The discrete Feynman integral

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Abstract

We construct a genuine Radon measure with values in $\mathcal{B}(\ell^2(\mathbf{Z}^d))$ on the set of paths in $\mathbf{Z}^d$ representing Feynman's integral for the discrete Laplacian on $\ell^2(\mathbf{Z}^d)$, and we prove the Feynman integral formula for the solutions of the Schrödinger equation with Hamiltonian $H = -\frac{1}{2}\Delta + V$, where Δ is the discrete Laplacian and V is an arbitrary bounded potential.

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1. A MEASURE ON ALL PATHS WITH VALUES IN $\mathbb{Z}$.

In [1], Feynman defined his path 'integral' as a sequential limit

$$\int_{x(0)=x_0}^{x(t)=x_n} e^{iS[x(t)]/\hbar} \text{“}dx\text{”} = \lim_{n\to\infty} \int_{\mathbf{R}^{n-1}} e^{iS(x_n,\ldots,x_0)/\hbar} dx_1 \ldots dx_{n-1}, \tag{1}$$

where the *action functional* $S[x(t)]$ of the path x is given by

$$S[x(t)] = \int_0^t L[x(t')]dt' = \int_0^t \left[\frac{m}{2}\dot{x}(t')^2 - V(x(t'))\right] dt' \tag{2}$$

for a given potential V. (Here we consider for simplicity the 1-dimensional case.) The approximating action is

$$S(x_n,\ldots,x_0) = \sum_{k=1}^n \left[\frac{m}{2}\left(\frac{x_k - x_{k-1}}{t/n}\right)^2 - V(x_k)\right]\frac{t}{n} \tag{3}$$

and $x_k = x(t/k)$. Feynman then argued that this integral over paths is a solution of the Schrödinger equation

$$i\hbar\frac{d\psi}{dt} = -\frac{\hbar^2}{2m}\Delta\psi + V\psi, \tag{4}$$

where Δ is the Laplacian. It is known [2] that, unlike the analogous Wiener measure corresponding to the heat equation, there exists no measure on the space of continuous paths which corresponds to the limit (1). Instead, various other approaches have been proposed (see e.g. [3–5]), none entirely satisfactory.

In this article we consider the discrete analogue of Feynman's path integral for a particle moving on a lattice, and show that one can define a genuine (Radon) measure on a space of paths on a d-dimensional lattice corresponding to this integral. Obviously, the Hamiltonian being defined on $\ell^2(\mathbf{Z}^d)$, the paths will have values in $\mathbf{Z}^d$ and cannot be continuous. This work is an extension of [6], where the path integral on a finite set was defined in an analogous fashion. Similar results appear in R. Carmona and J. Lacroix [8] in Propositions II.3.4 and II.3.12, which are attributed to Molchanov, see [7]. See also Remark 1.2 below.

We denote $H_0 = -\frac{1}{2}\Delta$ the free Hamiltonian, where Δ is the discrete Laplacian on $\mathcal{H} = \ell^2(\mathbf{Z}^d)$, i.e.

$$(H_0\psi)(\xi) = \sum_{i=1}^{d} \left(\psi(\xi) - \frac{1}{2}(\psi(\xi - e_i) + \psi(\xi + e_i)) \right), \tag{5}$$

where $e_1, \dots, e_d$ are the unit basis vectors in $\mathbf{R}^d$. This operator is bounded and has spectrum $\sigma(H_0) = [0, 2d]$. It can be diagonalised by Fourier transformation, i.e. its generalised eigenvectors are $\psi_k(\xi) = \frac{e^{ik\xi}}{\sqrt{2\pi}}$, where $k \in (-\pi, \pi]^d$, with corresponding eigenvalues $\lambda(k) = \sum_{i=1}^d (1 - \cos k_i)$. It follows that the time-evolution operator (or *propagator*) $U_t^0 = e^{-itH_0}$ has kernel given by

$$U_t^0(\xi', \xi) = \int_{-\pi}^{\pi} \frac{dk_1}{2\pi} \cdots \int_{-\pi}^{\pi} \frac{dk_d}{2\pi} e^{-it\lambda(k)} e^{ik(\xi' - \xi)}. \tag{6}$$

