Lambda) = \frac{1}{2} e^{-\frac{1}{2}W_A(s_\Lambda)} (F(s_\Lambda, q_\Lambda) - F(s_\Lambda^A, q_\Lambda)).$$

It is now convenient to introduce the new scalar product

$$\begin{aligned} (F_1, F_2)_U &= \left( e^{-\frac{1}{2}U(S_\Lambda^3)} F_1, e^{-\frac{1}{2}U(S_\Lambda^3)} F_2 \right) \\ &= \sum_{s_\Lambda} \int F_1(q_\Lambda, s_\Lambda) F_2(q_\Lambda, s_\Lambda) e^{-U(s_\Lambda)} dq_\Lambda \\ &= \int \left( (e^{-\frac{1}{2}U(S_\Lambda^3)} F_1(q_\Lambda), e^{-\frac{1}{2}U(S_\Lambda^3)} F_2(q_\Lambda)) \right)_0 dq_\Lambda \\ &=: \int (F_1(q_\Lambda), F_2(q_\Lambda))_U^0 dq_\Lambda. \end{aligned}$$

The operator  $P_A^+$  is symmetric with respect to the new scalar product since

$$(P_A^+ F_1(q_\Lambda), F_2(q_\Lambda))_U^0 = (P_A e^{-\frac{1}{2}U(S_\Lambda^3)} F_1(q_\Lambda), e^{-\frac{1}{2}U(S_\Lambda^3)} F_2(q_\Lambda))_0. \quad (3.7)$$

It is not difficult to check that

$$\begin{aligned} (P_A^+ F(q_\Lambda), F(q_\Lambda))_U^0 &= \frac{1}{2} \sum_{s_\Lambda} e^{-\frac{1}{2}[U(s_\Lambda) + U(s_\Lambda^A)]} (F(s_\Lambda, q_\Lambda) - F(s_\Lambda^A, q_\Lambda)) F(s_\Lambda, q_\Lambda) \\ &= \frac{1}{4} \sum_{s_\Lambda} e^{-\frac{1}{2}[U(s_\Lambda) + U(s_\Lambda^A)]} (F(s_\Lambda, q_\Lambda) - F(s_\Lambda^A, q_\Lambda))^2 \geq 0. \end{aligned} \quad (3.8)$$



Here we took into account that the function under the sign of exponent is invariant under changing signs of the spin variables  $s_A$ . It now follows that  $P_A \geq 0$ . Q.E.D.

It follows from this proposition that  $H'_\Lambda$  is positive definite since for negative  $J_A$  we have

$$(H'_\Lambda F, F) = - \sum_{A \subset \Lambda} \int J_A(q_\Lambda) (P_A F(q_\Lambda), F(q_\Lambda))_0 dq_\Lambda \geq 0.$$

Q.E.D.

**Proof of III.** This is based on an application of the following proposition (see Example X.9.3 in [RS]):

**Proposition 3.2.** *The operator  $-\Delta + \mu^2 x^2 + V + (y, x)$ , where  $(y, x)$  is the Euclidean scalar product of  $x \in \mathbf{R}^n$  with the constant vector  $y \in \mathbf{R}^n$ ,  $\Delta = \sum_{j=1}^n \frac{\partial}{\partial x_j}$ ,  $\mu \neq 0$  if  $y \neq 0$  and  $V$  is the operator of multiplication by a non-negative function  $V(x) \in L^2(\mathbf{R}^n, e^{-x^2} dx)$ , where  $x^2 = (x, x)$ , is essentially self-adjoint on  $C_0^\infty(\mathbf{R}^n)$ .*

