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Fractional Quantum Hall Hierarchy and the Second Landau Level

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We generalize the Haldane-Halperin hierarchy picture to apply to arbitrary, possibly non-Abelian, fractional quantum Hall states. Using this, we propose trial wave functions to describe the observed Hall conductance plateaus in the second Landau level. These hierarchy states are constructed over the Moore-Read state, the expected description of the $\nu = 5/2$ plateau, such that the quasiparticle gases generating the hierarchization only involve excitations from the electric charge sector. These proposed states all have electron pairing in the ground state and an excitation spectrum that includes non-Abelian anyons of the Ising model σ-vortex type.

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The fractional quantum Hall (FQH) plateaus in the first Landau level are described rather well by the Laughlin states [1] and the Abelian hierarchy states constructed over them [2, 3, 4, 5]. The observed filling fractions, the measured fractional charge of quasiparticles [6, 7, 8], and recent results from interferometric experiments [9, 10] all support this picture, and are backed up by a wealth of theoretical and numerical evidence. The physics of the second Landau level, however, remains far more perplexing, with the prominence of an even-denominator $\nu = 5/2$ FQH state [11, 12] that cannot be explained by the standard hierarchy. Fully developed FQH plateaus have been observed at $\nu = 7/3$, $12/5$, $5/2$, $8/3$, and $14/5$ [12, 13], but advances in experiments and sample quality may find additional plateaus developing where "features" have been observed, including at another intriguing even-denominator $\nu = 19/8$. The currently held view is that the $\nu = 5/2$ plateau is characterized by p-wave electron pairing and described by the Moore–Read (MR) state, which gives the dramatic prediction of quasiparticles with non-Abelian braiding statistics [14, 15]. Numerical evidence favoring the MR state has been provided in [16, 17] and empirical verification is currently being pursued. Though the remaining observed filling fractions have odd denominators, the electron correlations for $7/3 \leq \nu \leq 8/3$ have a non-Laughlin character similar to that of $\nu = 5/2$, and only $\nu = 14/5$ is expected to be a Laughlin type state [18]. Other than the Abelian hierarchy, the Read–Rezayi (RR) k-body clustered states [19] (which include MR) and their particle-hole conjugates are essentially the only single layer, spin-polarized descriptions proposed for these FQH plateaus. We introduce a generalization of the Haldane–Halperin (HH) hierarchy construction, allowing as its building blocks not just Laughlin-like states, but also more general FQH states such as the MR and other non-Abelian states. Using this we propose a hierarchy of states based on the MR state that provides candidate wave functions for all observed plateaus (and even for the weaker features) in the second Landau level.

In the HH picture, hierarchization is carried out by forming a gas of fundamental quasiholes or quasielectrons in an Abelian FQH state that is projected into a Laughlin-type state. We generalize this hierarchical construction by forming a gas of quasiparticle excitations of specified type in an arbitrary FQH state that is projected into another FQH-type state. The k^{th} level hierarchy wavefunction Ψ_k, with electron coordinates z_j and quasiparticle excitations of arbitrary type (which will be left implicit) at the coordinates w_j, is obtained from the $(k-1)^{th}$ level state Ψ_{k-1} by taking the inner product

$$\begin{aligned}
&\Psi_k\left(z_1,\ldots,z_{N_0};w_1,\ldots,w_{n_k}\right)\\
&\quad= \int \prod_{j=1}^{N_k} d^2u_j \Phi_k^*\left(u_1,\ldots,u_{N_k};w_1,\ldots,w_{n_k}\right)\\
&\qquad\times\Psi_{k-1}\left(z_1,\ldots,z_{N_0};u_1,\ldots,u_{N_k},w_1,\ldots,w_{n_k}\right) \quad (1)
\end{aligned}$$

where u_j are the coordinates of excitations in the $(k-1)^{th}$ level state that form a quasiparticle gas which is projected onto a FQH-type state given by the wavefunction Φ_k. These $(k-1)^{th}$ level quasiparticles are matched up with the "electrons" of the Φ_k state, which must therefore be designed to have the same braiding statistics (up to a bosonic factor). Furthermore, the quasiparticle gas excitations (and thus their corresponding "electrons") should be Abelian. Together, these ensure single-valuedness of the integrand in the integration coordinates, and thus a well-defined inner product and unique lowest energy k^{th} level ground state ($n_k = 0$) wavefunction. The specific Abelian excitation type of these quasiparticles, as well as the wavefunctions Φ_k, should be determined by physical arguments, possibly involving energetic considerations and charge minimality.