If we assume that the potential V is time-dependent and localised in time, i.e. it depends only on $x(t_k)$ for a finite number of instants t_k in time, then we can perform the integral over intermediate times and define for such potentials

$$\int_{x(0)=x_0}^{x(t)=x} e^{-i\sum_{k=1}^n V(x(t_k))} dF(x) = U_{t-t_n}^0 e^{-iV(x(t_n))} U_{t_n - t_{n-1}}^0 \cdots e^{-iV(x(t_1))} U_{t_1}^0.$$

Here $0 < t_1 < \cdots < t_n < t$ is an arbitrary subdivision. This is the starting point of our definition. We first define the measure on the set of all paths $x : [0, t] \to \dot{\mathbf{Z}}^d$, where $\dot{\mathbf{Z}}^d$ is the one-point compactification of $\mathbf{Z}^d$. We denote a subdivision $t_1 < \cdots < t_n$ of $[t, t']$ by σ, and the corresponding projection by $\pi_\sigma : (\dot{\mathbf{Z}}^d)^{[t,t']} \to (\dot{\mathbf{Z}}^d)^\sigma$. In particular, π_t is the projection $x \mapsto x(t)$. We also let $\pi_{t',t} : (\dot{\mathbf{Z}}^d)^{[t,t'']} \to (\dot{\mathbf{Z}}^d)^{[t,t']}$ be the restriction map $x \mapsto x\big|_{[t,t']}$ if $t < t' < t''$.

Theorem 1.1 *There exists a unique family of Radon measures $F_{t',t}$ on $(\dot{\mathbf{Z}}^d)^{[t,t']}$ with values in $\mathcal{B}(\ell^2(\mathbf{Z}^d))$ (with the strong operator topology) having the following properties:*

$$\int (\Phi_2 \circ \pi_{t'',t'}) (\Phi_1 \circ \pi_{t',t}) \, dF_{t'',t} = \int \Phi_2 dF_{t'',t'} \int \Phi_1 dF_{t',t}, \tag{7}$$

if Φ_1 is a continuous function on $(\dot{\mathbf{Z}}^d)^{[t,t']}$ and Φ_2 a continuous function on $(\dot{\mathbf{Z}}^d)^{[t',t'']}$; and

$$\int dF_{t',t} = U_{t'-t}^0, \tag{8}$$

and

$$\int (\varphi \circ \pi_t) dF_{t,t} = \mathcal{M}_\varphi, \tag{9}$$

the multiplication operator with the function φ.

Proof. We first remark that the conditions in the theorem imply that for any finite subdivision $\sigma:\ t \leq t_1 < t_2 < \cdots < t_n \leq t'$, and continuous functions $\varphi_i : \dot{\mathbf{Z}}^d \to \mathbf{C}$,

$$\int (\varphi_n \circ \pi_{t_n}) \dots (\varphi_1 \circ \pi_{t_1}) dF_{t',t} = U^0_{t'-t_n} \mathcal{M}_{\varphi_n} U^0_{t_n - t_{n-1}} \cdots U^0_{t_2-t_1} \mathcal{M}_{\varphi_1} U^0_{t_1-t}. \tag{10}$$

(Notice that if $\varphi : \dot{\mathbf{Z}}^d \to \mathbf{C}$ is a continuous function then $\lim_{|\xi|\to\infty} \varphi(\xi)$ exists so φ is certainly bounded. In defining $\mathcal{M}_\varphi$ we obviously restrict φ to $\mathbf{Z}^d$.)
This expression determines a consistent system of measures $F^\sigma_{t',t}$ on $(\dot{\mathbf{Z}}^d)^\sigma$ with values in $\mathcal{B}(\mathcal{H})$ through

$$\int (\varphi_n \otimes \cdots \otimes \varphi_1) dF^\sigma_{t',t} = U^0_{t'-t_n} \mathcal{M}_{\varphi_n} U^0_{t_n - t_{n-1}} \cdots U^0_{t_2-t_1} \mathcal{M}_{\varphi_1} U^0_{t_1-t}. \tag{11}$$