**Proof.** The following inequality holds:

$$\mu^2 \|x^2 \psi\| \leq \|(-\Delta + V + \mu^2 x^2) \psi\| + 2\mu^2 n \|\psi\|, \quad (3.9)$$

where  $\mu$  is a real number, and we tacitly assume that  $x^2$  means the operator of multiplication by  $x^2 = (x, x)$ . This follows from the inequalities ( $\partial_j = \frac{\partial}{\partial x_j}$ )

$$\begin{aligned} (-\Delta + V + \mu^2 x^2)^2 &= (-\Delta + V)^2 + \mu^2 [x^2(-\Delta + V) + (-\Delta + V)x^2] + (\mu^2 x^2)^2 \\ &\geq (\mu^2 x^2)^2 - \mu^2 (\Delta x^2 + x^2 \Delta) \\ &= (\mu^2 x^2)^2 - \mu^2 \sum_{i=1}^n [x_i, [x_i, \Delta]] - 2\mu^2 \sum_{i=1}^n x_i \delta x_i \\ &\geq (\mu^2 x^2)^2 - \mu^2 \sum_{i=1}^n [x_i, [x_i, \Delta]] = (\mu^2 x^2)^2 + 2\mu^2 n. \end{aligned}$$

Here we took into account that  $V$  is positive and  $[\partial_j, x_k] = \partial_j x_k - x_k \partial_j = \delta_{j,k}$ . Then using the simple inequality

$$|(y, x)| \leq \epsilon x^2 + \frac{n}{4\epsilon} |y|,$$

where  $|y| = \max_j |y_j|$ , we get

$$\|(y, x) \psi\| \leq a \|(-\Delta + V + \mu^2 x^2) \psi\| + b \|\psi\|, \quad a = \mu^{-2} \epsilon, \quad b = n(2 + \frac{|y|}{4\epsilon}). \quad (3.10)$$

For  $\mu \neq 0$  the number  $a$  can be arbitrary small and hence it follows from the Kato-Rellich Theorem [RS] that  $-\Delta + V + \mu^2 x^2 + (y, x)$  is essentially self-adjoint on any core for  $-\Delta + V + \mu^2 x^2$ , in particular  $C_0^\infty(\mathbf{R}^n)$  (Theorem X.59 in [RS]). (Note that the inequality (3.9) is similar to the one used in Example 3.X in [RS] to prove the essential self-adjointness of  $-\Delta + V$ .) Q.E.D.

Since  $S_\Lambda^3$  is a diagonal operator on  $(\otimes \mathbf{C}^2)^{|\Lambda|}$ , the operator

$$- \sum_{x \in \Lambda} \partial_x^2 + \mu^2 \sum_{x \in \Lambda} q_x^2 + V_\Lambda + 2\eta \mu^2 \sum_{x \in \Lambda} q_x \phi_x(S_\Lambda^3)$$

is a direct sum of  $2^{|\Lambda|}$  copies of the operator in Proposition 3.2 with  $n = |\Lambda|$ . From Proposition 3.2 it follows that this operator is essentially self-adjoint on the set  $\mathbf{C}_0^\infty(\mathbf{R}^{|\Lambda|}) \otimes (\otimes \mathbf{C}^2)^{|\Lambda|}$ . The same is true for the complete Hamiltonian since the operator

$$(\eta\mu)^2 \sum_{x \in \Lambda} \phi_x^2(S_\Lambda^3) + \sum_{A \subseteq \Lambda} J_A S_{[A]}^1$$

is bounded. Q.E.D.

**Proof of IV.** The operators  $h_x, H'_\Lambda$  are positive definite on  $(\otimes \mathbf{C}^2)^{|\Lambda|} \otimes \mathbf{C}_0^\infty(\mathbf{R}^{|\Lambda|})$ . This and Statement III imply that  $H_\Lambda$  is positive definite on its domain  $D(H_\Lambda)$  and  $\Psi_\Lambda$  is its ground state. Q.E.D.

**Proof of V.** By Theorem XIII.44 in [RS] it suffices to prove that the symmetric semigroup  $P_\Lambda^t$ , generated by  $-H_\Lambda$ , maps non-negative functions into (strictly) positive functions (is positivity improving) to conclude that the ground state is unique. We will establish this property with the help of a perturbation expansion.