In order to obtain a homogeneous electron wavefunction, the number of excitations N_k in the quasiparticle gas must be chosen such that the highest power of u_α in Ψ_{k-1} is equal to that in Φ_k (with u_α^* counting as a negative power). One may think of this as the $(k-1)^{th}$ level quasiparticle gas determining how many induced "flux" quanta are felt by the "electrons" in Ψ_{k-1} (where the 0^{th} level "flux" and "electrons" are of course the actual magnetic flux and electrons of the system). This gives a system of equations relating the number of

flux quanta N_ϕ, electrons N_0, quasiparticle gas excitations of each level N_j ($j=1,\ldots,k$), and additional quasiparticles n_k. This may be immediately solved to obtain the filling fraction [35] and shift from the the expression: $N_\phi = \nu^{-1}N_0 - S$.

The resulting quasiparticle excitation spectrum of a k^{th} level hierarchy state contains a charge $2e$ boson B_0 and chargeless bosons B_j associated with each level of hierarchization ($j=1,\ldots,k$). These are identified with the vacuum in the anyonic charge spectrum (i.e. quasiparticles that differ only by these bosons have the same anyonic charge), and all permissible quasiparticle excitations must be mutually local (i.e. have trivial monodromy) with them. Allowed excitations much also be mutually local with the electrons, or equivalently with the charge e fermionic hole h_0 of the excitation spectrum (two of which combine to give B_0).

A natural method of generating wavefunctions for FQH states is to use conformal field theory (CFT) correlators with appropriately chosen vertex operator insertions for the various excitations present [14]. Excitations from a particular layer can be written as a vertex operator insertion in that layer, but general excitations may involve insertions of operators in multiple layers. To produce the ground state for a hierarchization in which the j^{th} level quasiparticle gas is always formed from excitations belonging only to the j^{th} layer (in that they arise only in Φ_j) [36] we use the CFT correlators

$$\Phi_j\left(u_1^{(j)},\ldots,u_{N_j}^{(j)};u_1^{(j+1)},\ldots,u_{N_{j+1}}^{(j+1)}\right) = \left\langle \prod_{\alpha=1}^{N_j} V_{e_j}\left(u_\alpha^{(j)}\right)\prod_{\beta=1}^{N_{j+1}} V_{c_j}\left(u_\beta^{(j+1)}\right)\right\rangle, \quad (2)$$

where V_{e_j} and V_{c_j} are respectively vertex operators for the "electrons" and quasiparticle gas excitations of the j^{th} layer, and we employ (throughout this letter) the standard convention of leaving the neutralizing background charge operators implicit (as well as the Gaussian factors to which they gives rise). Using this expression in Eq. (1), we take $\Psi_0 = \Phi_0$ with $z = u^{(0)}$, and Φ_k has no quasiparticle gas and hence no coordinates $u^{(k+1)}$ nor vertex operators V_{c_k}.

The particle-hole conjugate [20] of an arbitrary FQH state ψ is obtained by projecting holes of a $\nu=1$ quantum Hall wavefunction (i.e. $V_{e_0} = V_{c_0} = e^{i\varphi_0}$) onto this state, which uses Eq. (1) with (leaving index ranges implicit from now on)

$$\Psi_0 = \prod_{\alpha<\beta}(z_\alpha - z_\beta)\prod_{\alpha,\beta}(z_\alpha - u_\beta)\prod_{\alpha<\beta}(u_\alpha - u_\beta), \quad (3)$$
$$\Phi_1(u_1,\ldots,u_{N_1}) = \psi(u_1,\ldots,u_{N_1}). \quad (4)$$

The resulting state has $\nu = 1-\nu_\psi$ and $S = \frac{1-\nu_\psi S_\psi}{1-\nu_\psi}$.