Note that the tensor products $\varphi_n \otimes \cdots \otimes \varphi_1$ form a total system of functions in $\mathcal{C}((\dot{\mathbf{Z}}^d)^\sigma)$. It follows immediately from the group property of U^0 that this is a consistent (projective) system of measures, in the sense that if σ' is a refinement of σ (i.e. it contains all the points of σ) then the restriction of $F^{\sigma'}_{t',t}$ to the functions depending only on the points of σ is equal to $F^\sigma_{t',t}$:

$$F^{\sigma'}_{t',t} \circ \pi^{-1}_{\sigma,\sigma'} = F^\sigma_{t',t}. \tag{12}$$

We presently set out to prove that the measures $F^\sigma_{t',t}$ satisfy a uniform bound of the type

$$||F^\sigma_{t',t}(\Phi)|| \leq C(t,t')\, ||\Phi||_\infty, \tag{13}$$

where the constant $C(t,t')$ is independent of σ. Given such a bound, we can extend the measures $F^\sigma_{t',t}$ continuously to a functional $F_{t',t}$ on $\mathcal{C}\big((\dot{\mathbf{Z}}^d)^{[t,t']}\big)$. Indeed, if we define for a function $\Phi \in \mathcal{C}\big((\dot{\mathbf{Z}}^d)^{[t,t']}\big)$ of the form $\Phi = \Psi \circ \pi_\sigma$, $\int \Phi\, dF_{t',t} = \int \Psi\, dF^\sigma_{t',t}$, then

$$\left\| \int \Phi\, dF_{t',t} \right\| = \left\| F^\sigma_{t',t}(\Psi) \right\| \leq C(t,t') ||\Psi||_\infty = C(t,t') ||\Phi||_\infty.$$

By the Stone-Weierstrass theorem, the functions Φ of the form $\Phi = \Psi \circ \pi_\sigma$ for some subdivision σ are seen to be dense in $\mathcal{C}\big((\dot{\mathbf{Z}}^d)^{[t,t']}\big)$, so that $F_{t',t}$ thus defined extends uniquely to a continuous linear functional on $\mathcal{C}\big((\dot{\mathbf{Z}}^d)^{[t,t']}\big)$.

Remark 1.1 The Riesz-Markov theorem does not hold in general for vector-valued measures. However, the functionals $F^{\sigma}_{t',t}$ are indeed $\mathcal{B}(\mathcal{H})$-valued Radon measures on $\dot{\mathbf{Z}}^{\sigma}$ provided the former is equipped with the strong operator topology. This is a consequence of the fact that the weak topology induced on $\mathcal{B}(\mathcal{H})$ by the dual of $\mathcal{B}(\mathcal{H})$ with the strong operator topology, is the same as the weak operator topology: see below.

To prove the bound (13), we need to prove:

$$\sum_{\xi_1,\dots,\xi_n\in\mathbf{Z}} \left|U^0_{t',t_n}(\xi',\xi_n)\dots U^0_{t_1,t}(\xi_1,\xi)\right| \le C(t,t'). \tag{14}$$

In fact, we need a norm estimate on the operator Q_t with kernel $Q_t(\xi',\xi) = |U^0_t(\xi',\xi)|$:

Lemma 1.1 *The operator Q_t with kernel $Q_t(\xi',\xi) = |U^0_t(\xi',\xi)|$ satisfies the bounds*

$$Q_t(\xi',\xi) \le e^{2dt} \text{ and } ||Q_t|| \le e^{2dt}.$$

Proof. Define

$$\lambda(k) = \sum_{i=1}^{d} (1-\cos k_i). \tag{15}$$

By the Taylor expansion with integral remainder, we have,

$$\begin{aligned} U^0_t(\xi',\xi) \;=\; & \int_{-\pi}^{\pi}\frac{dk_1}{2\pi}\dots\int_{-\pi}^{\pi}\frac{dk_d}{2\pi}e^{ik(\xi'-\xi)} \\ & -it\sum_{j=1}^{d}\int_{-\pi}^{\pi}\frac{dk_1}{2\pi}\dots\int_{-\pi}^{\pi}\frac{dk_d}{2\pi}(1-\cos k_j)e^{ik(\xi'-\xi)} \\ & -\int_0^t dt'\,t'^2\int_{-\pi}^{\pi}\frac{dk_1}{2\pi}\dots\int_{-\pi}^{\pi}\frac{dk_d}{2\pi}\lambda(k)^2e^{-it'\lambda(k)}e^{ik(\xi'-\xi)}. \end{aligned}$$