The kernel of the semigroup  $P_0^t$ , generated by  $h_\Lambda + V_\Lambda$ , is expressed in terms of the Feynman-Kac (FK) formula [RS, Gi] as follows

$$P_0^t(q_\Lambda, s_\Lambda; q'_\Lambda, s'_\Lambda) = \delta_{s_\Lambda, s'_\Lambda} \int P_{q_\Lambda, q'_\Lambda}^t(dw_\Lambda) \exp \left\{ - \int_0^t V_\Lambda^+(w_\Lambda(\tau), s_\Lambda) d\tau \right\}, \quad (3.11)$$

where  $V_\Lambda^+(q_\Lambda; s_\Lambda) = V_\Lambda(q_\Lambda; s_\Lambda) + \sum_{x \in \Lambda} [\mu^2(q_x + \eta\phi_x(s_\Lambda))^2 - \mu]$ ,  $P_{q_\Lambda, q'_\Lambda}^t(dw_\Lambda) = \prod_{x \in \Lambda} P_{q_x, q'_x}^t(dw_x)$  is a product of conditional Wiener measures on collections of continuous paths  $w_\Lambda(t)$ . The semigroup  $P^t$  generated by the complete Hamiltonian can then be represented as a perturbation series in powers of

$$V_0 = - \sum_{A \subseteq \Lambda} J_A S_{[A]}^1.$$

This series is convergent in the uniform operator norm [Ka] since  $V_0$  is a bounded operator. The perturbation expansion is given by

$$P^t = \sum_{n=0}^{\infty} \int_{0 \leq \tau_1 \leq \tau_2 \leq \dots \leq \tau_n \leq t} d\tau_1 \dots d\tau_n P_0^{\tau_1} \prod_{j=2}^{n+1} (V_0 P_1 0^{\tau_j - \tau_{j-1}}), \quad (3.12)$$

where  $\tau_{n+1} = t$ . We now use the following simple inequality

$$\begin{aligned} \int_0^t |V_\Lambda^+(w_\Lambda(\tau), s_\Lambda)| d\tau &\leq \bar{V}(w_\Lambda) := |J|_\Lambda \int_0^t \left[ \exp\{\alpha \bar{U}_0 + 2\eta\mu \sum_{x \in \Lambda} \bar{\phi}_x |w_x(\tau)|\} \right. \\ &\quad \left. + \sum_{x \in \Lambda} (\mu^2(|w_x(\tau)| + \eta\bar{\phi}_x)^2 - \mu) \right] d\tau, \end{aligned}$$

where  $\bar{U}_0 = \max_{s_\Lambda} U_0(s_\Lambda)$ ,  $|J|_\Lambda = \sup_{q_\Lambda} \sum_{A \subseteq \Lambda} |J_A|$ , and  $\bar{\phi}_x = \max_{s_\Lambda} |\phi_x(s_\Lambda)|$ . Let

$$\bar{P}_1^t(q_\Lambda; q'_\Lambda) = \int P_{q_\Lambda, q'_\Lambda}^t(dw_\Lambda) e^{-\bar{V}(w_\Lambda)},$$

and

$$V_- = -J_- \sum_{x \in \Lambda} S_x^1$$

then it follows from the positivity of the kernel  $P_1^t$  and the fact that  $V_0 \geq V_- \geq 0$ , that

$$P^t(q_\Lambda, s_\Lambda; q'_\Lambda, s'_\Lambda) \geq \bar{P}_1^t(q_\Lambda; q_\Lambda) \sum_{n \geq 0} \frac{t^n}{n!} V_-^n(s_\Lambda; s'_\Lambda). \quad (3.13)$$

Here we utilized the semigroup property of  $\bar{P}^t$ , and the inequality

$$(V_0 P_0^{\tau_j - \tau_{j-1}})(q_\Lambda, s_\Lambda; q'_\Lambda, s'_\Lambda) \geq \bar{P}^{\tau_j - \tau_{j-1}}(q_\Lambda; q'_\Lambda) V_-(s_\Lambda; s'_\Lambda).$$

Now, it is easily proved as in [DS] that the matrix  $V_-$  is irreducible. As a result there exists a positive integer  $n$  such that  $V_-^n$  has positive non-diagonal elements [[GI, GII]. Hence the kernel in the left-hand side of (3.13) is positive. Q.E.D.

## 4 Long range order

In this section we will prove the occurrence of different types of long-range order (lro). In the proof of the occurrence of lro in the oscillator variables and in the third component of the spin we rely on the following theorem.