The usual HH hierarchy, combined with particle-hole conjugation, may be used to obtain all the FQH states observed in the first Landau level. This hierarchy is most concisely described as $\mathrm{U}(1)_K$ [21], where the coupling constant K-matrix has non-zero elements K_{00} odd, K_{jj} even for $j>0$, and $K_{j,j+1} = K_{j+1,j} = \pm 1$. To make contact between this and explicit wavefunctions using CFT correlators, we use $m_0 = K_{00}$ and $m_j = K_{jj} - \frac{1}{m_{j-1}}$ for $j>0$ with

$$V_{e_0} = e^{i\sqrt{m_0}\,\varphi_0}, \quad V_{\lambda q_0} = e^{i\frac{\lambda}{\sqrt{m_0}}\varphi_0} \quad (5)$$
$$\begin{cases} V_{e_j} = e^{i\sqrt{m_j}\,\overline{\varphi}_j}, \quad V_{\lambda q_j} = e^{i\frac{\lambda}{\sqrt{m_j}}\overline{\varphi}_j} & \text{for } m_j>0 \\ V_{e_j} = e^{i\sqrt{-m_j}\,\varphi_j}, \quad V_{-\lambda q_j} = e^{i\frac{\lambda}{\sqrt{-m_j}}\varphi_j} & \text{for } m_j<0 \end{cases} \quad (6)$$
$$V_{c_j} = V_{K_{j,j+1}q_j} \quad (7)$$

in Eqs. (1,2). Vertex operators with negative λ generate singular terms with negative exponents in the resulting wavefunction. To correct this, it will be understood that negative powered factors in the wavefunction are schematically meant to be replaced with matching positive powers of their complex-conjugate, and a projection of the wavefunction into the lowest Landau level applied at the end (replacing z^* with $2\partial/\partial z$). This essentially matches the wavefunctions given in [5, 22], possibly up to short-ranged corrections. The quasiparticle gas excitations $c_j = \pm q_j$ are either the fundamental quasiholes or quasielectrons of the j^{th} level state Ψ_j, and should be determined by whether the filling fraction is respectively decreased or increased in going to the next level. Hence, one ought to use $K_{j,j+1} = -\mathrm{sgn}\{K_{j+1,j+1}\}$. The filling fraction and shift are determined from K to be (see, e.g. [21])

$$\nu = \left[K^{-1}\right]_{00}, \quad S = \frac{1}{\nu}\sum_j \left[K^{-1}\right]_{0j} K_{jj}. \quad (8)$$

An arbitrary HH quasiparticle excitation is specified by the number of "fluxes" (vortices) $a_j \in \mathbb{Z}$ in the j^{th} layer. An $\overrightarrow{a}$ excitation produces a factor of $\prod_{j=0}^{k}\prod_{\alpha_j}\left(w - u_{\alpha_j}^{(j)}\right)^{a_j}$ in the wavefunction, and is obtained by inserting

$$V_{\overrightarrow{a}}(w) = \prod_{j=0}^{k} V_{\lambda_j q_j}(w), \quad (9)$$

where $\lambda_0 = a_0$ and $\lambda_{j>0} = a_j - \frac{K_{j,j-1}\lambda_{j-1}}{m_{j-1}}$, in the CFT correlator. The electric charges and braiding statistics (in terms of R-matrices) of such excitations are given by

$$Q_{\overrightarrow{a}} = e\;\hat{t}_0 \cdot K^{-1}\cdot \overrightarrow{a} \quad (10)$$
$$R^{\overrightarrow{a},\overrightarrow{b}} = \exp\left(i\pi\,\overrightarrow{a}\cdot K^{-1}\cdot\overrightarrow{b}\right), \quad (11)$$

where $\hat{t}_j$ is the unit vector with a 1 in the j^{th} row. In the HH hierarchy, we have $h_0 = K\cdot\hat{t}_0$, $B_0 = 2h_0$, and $B_{j>0} = K\cdot\hat{t}_j$. Using the appropriate identifications, the entire excitation spectrum in this case is generated, through repeated fusion, by the fundamental quasihole excitation in the highest hierarchy layer, so arbitrary excitations may be written as $n\hat{t}_k$.

The general hierarchy prescription in Eq. (1) can generate a multitude of states at any given filling fraction, so we will restrict our attention to the constructions that seem most physically relevant and tenable. Specifically, we build the simplest possible hierarchy involving the MR state, which is closely

analogous to the HH hierarchy in that the hierarchization occurs only in the $\mathrm{U}(1)$ charge sector of the theory. This is perhaps the most natural way to form hierarchies with non-Abelian states in general, because it treats the mechanism giving rise to the non-Abelian sector (in the MR case: pairing giving rise to the Ising sector) as a ubiquitous property of the class of states, while the charge sector is allowed to form a hierarchy as it is already known to do for Abelian states.