The first two terms evaluate to

$$\delta_{\xi',\xi} - it\sum_{j=1}^{d}\left(\delta_{\xi',\xi}-\frac{1}{2}(\delta_{\xi',\xi-e_j}+\delta_{\xi',\xi+e_j})\right).$$

In the remainder term we define

$$g(\lambda,t) = \lambda^2 e^{-it\lambda} \tag{16}$$

so that the integrand is $g(\lambda(k),t)e^{ik(\xi'-\xi)}$. We now want to integrate by parts twice in each variable k_i for which $\xi'_i \ne \xi_i$. We have, first of all, for $r \le d$,

$$\frac{\partial}{\partial k_1}\dots\frac{\partial}{\partial k_r}g\left(\lambda(k)\right) = \prod_{j=1}^{r}\sin k_j\,\frac{\partial^r}{\partial\lambda^r}\bigg|_{\lambda=\lambda(k)} g(\lambda,t).$$

Differentiating again with respect to $k_1, \ldots, k_s$ ($s \le r$) yields

$$\begin{aligned}
&\frac{\partial^2}{\partial k_1^2} \cdots \frac{\partial^2}{\partial k_s^2} \frac{\partial}{\partial k_{s+1}} \cdots \frac{\partial}{\partial k_r} g\left(\lambda(k), t\right) \\
&= \sum_{J \subset \{1,\ldots,s\}} \left(\prod_{j \in J} \sin^2 k_j \right) \left(\prod_{j \in \{1,\ldots,s\} \setminus J} \cos k_j \right) \left(\prod_{i=s+1}^{r} \sin k_i \right) \\
&\qquad \times \left. \frac{\partial^{r+|J|}}{\partial \lambda^{r+|J|}} \right|_{\lambda=\lambda(k)} g(\lambda, t).
\end{aligned}$$

Note that in particular if $s < r$ all these are zero at the integration bounds $k_i = \pm\pi$ for $i > r$. The derivatives of g are given by

$$\frac{\partial^n}{\partial \lambda^n} g(\lambda, t) = \left[n(n-1)(-it)^{n-2} + 2n\lambda(-it)^{n-1} + \lambda^2(-it)^n \right] e^{-it\lambda},$$

and can be bounded by $n(n-1) + 2n\lambda + \lambda^2$ for $t \le 1$. Since $0 \le \sum_{j=1}^{d}(1 - \cos k_j) \le 2d$, we have

$$\begin{aligned}
&\left| \frac{\partial^2}{\partial k_1^2} \cdots \frac{\partial^2}{\partial k_r^2} g\left(\lambda(k), t\right) \right| \\
&\le \sum_{p=0}^{r} \binom{r}{p} \left((r+p)(r+p-1) + 4(r+p)d + 4d^2 \right) \\
&= [r(r-1) + r^2 + 6rd + 4d^2]2^r + r(r-1)2^{r-2} \\
&\le (12d^2 - d)2^d + d(d-1)2^{d-2} =: c_d. \qquad (17)
\end{aligned}$$

We only integrate by parts with respect to those k_i such that $\xi_i' \ne \xi_i$. This yields

$$\begin{aligned}
&\int_{-\pi}^{\pi} \frac{dk_1}{2\pi} \cdots \int_{-\pi}^{\pi} \frac{dk_d}{2\pi} g(k,t) e^{ik(\xi'-\xi)} = \\
&= \sum_{I \subset \{1,\ldots,d\}} \prod_{i \in I} \delta_{\xi_i', \xi_i} \prod_{i \in I^c} (1 - \delta_{\xi_i', \xi_i}) \prod_{i \in I^c} \frac{-1}{(\xi_i' - \xi_i)^2} \\
&\qquad \times \int_{-\pi}^{\pi} \frac{dk_1}{2\pi} \cdots \int_{-\pi}^{\pi} \frac{dk_d}{2\pi} \left(\prod_{i \in I^c} \frac{\partial^2}{\partial k_i^2} g(\lambda(k), t) \right) e^{ik(\xi'-\xi)}
\end{aligned}$$