**Theorem 4.1. (Basic Theorem)** *Let the potential energy  $U$  of a classical ferromagnetic Ising model on the hyper-cubic lattice  $\mathbf{Z}^d$  with the partition function  $Z_\Lambda = \sum_{s_\Lambda} \exp\{-\beta U(s_\Lambda)\}$  be given by*

$$U(s_\Lambda) = - \sum_{A \subseteq \Lambda} J_A s_{[A]}, \quad (4.1)$$

where  $J_A \geq 0$ . Assume that  $J_A = 0$  for  $|A|$  odd, and  $J_{\langle x,y \rangle} \geq J_1 > 0$ . Then for sufficiently large  $\beta J_1$ , and dimension  $d \geq 2$ , there is ferromagnetic lro, that is, for the Gibbsian two point spin average the following bound holds:

$$\langle s_x s_y \rangle_\Lambda > 0 \quad (4.2)$$

uniformly in  $\Lambda$ , and the magnetization (the order parameter)  $M_\Lambda = |\Lambda|^{-1} \sum_{x \in \Lambda} s_x$  is non-zero in the thermodynamic limit  $\Lambda \rightarrow \mathbf{Z}^d$ .

The proof of this theorem is based on an application of the generalized Peierls principle (argument). It will be given at the end of this section (see, also, [SI, SII]). The next theorem is a consequence of the Basic Theorem.

**Theorem 4.2.** *Consider the Hamiltonian (1.3) where  $U_*$  is the same as  $U$  in the Basic Theorem and  $\phi_x$  has the general form (1.2) with  $J'_{x,A} \geq 0$  substituted for  $J_A$  and with the same assumptions, in particular  $J'_{\langle x,y \rangle} \geq J_1$  for nearest neighbour pairs  $\langle x,y \rangle$ . Then, for a sufficiently large  $\beta = (\eta^2 \mu + \alpha) J_1$ , there is lro in the ground state for the third spin component and the oscillator variable, that is, uniformly in  $\Lambda$ ,  $x$  and  $y$ , the following bounds hold:  $\langle S_x^3 S_y^3 \rangle_\Lambda > 0$  and  $\langle \hat{q}_x \hat{q}_y \rangle_\Lambda > 0$ .*

**Proof.** Equality (1.7) of Theorem 1.1 together with the first Griffiths inequality imply that

$$\langle \hat{q}_x \hat{q}_y \rangle_\Lambda = \eta^2 \langle \phi_x(S_\Lambda^3) \phi_y(S_\Lambda^3) \rangle_{*\Lambda} \geq \eta^2 \sum_{z, z' \in \Lambda} J'_{x,z} J'_{y,z'} \langle s_x s_y \rangle_{*\Lambda} > 0. \quad (4.3)$$

Obviously, we also have

$$\langle S_x^3 S_y^3 \rangle_\Lambda = \langle s_x s_y \rangle_{*\Lambda}. \quad (4.4)$$

$U_*$  admits representation (4.1) with  $J_{*A}$  substituted instead of  $J_A$  such that  $J_{*x,y} \geq (\eta^2 \mu + \alpha) J_1$  (the expression for  $U_*$  contains constant a term -  $\sum_{x,A \subseteq \Lambda} J_{x,A}^2$ ). The Basic Theorem and (4.4),

(4.3) imply occurrence of ferromagnetic lro in  $S^3$  and the oscillator variables. Q.E.D.

The next theorem does not rely on the Basic Theorem.

**Theorem 4.3.** *Assume that there exist positive constants  $C, B_j, j = 0, 1, 2$  independent of  $\Lambda$  such that  $|\phi_x(s_\Lambda)| \leq C$  and for  $|A| = 2$*

$$|W_{0A}(s_\Lambda)| \leq B_0, \quad W_A^{(j)}(s_\Lambda) \leq B_j, \quad W_A^{(j)}(s_\Lambda) = \sum_{x' \in \Lambda} |\phi_{x'}^j(s_\Lambda^A) - \phi_{x'}^j(s_\Lambda)|, \quad j = 1, 2, \quad (4.5)$$

where  $W_{0A}$  is calculated by the same rule as  $W_A$ . Then there is ferromagnetic lro in the ground state in the variable  $S^1$  for arbitrary dimension  $d$ .