We begin by describing the MR state, which is used for the 0^{th} level. The CFT describing MR may be written as $\text{Ising} \times \mathrm{U}\left(1\right)_2|_{\mathcal{C}}$, where the anyonic charge spectrum restriction is given by $\mathcal{C} = \{(I,n), (\psi,n), (\sigma, n+1/2) : n \in \mathbb{Z}_4\}$ [37]. The entire anyonic charge spectrum is generated by the fundamental quasihole, $(\sigma, 1/2)$. The corresponding electron and fundamental quasihole vertex operators are respectively

$$V_{e_0} = \psi e^{i\sqrt{2}\varphi_0}, \quad V_{qh_0} = \sigma e^{i\frac{1}{\sqrt{8}}\varphi_0}. \tag{12}$$

To form a hierarchy over the MR state, we must first specify the 0^{th} level quasiparticle gas. The physical picture we envision here is that forming a gas of fundamental quasiholes/quasielectrons $(\sigma, \pm 1/2)$ of the MR state forces them to pair up into preferential Abelian bound state excitations that can no longer be recoupled. A pair of σ Ising charges have two possible fusion channels, I and ψ, that describe their combined anyonic charge. These are degenerate at large distances, but short range interactions will break the energetic degeneracy, with physical intuition and some numerical evidence suggesting that I is the energetically favored channel. Because of this, we expect the 0^{th} quasiparticle gas to be composed of excitations with anyonic charge (I, K_{01}), where $K_{01} = \pm 1$ indicates paired quasiholes/quasielectrons. The corresponding vertex operator and resulting wavefunction are

$$\begin{aligned} V_{c_0} &= I e^{i\frac{K_{01}}{\sqrt{2}}\varphi_0}, \qquad (13)\\ \Psi_0 &= \mathrm{Pf}\left\{\frac{1}{z_\alpha - z_\beta}\right\} \prod_{\alpha<\beta} \left(z_\alpha - z_\beta\right)^2 \\ &\quad \times \prod_{\alpha,\beta} \left(z_\alpha - u_\beta\right)^{K_{01}} \prod_{\alpha<\beta} \left(u_\alpha - u_\beta\right)^{1/2}. \qquad (14)\end{aligned}$$

In order to build the simplest hierarchy over MR, we take all higher layers to be Abelian $\mathrm{U}\left(1\right)$ Hall fluids, with the minimal charge excitations of each level comprising its quasiparticle gas. It follows that each level's quasiparticle gas excitations are trivial in the Ising sector, and hence we may again use the K-matrix formalism to describe the resulting hierarchy states as $\text{Ising} \times \mathrm{U}\left(1\right)_K|_{\mathcal{C}}$, where now $K_{00} = 2$ (rather than the usual restriction that K_{00} be odd) and, as before, the other non-zero elements of K are K_{jj} even for $j > 0$ and $K_{j,j+1} = K_{j+1,j} = \pm 1$. The anyonic charges in the spectrum $\mathcal{C}$ are given by $A = (a_{\mathrm{I}}, \overrightarrow{a})$ where a_{I} is the Ising charge (I, ψ, or σ) and $\overrightarrow{a}$ is the $\mathrm{U}(1)_K$ flux vector. Given $h_0 = \left(\psi, K \cdot \hat{t}_0\right)$, $B_0 = \left(I, 2K \cdot \hat{t}_0\right)$, and $B_{j>0} = \left(I, K \cdot \hat{t}_j\right)$, the fluxes may take the values: $a_0 \in \mathbb{Z}$ for $a_{\mathrm{I}} = I$ or ψ, $a_0 \in \mathbb{Z} + \frac{1}{2}$ for $a_{\mathrm{I}} = \sigma$, and $a_{j>0} \in \mathbb{Z}$. The resulting anyonic charge spectrum has $|\mathcal{C}| = 6 \det K$ particle types (or torus degeneracy of $3 \det K$). Two quasiparticle excitation types are needed to generate the entire charge spectrum: the fundamental quasiholes/quasielectrons in the lowest and highest layers: $qh_0 = \left(\sigma, \frac{1}{2}\hat{t}_0\right)$ and $q_k = \left(I, \hat{t}_k\right)$. The filling fraction is the same as in Eq. (8), and $S = S_K + 1$, where S_K is the shift given in Eq. (8) and the $+1$ is due to the Pfaffian from the Ising sector. Quasiparticle excitations have the same electric charge $Q_A = Q_{\overrightarrow{a}}$ as in Eq. (10) and the braiding R-matrices are given by $R^{A,B}_C = R^{a_{\mathrm{I}},b_{\mathrm{I}}}_{c_{\mathrm{I}}} R^{\overrightarrow{a},\overrightarrow{b}}$, where $R^{\overrightarrow{a},\overrightarrow{b}}$ is given in Eq. (11), and the Ising sector's $R^{a_{\mathrm{I}},b_{\mathrm{I}}}_{c_{\mathrm{I}}}$ are