and hence

$$\begin{aligned}
&\left| \int_{-\pi}^{\pi} \frac{dk_1}{2\pi} \cdots \int_{-\pi}^{\pi} \frac{dk_d}{2\pi} g(k,t) e^{ik(\xi'-\xi)} \right| \\
&\le \sum_{I \subset \{1,\ldots,d\}} \prod_{i \in I} \delta_{\xi_i', \xi_i} \prod_{i \in I^c} (1 - \delta_{\xi_i', \xi_i}) \frac{c_d}{|\xi_i' - \xi_i|^2} \\
&\le 2^d c_d \prod_{i=1}^{d} \frac{1}{|\xi_i' - \xi_i|^2 + 1}.
\end{aligned}$$

We now introduce an operator Γ on $\ell^2(\mathbf{Z}^d)$ and an operator M on $\ell^2(\mathbf{Z})$ with kernels

$$\Gamma(\xi',\xi) = \sum_{j=1}^{d} \left(\delta_{\xi',\xi} + \frac{1}{2}(\delta_{\xi',\xi-e_j} + \delta_{\xi',\xi+e_j}) \right) \tag{18}$$

and

$$M(\xi',\xi) = \frac{2}{|\xi'-\xi|^2+1} \tag{19}$$

and write $M_d = M \otimes \cdots \otimes M$ on $\ell^2(\mathbf{Z}^d)$. (Note that $\Gamma = 2d\mathbf{1} - H_0$ so $0 \le \Gamma \le 2d\mathbf{1}$.) Then, for $t \le 1$,

$$|U_t^0(\xi',\xi)| \le \delta_{\xi',\xi} + t\Gamma(\xi',\xi) + t^2 c_d M_d(\xi',\xi). \tag{20}$$

Dividing, for arbitrary $t > 0$, the interval $[0,t]$ into n equal parts such that the length of each is at most 1, we obtain

$$\begin{aligned}
&Q_t(\xi',\xi) = \left| \sum_{\xi_1,\ldots,\xi_{n-1}\in\mathbf{Z}} U^0_{t-t_{n-1}}(\xi',\xi_{n-1})\ldots U^0_{t_1}(\xi_1,\xi) \right| \\
&\le \sum_{\xi_1,\ldots,\xi_{n-1}} |U^0_{t-t_{n-1}}(\xi',\xi_{n-1})\ldots U^0_{t_1}(\xi_1,\xi)| \\
&\le \sum_{\xi_1,\ldots,\xi_{n-1}} \big(\delta_{\xi',\xi_{n-1}} + (t-t_{n-1})\Gamma(\xi',\xi_{n-1}) + (t-t_{n-1})^2 c_d M_d(\xi',\xi_{n-1})\big) \times \\
&\qquad \ldots \big(\delta_{\xi_1,\xi} + t_1\Gamma(\xi_1,\xi) + t_1^2 c_d M_d(\xi_1,\xi)\big) \\
&\le \sum_{\xi_1,\ldots,\xi_{n-1}} \left(e^{(t-t_{n-1})\Gamma+(t-t_{n-1})^2 c_d M_d}\right)(\xi',\xi_{n-1})\ldots\left(e^{t_1\Gamma+t_1^2 c_d M_d}\right)(\xi_1,\xi) \\
&= \left(e^{t\Gamma+\frac{1}{n}t^2 c_d M_d}\right)(\xi',\xi).
\end{aligned} \tag{21}$$

By Fourier transformation it is easy to see that the operator M is bounded. Indeed, $||M\psi|| = ||\widehat{M\psi}||$ and

$$\begin{aligned}
|(\widehat{M\psi})(k)| &= \left| \sum_{\xi\in\mathbf{Z}} (M\psi)(\xi')e^{ik\xi'} \right| \\
&= \left| \sum_{\xi\in\mathbf{Z}} \left(\sum_{\xi'\in\mathbf{Z}} M(\xi',\xi)e^{ik(\xi'-\xi)} \right) \psi(\xi)e^{ik\xi} \right| \\
&= \left| \sum_{\xi''\in\mathbf{Z}} \frac{2}{|\xi''|^2+1} e^{ik\xi''} \right| |\hat{\psi}(k)| \\
&\le \sum_{\xi\in\mathbf{Z}} \frac{2}{\xi^2+1} |\hat{\psi}(k)|.
\end{aligned}$$