**Proof.** The definition of  $S^1$  implies that

$$S_x^1 S_y^1 \Psi_\Lambda(q_\Lambda) = \frac{1}{4} \sum_{s_\Lambda} e^{-\frac{1}{2}U(s_\Lambda; q_\Lambda)} \psi_\Lambda^0(s_\Lambda^{x,y}) \psi_0(q_\Lambda) = \frac{1}{4} \sum_{s_\Lambda} e^{-\frac{1}{2}U(s_\Lambda^{x,y}; q_\Lambda)} \psi_\Lambda(s_\Lambda; q_\Lambda).$$

Taking into account also the orthonormality of the basis one obtains

$$\begin{aligned} 4 \langle S_x^1 S_y^1 \rangle_\Lambda &= Z_\Lambda^{-1} \sum_{s_\Lambda} \int e^{-\frac{1}{2}[U(s_\Lambda^{x,y}; q_\Lambda) + U(s_\Lambda; q_\Lambda)]} \psi_{0\Lambda}^2(q_\Lambda) dq_\Lambda \\ &= Z_\Lambda^{-1} \sum_{s_\Lambda} e^{\frac{\eta^2 \mu}{4} \sum_{x' \in \Lambda} (\phi_{x'}(s_\Lambda) + \phi_{x'}(s_\Lambda^{x,y}))^2} e^{-\frac{\alpha}{2}[U_0(s_\Lambda^{x,y}) + U_0(s_\Lambda)]} \\ &\geq e^{-\frac{\alpha}{2} B_0} Z_\Lambda^{-1} \sum_{s_\Lambda} e^{\frac{\eta^2 \mu}{4} \sum_{x' \in \Lambda} (\phi_{x'}(s_\Lambda) + \phi_{x'}(s_\Lambda^{x,y}))^2} e^{-\alpha U_0(s_\Lambda)}. \end{aligned} \quad (4.6)$$

We also have

$$\begin{aligned} &\sum_{x' \in \Lambda} (\phi_{x'}(s_\Lambda) + \phi_{x'}(s_\Lambda^{x,y}))^2 = \\ &= \sum_{x' \in \Lambda} [3\phi_{x'}^2(s_\Lambda) + \phi_{x'}^2(s_\Lambda^{x,y})] + 2 \sum_{x' \in \Lambda} \phi_{x'}(s_\Lambda) [-\phi_{x'}(s_\Lambda) + \phi_{x'}(s_\Lambda^{x,y})] \\ &= 4 \sum_{x' \in \Lambda} \phi_{x'}^2(s_\Lambda) + \sum_{x' \in \Lambda} [-\phi_{x'}^2(s_\Lambda) + \phi_{x'}^2(s_\Lambda^{x,y})] + 2 \sum_{x' \in \Lambda} \phi_{x'}(s_\Lambda) [-\phi_{x'}(s_\Lambda) + \phi_{x'}(s_\Lambda^{x,y})] \\ &\geq 4 \sum_{x' \in \Lambda} \phi_{x'}^2(s_\Lambda) - B_2 - 2CB_1. \end{aligned}$$

This yields

$$\begin{aligned} \langle S_x^1 S_y^1 \rangle_\Lambda &\geq \frac{1}{4} e^{-\frac{\eta^2 \mu}{4}(B_2 + 2B_1)} e^{-\frac{\alpha}{2} B_0} Z_\Lambda^{-1} \sum_{s_\Lambda} e^{\frac{\eta^2 \mu}{4} \sum_{x' \in \Lambda} \phi_{x'}^2(s_\Lambda)} e^{-\alpha U_0(s_\Lambda)} \\ &= \frac{1}{4} e^{-\frac{\eta^2 \mu}{4}(B_2 + 2CB_1)} e^{-\frac{\alpha}{2} B_0} > 0. \end{aligned}$$

Q.E.D.

**Corollary 4.1.** Let  $\phi_x(s_\Lambda) = \sum_{y \in \Lambda} J(x-y)s_y$  and  $\|J\|_1 = \sum_{x \in \mathbf{Z}^d} |J(x)| < \infty$ . Then there is ferromagnetic lro in the ground state in the variable  $S^1$  in an arbitrary dimension  $d$ , and the magnetization in the  $x$ -direction  $M_\Lambda^1$  is non-zero in the thermodynamic limit.