$$\begin{aligned} &R^{I,I}_I = R^{I,\psi}_\psi = R^{\psi,I}_\psi = R^{I,\sigma}_\sigma = R^{\sigma,I}_\sigma = 1, \quad R^{\psi,\psi}_I = -1, \\ &R^{\psi,\sigma}_\sigma = R^{\sigma,\psi}_\sigma = -i, \quad R^{\sigma,\sigma}_I = e^{-i\frac{\pi}{8}}, \quad R^{\sigma,\sigma}_\psi = e^{i\frac{3\pi}{8}}. \end{aligned} \tag{15}$$

We obtain explicit $\Phi_{j>0}$ for use with Eqs. (1,14), by simply applying Eqs. (2,6,7) for the new K. An $A = (a_{\mathrm{I}}, \overrightarrow{a})$ excitation corresponds to insertion of the vertex operator

$$V_A\left(w\right) = a_{\mathrm{I}}\left(w\right) V_{\overrightarrow{a}}\left(w\right) \tag{16}$$

with $V_{\overrightarrow{a}}$ from Eq. (9), but permitting half-integral a_0.

We obtain a $\nu = 2/3$ ground state wavefunction at one level of hierarchy by using $K_{11} = 2$, for which

$$\Phi_1\left(u_1, \dots, u_{N_1}\right) = \prod_{\alpha<\beta} \left(u^*_\alpha - u^*_\beta\right)^{3/2}. \tag{17}$$

This state has $S = 4$ (for $K_{01} = -1$), $|\mathcal{C}| = 18$, and the spectrum generating excitations qh_0 and q_1 have minimal electric charge $e/3$. With particle-hole conjugation, this provides candidate states for $\nu = 7/3$ and $8/3$.

Alternatively, we obtain a $\nu = 2/5$ ground state wavefunction at one level of hierarchy by using $K_{11} = -2$, for which

$$\Phi_1\left(u_1, \dots, u_{N_1}\right) = \prod_{\alpha<\beta} \left(u_\alpha - u_\beta\right)^{5/2}. \tag{18}$$

This state has $S = 2$ (for $K_{01} = 1$), $|\mathcal{C}| = 30$, and the spectrum generating excitations qh_0 and q_1 have minimal electric charge $e/5$. This provides a candidate state for $\nu = 12/5$.

We can obtain a $\nu = 3/8$ state at two levels of hierarchy using $K_{11} = K_{22} = -2$ (i.e. built on the $\nu = 2/5$ state from above). This could describe what may be a FQH state developing at $\nu = 19/8$ seen in [13]. We also note that a $\nu = 4/5$ state is produced at three levels of hierarchy using $K_{11} = K_{22} = K_{33} = 2$, but, as it passes through an unobserved $\nu = 3/4$ state at the second hierarchy level, this is rather unlikely to be the correct description for the observed $\nu = 14/5$ plateau, which is expected to be a Laughlin state anyway.