Hence

$$||M|| \leq \sum_{\xi \in \mathbf{Z}} \frac{2}{\xi^2+1} = 2\pi \coth(\pi). \tag{22}$$

Taking $n \to \infty$ in (21), we have

$$Q_t(\xi', \xi) \leq \left(e^{t\Gamma}\right)(\xi', \xi) \leq e^{2dt}. \tag{23}$$

Moreover, since $||\Gamma|| = 2d$,

$$||Q_t|| \leq e^{2dt}. \tag{24}$$

□

This bound implies that

$$||F^{\sigma}_{t',t}|| \leq e^{2d(t'-t)}. \tag{25}$$

Indeed,

$$\begin{aligned}
||F^{\sigma}_{t',t}|| &= \sup_{||\Phi||_\infty=1} ||F^{\sigma}_{t',t}(\Phi)|| \\
&= \sup_{||\Phi||_\infty=1} \sup_{||\varphi||_2=1} \left\| \sum_{\xi,\xi_1,\ldots,\xi_n} U_{t'-t_n}(\cdot, \xi_n) \ldots U_{t_1-t}(\xi_1, \xi) \Phi(\xi_n, \ldots, \xi_1) \varphi(\xi) \right\|_2 \\
&\leq \sup_{||\varphi||_2=1} \left\| \sum_{\xi,\xi_1,\ldots,\xi_n} Q_{t'-t_n}(\cdot, \xi_n) \ldots Q_{t_1-t}(\xi_1, \xi) |\varphi(\xi)| \right\|_2 \\
&\leq \left\| e^{(t'-t)\Gamma} |\varphi| \right\|_2 \leq ||e^{(t'-t)\Gamma}|| \leq e^{2d(t'-t)}.
\end{aligned}$$

Fixing Φ, we also have, for any $\varphi, \psi \in \ell^2(\mathbf{Z}^d)$,

$$\begin{aligned}
&|\langle \psi \mid F^{\sigma}_{t',t}(\Phi)\varphi \rangle| \\
&\quad = \left| \sum_{\xi,\xi',\xi_1,\ldots,\xi_n} \overline{\psi(\xi')}\, U_{t'-t_n}(\xi', \xi_n) \ldots U_{t_1-t}(\xi_1, \xi) \Phi(\xi_n, \ldots, \xi_1) \varphi(\xi) \right| \\
&\quad \leq \sum_{\xi,\xi',\xi_1,\ldots,\xi_n} |\psi(\xi')| Q_{t'-t_n}(\xi', \xi_n) \ldots Q_{t_1-t}(\xi_1, \xi) |\varphi(\xi)| \\
&\quad \leq \nu^{\sigma}_{\psi,\varphi}(|\Phi|),
\end{aligned}$$

where the measure $\nu^{\sigma}_{\psi,\varphi}$ is defined by

$$\begin{aligned}
\nu^{\sigma}_{\psi,\varphi}(\Phi) = \sum_{\xi,\xi',\xi_1,\ldots,\xi_n} &|\psi(\xi')| \left(e^{(t'-t_n)\Gamma}\right)(\xi', \xi_n) \ldots \\
&\ldots \left(e^{(t_1-t)\Gamma}\right)(\xi_1, \xi) \Phi(\xi_n, \ldots, \xi_1) |\varphi(\xi)|.
\end{aligned} \tag{26}$$

This measure is clearly positive and uniformly bounded by

$$||\nu^{\sigma}_{\psi,\varphi}|| = \sum_{\xi,\xi'} |\psi(\xi')| \left(e^{(t'-t)\Gamma} \right) (\xi',\xi) |\varphi(\xi)| \le e^{2d(t'-t)} ||\psi|| \, ||\varphi||. \tag{27}$$

It follows that both $F^{\sigma}_{t',t}$ and $F_{t',t}$ are indeed Radon measures on $\dot{\mathbf{Z}}^{[t,t']}$ with values in $\mathcal{B}(\mathcal{H})$. In fact, a continuous map on $\mathcal{C}(X)$ with values in a quasi-complete locally convex topological Hausdorff space is a Radon measure if it is weakly compact [9–11]. We have