**Proof.** We check that the conditions of Theorem 4.3 for the potential  $\phi_x$  are valid. It is obvious that  $|\phi_x(s_\Lambda)| \leq \|J\|_1$  and that

$$|\phi_{x'}(s_\Lambda^{x,y}) - \phi_{x'}(s_\Lambda)| = |-2[s_y J(y-x') + s_x J(x-x')]| \leq 2[|J(y-x')| + |J(x-x')|].$$

As a result  $W_{x,y}^{(1)}(s_\Lambda) \leq 4\|J\|_1$ . Moreover,

$$\begin{aligned} |\phi_{x'}^2(s_\Lambda^{x,y}) - \phi_{x'}^2(s_\Lambda)| &= \left| \left[ \sum_{z \in \Lambda \setminus (x,y)} J(z-x')s_z - s_y J(y-x') - s_x J(x-x') \right]^2 \right. \\ &\quad \left. - \left[ \sum_{z \in \Lambda \setminus (x,y)} J(z-x')s_z + s_y J(y-x') + s_x J(x-x') \right]^2 \right| \\ &= 4 \left| \sum_{z \in \Lambda \setminus (x,y)} J(z-x')s_z (s_y J(y-x') + s_x J(x-x')) \right| \\ &\leq 4 \sum_{z \in \Lambda} |J(z-x')| (|J(y-x')| + |J(x-x')|) \\ &\leq 4\|J\|_1 (|J(y-x')| + |J(x-x')|). \end{aligned}$$

Hence  $W_{x,y}^{(2)}(s_\Lambda) \leq 8\|J\|_1^2$ . Q.E.D.

**Proof of the Basic Theorem.** Let  $\chi_x^\pm = \frac{1}{2}(1 \pm 2s_x)$ . Then one has

$$4\langle \chi_x^+ \chi_y^- \rangle_\Lambda = 1 + 2\langle s_x \rangle_\Lambda - 2\langle s_y \rangle_\Lambda - 4\langle s_x s_y \rangle_\Lambda.$$

Since the systems are invariant under spin flip, the second and third terms in the right-hand side of last equality are equal to zero and

$$\langle s_x s_y \rangle_\Lambda = \frac{1}{4} - \langle \chi_x^+ \chi_y^- \rangle_\Lambda. \quad (4.7)$$

Hence if

$$\langle \chi_x^+ \chi_y^- \rangle_\Lambda < \frac{1}{4} - a \quad (4.8)$$

then there is ferromagnetic lro, i.e.

$$\langle s_x s_y \rangle_\Lambda \geq a > 0, \quad (4.9)$$

where  $a < \frac{1}{4}$  is independent of  $\Lambda$ ,  $x$  and  $y$ . Clearly, it suffices to prove that there exists a positive function  $E_0(\beta)$  and a positive constant  $a'$  such that

$$\langle \chi_x^+ \chi_y^- \rangle_\Lambda \leq a' e^{-E_0(\beta)} \quad (4.10)$$

and that  $E_0$  is increasing to infinity as  $\beta \rightarrow \infty$  then (3.5) will hold for a sufficiently large inverse temperature  $\beta$ . The Peierls principle reduces the derivation of (3.7) to the derivation of the contour bound.

**Peierls principle:** Assume that the following contour bound holds:

$$\left\langle \prod_{\langle x,y \rangle \in \Gamma} \chi_x^+ \chi_y^- \right\rangle_{\Lambda} \leq e^{-|\Gamma|E}, \quad (4.11)$$

where  $\langle \cdot \rangle_{\Lambda}$  denotes the Gibbs average for the spin system in the region  $\Lambda \subseteq \mathbf{Z}^d$ ,  $\Gamma$  is a set of nearest neighbors, adjacent to the (connected) contour, i.e. a boundary of the connected set of unit hypercubes centered at lattice sites. Then (4.10) is valid with  $E_0 = a''E$ , where  $a''$  is a positive constant independent of  $\Lambda$ .