If the MR quasiholes/quasielectrons were instead to pair up in the ψ-channel to form a hierarchy's 0^{th} level quasiparticle gas of (ψ, K_{01}) excitations, we would have

$$\begin{aligned} V_{c_0} &= \psi e^{i\frac{K_{01}}{\sqrt{2}}\varphi_0}, \qquad (19)\\ \Psi_0 &= \mathrm{Pf}\left\{\frac{1}{Z_\alpha - Z_\beta}\right\} \prod_{\alpha<\beta} \left(z_\alpha - z_\beta\right)^2 \\ &\quad \times \prod_{\alpha,\beta} \left(z_\alpha - u_\beta\right)^{K_{01}} \prod_{\alpha<\beta} \left(u_\alpha - u_\beta\right)^{1/2} \qquad (20)\end{aligned}$$

instead of Eqs. (13,14), where $Z_\alpha = z_\alpha$ for $\alpha = 1, \ldots, N_0$ and $Z_{\alpha+N_0} = u_\alpha$ for $\alpha = 1, \ldots, N_1$. Taking all higher layers to be Abelian again gives a hierarchy described by Ising $\times$ U $(1)_K|_{\mathcal{C}}$ and Eqs. (6,7), but now with K_{11} odd in order to match the braiding statistic of the c_0 excitations. In this case, the first layer's chargeless boson is $B_1 = \left(\psi, K \cdot \hat{t}_1\right)$ and excitations with $a_{\mathrm{I}} = \sigma$ must have $a_0, a_1 \in \mathbb{Z} + \frac{1}{2}$ (and hence cannot be written as a single layer excitation).

We obtain a $\nu = 1/3$ ground state wavefunction at one level of hierarchy using $K_{11} = -1$, for which

$$\Phi_1\left(u_1, \ldots, u_{N_1}\right) = \prod_{\alpha<\beta} \left(u_\alpha - u_\beta\right)^{3/2}. \tag{21}$$

This state has $S = 3$ (for $K_{01} = 1$), $|\mathcal{C}| = 18$, and the spectrum is generated by two minimal electric charge $e/3$ excitations, $(\sigma, 1/2, 1/2)$ and $(I, 0, 1)$.

We have shown how to perform general hierarchical constructions of FQH states, and explicitly constructed hierarchy states over the MR state that exhibit the same type of pairing structure and occur at all the experimentally observed FQH filling fractions in the second Landau level to date. Remarkably, these hierarchies over MR can also produce states (though we did not list them all explicitly) at all second Landau level filling fractions which have experimentally exhibited features suggestive of developing FQH states. It is also worth noting that it is possible to obtain states at all these filling fractions without using particle-hole conjugation, which is a less exact symmetry in the second Landau level where effects such as level mixing and multi-body interactions are more prevalent. Though obtaining some of the filling fractions from these hierarchies may seem less natural, and there is certainly competition from higher Landau level analogs of Abelian hierarchy states, as well as from the RR and possibly other proposed states, our hierarchy engenders an attractive picture in which MR-type pairing is the predominant characteristic of the second Landau level, or at least some region of it. The relative strengths of measured energy gaps in the second Landau level (see Table I) also lends credence to this hierarchical picture [23], though, just as with the Abelian hierarchy, one should not expect too much predictive power in regards to the strength of states. Additionally, there is some evidence from numerics [24] for a non-Abelian state at $\nu = 12/5$ with $S = 2$, which is fitting with our hierarchy state, and neither the HH/Jain nor the RR states (which respectively have $S = 4$ and -2). In any case, producing wavefunctions supported by circumstantial evidence does not guarantee their physical relevance, and the true nature of all physical FQH states must ultimately be settled by experiments, such as those that probe scaling behavior [25, 26, 27] and braiding statistics [28, 29, 30, 31, 32, 33].

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ν	$\frac{7}{3}$	$\frac{12}{5}$	$\frac{5}{2}$	$\frac{8}{3}$	$\frac{14}{5}$
Δ_1	100	$\star$	110	55	
Δ_2	$\star$		310	$\star$	$\star$
Δ_3	~ 600	70	$\star$	$\star$	$\star$
Δ_4	584	$\star$	544	562	252
Δ_4'	206		272	150	$\star$

TABLE I: Filling fractions and excitation gaps (in mK) for the second Landau level. Gaps $\Delta_1, \Delta_2, \Delta_3$ were reported in Refs. [12, 13, 34] respectively. Δ_4 and Δ_4' are gaps for the two samples studied in [23]. ($\star$ indicates an observed plateau, but no gap value reported.)

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[36] It is straightforward to similarly construct hierarchies with gases of multi-layer excitations, but it is cumbersome and unnecessary for the examples in this letter.

[37] Normally the Abelian charge sector for the MR state would be written as $\mathrm{U}(1)_4$, but we can instead write it as $\mathrm{U}(1)_2$ by allowing the spectrum to include half-integer fluxes.