Lemma 1.2 *Denote the strong operator topology on* $\mathcal{B}(\mathcal{H})$ *as* $\mathcal{T}_s$. *Then the weak topology* $\sigma(\mathcal{B}(\mathcal{H}), (\mathcal{B}(\mathcal{H})_s)')$ *induced on* $\mathcal{B}(\mathcal{H})$ *by the strong dual agrees with the weak operator topology. Moreover, bounded subsets of* $\mathcal{B}(\mathcal{H})$ *are weakly compact.*

Proof. It is known (see [12], Chapter IV, §2, Prop. 11, or [13], Theorem 4.2.6) that the strongly continuous linear forms on $\mathcal{B}(\mathcal{H})$ are of the form

$$\ell(A) = \sum_{j=1}^{n} \langle \psi_j \,|\, A\phi_j \rangle$$

for finite sets of vectors $\psi_j, \phi_j \in \mathcal{H}$, and are therefore weakly continuous. It follows that the weak topology induced by $\mathcal{B}(\mathcal{H})'_s$ is just the weak operator topology. But the weak operator topology is weaker than the ultra-weak topology, which is the weak-* topology induced by the predual of $\mathcal{B}(\mathcal{H})$, i.e. the trace-class operators $\mathcal{L}^1(\mathcal{H})$: see [14], Theorem 1 of Part I, Chapter 3, or [13], Theorem 4.2.3. By the Banach-Alaoglu theorem, bounded subsets are compact in the latter topology, and hence also in the weak operator topology. □

It remains to remark that the last two conditions ((8) and (9)) are fulfilled by construction, and the first condition (7) is easily proved by approximating Φ_1 and Φ_2 by functions of the form $\Psi_1 \circ \pi_\sigma$ and $\Psi_2 \circ \pi_\sigma$, where σ is a subdivision including the intermediate point t'. □

Remark 1.2 The proof shows that the measures are absolutely continuous with respect to the positive measure corresponding to the random walk on $\mathbf{Z}^d$. Indeed, $e^{t\Gamma} = e^{2dt} e^{-H_0 t}$, and e^{-tH_0} is the generator of the random walk. This fact was used by Molchanov [7] to formulate a version of Feynman's path integral in terms of random walks as follows:

$$\left(e^{-itH} \psi \right)(\xi) = e^{2dt} \mathbf{E}^{(d)}_{\xi} \left[\psi(x(t)) i^{N(t)} \exp\left(-\int_0^t V(x(s)) ds \right) \right], \tag{28}$$

where $N(t)$ is the number of jumps of the path before time t. (See Prop. II.3.12 of [8].)

2. REGULARITY OF THE PATHS.

We now show that the measures $F_{t',t}$ are in fact concentrated on paths with values in $\mathbf{Z}^d$ which are almost everywhere constant. First consider the Skorokhod space of functions $x : [t,t'] \to \mathbf{R}^d$ which are right-continuous and have limits on the left as well as being continuous at t'. This is usually denoted $D([t,t'],\mathbf{R}^d)$.

Lemma 2.1 *The Skorokhod space $D([t,t'],\mathbf{R}^d)$ is a Borel set in $(\mathbf{R}^d)^{[t,t']}$. Moreover, any Borel subset of $D([t,t'],\mathbf{R}^d)$ is also a Borel subset of $(\mathbf{R}^d)^{[t,t']}$.*

Proof. This theorem follows in fact from a general theorem (Theorem 5 and Corollary 1 of [15]), which states that if X is a Polish space, continuously embedded into a Hausdorff space Y then X is a Borel subset of Y. However, for completeness, we provide a simple direct proof here along the lines of [3]. For $\epsilon, \delta > 0$ define the set $D_{\delta,\epsilon}[t,t']$ by

$$\begin{aligned} D_{\delta,\epsilon}[t,t'] &= \left\{ x \in (\mathbf{R}^d)^{[t,t']} : \sup_{s\in[t,t')} \sup_{s'\in(s,s+\delta)} |x(s') - x(s)| \le \epsilon \right\} \\ & \bigcap \left\{ x \in (\mathbf{R}^d)^{[t,t']} : \sup_{s\in(t,t')} \sup_{s_1,s_2\in(t_1-\delta,t_1)} |x(s_2) - x(s_1)| \le \epsilon \right\} \\ & \bigcap \left\{ x \in (\mathbf{R}^d)^{[t,t']} : \sup_{s\in(t'-\delta,t')} |x(t') - x(s)| \le \epsilon \right\}. \end{aligned} \tag{29}$$