Bricmont and Fontaine [BF] derived the contour bound for spin systems with the potential energy (3.1) with the help of the second Griffiths [KS] and Jensen inequalities [BF] (see also [?, SI, SII])

$$\langle s_{[A]} s_{[B]} \rangle_{\Lambda[\Gamma]} - \langle s_{[A]} \rangle_{\Lambda[\Gamma]} \langle s_{[B]} \rangle_{\Lambda[\Gamma]} \geq 0, \quad (4.12)$$

and

$$\int e^f d\mu \geq \exp\left\{ \int f d\mu \right\},$$

where  $d\mu$  is a probability measure on a measurable space. Their proof starts from the inequality

$$\chi_x^+ \chi_y^- \leq e^{-\frac{\beta'}{2} s_x s_y} \chi_x^+ \chi_y^- \leq e^{-\frac{\beta'}{2} s_x s_y}.$$

As a result (writing  $\beta' = \beta J_1$ )

$$\begin{aligned} \left\langle \prod_{\langle x,y \rangle \in \Gamma} \chi_x^+ \chi_y^- \right\rangle_{\Lambda} &\leq \left\langle e^{-\frac{\beta'}{2} \sum_{\langle x,y \rangle \in \Gamma} s_x s_y} \right\rangle_{\Lambda} \\ &= \left\langle e^{\frac{\beta'}{2} \sum_{\langle x,y \rangle \in \Gamma} s_x s_y} \right\rangle_{\Lambda[\Gamma]}^{-1} \\ &\leq e^{-\frac{\beta'}{2} \sum_{\langle x,y \rangle \in \Gamma} \langle s_x s_y \rangle_{\Lambda[\Gamma]}} = e^{-E_{\Gamma}}, \end{aligned} \quad (4.13)$$

where  $\langle \cdot, \cdot \rangle_{\Lambda[\Gamma]}$  is the average corresponding to the potential energy

$$U_{\Gamma}(q_{\Lambda}) = U(s_{\Lambda}) + \frac{J_1}{2} \sum_{\langle x,y \rangle \in \Gamma} s_x s_y. \quad (4.14)$$

In the last line we applied the Jensen inequality. From the second Griffiths inequality it follows that the average  $\langle s_x s_y \rangle_{\Lambda[\Gamma]}$  is a monotone increasing function in  $J_A$ . So, in the potential energy determining this average we can put  $J_A = 0$ , except for  $A = \langle x, y \rangle$ , without increasing the average. This leads to

$$\langle s_x s_y \rangle_{\Lambda[\Gamma]} \geq \langle s s' \rangle = Z_2^{-1}\left(\frac{\beta'}{2}\right) \sum_{s_1, s_2 = \pm 1} s_1 s_2 e^{\frac{\beta'}{2} s_1 s_2}, \quad Z_2(\beta) = \sum_{s_1, s_2 = \pm 1} e^{\beta s_1 s_2}.$$

That is,

$$E_{\Gamma} \geq |\Gamma|E, \quad E = 2^{-1} \beta' \langle s s' \rangle$$

or

$$E = \beta (e^{2^{-1}\beta'} - e^{-2^{-1}\beta'}) (e^{2^{-1}\beta'} + e^{-2^{-1}\beta'})^{-1} \geq 2^{-1} \beta' (1 - e^{-\beta'}). \quad (4.15)$$

Here we used in the denominator the inequality  $e^{-2^{-1}\beta'} \leq e^{2^{-1}\beta'}$ . Obviously,  $E$  tends to infinity if  $\beta'$  tends to infinity. This implies (4.10) and (4.8). Q.E.D.

**Remark.** *If only one-point sets are left in the sum for  $V_\Lambda$  then the expression for  $H_\Lambda$  can be rewritten in the following way*

$$H_\Lambda = \sum_{x \in \Lambda} H_x.$$

*The property of the ground state  $\Psi_\Lambda$  to be a ground state with the zero eigenvalue of a local Hamiltonian  $H_x$  was found earlier for special isotropic anti-ferromagnetic Heisenberg chains with valence bond ground state in [AKLT] and special  $XZ \frac{1}{2}$  spin systems with the Hamiltonian related in a simple way to a generator of a Markovian spin-flip process in [DS].*

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