We then claim that

$$D([t,t'],\mathbf{R}^d) = \bigcap_{\epsilon>0} \bigcup_{\delta>0} D_{\delta,\epsilon}[t,t']. \tag{30}$$

Indeed, suppose that $x \in (\mathbf{R}^d)^{[t,t']}$ and for all $\epsilon > 0$ there exists $\delta > 0$ such that $x \in D_{\delta,\epsilon}[t,t']$. Then for all $s \in [t,t')$, and all $\epsilon > 0$ there is $\delta > 0$ so that $|x(s') - x(s)| \le \epsilon$ whenever $s' \in (s, s+\delta)$, i.e. $\lim_{s'\downarrow s} x(s') = x(s)$, so x is right-continuous at s. Similarly, $\lim_{s'\uparrow s} x(s')$ exists for all $s \in [t,t')$ and $\lim_{s\uparrow t'} x(s) = x(t')$; the former because a Cauchy condition holds. Thus $x \in D([t,t'],\mathbf{R}^d)$.

Conversely, suppose $x \in D([t,t'],\mathbf{R}^d)$. Then, for any $s \in [t,t')$, $\lim_{s'\downarrow s} x(s')$ $= x(s)$, so for all $\epsilon > 0$ there exists $\delta_s > 0$ such that $|x(s') - x(s)| \le \epsilon/2$ for $s' \in (s, s+\delta_{t_1})$. Moreover, there also exists $\delta_{t'} > 0$ such that $|x(s) - x(t')| \le \epsilon/2$ if $s \in (t' - \delta_{t'}, t']$. We can now cover $[t,t']$ with a finite number of intervals $(s_k, s_k + \delta_{s_k}/2)$ (taking the first interval to be $[t, t+\delta_t/2)$ and the last $(t' - \delta_{t'}/2, t'])$. Let δ_1 be the minimum of the corresponding

$\delta_{s_k}/2$. Then, if $s \in [t, t']$ and $s' \in (s, s+\delta_1)$, there exists k such that $s \in [s_k, s_k + \delta_{s_k}/2)$ and hence

$$|x(s) - x(s')| \leq |x(s) - x(s_k)| + |x(s_k) - x(s')| \leq \epsilon$$

since $|s' - s_k| < \delta_{s_k}/2 + \delta' \leq \delta_{s_k}/2$. Similarly, for all $s \in (t, t']$ there exists $\delta'_s > 0$ such that for $s', s'' \in (s - \delta'_s, s]$, $|x(s') - x(s'')| \leq \epsilon/2$. Covering $[t, t']$ now with intervals $(s'_k - \delta'_{s'_k}/2, s'_k)$ together with $(t' - \delta_{t'}/2, t']$ we find in the same way that for every $s \in [t, t']$ and $s', s'' \in (s - \delta_2, s)$, $|x(s') - x(s'')| \leq \epsilon$, where $\delta_2 = \min \delta_{s'_k}$. Taking $\delta = \delta_1 \wedge \delta_2$ we find that $x \in D_{\epsilon,\delta}$.

It is obvious that the sets $D_{\delta,\epsilon}[t, t']$ are closed. Moreover, they are decreasing in δ and increasing in ϵ so we can restrict the intersection over ϵ and the union over δ to numbers of the form $1/n$ with $n \in \mathbf{N}$. It follows that $D([t, t'], \mathbf{R}^d)$ is a Borel set.

The second statement follows from Theorem 7.1 in [16]. □

Let us denote

$$\mathcal{S}^d[t, t'] = D([t, t'], \mathbf{R}^d) \cap (\mathrm{Fin}^d)[t, t'], \tag{31}$$

where $\mathrm{Fin}^d[t, t'] = \cup_{\Lambda \subset \mathbf{Z}^d \,\mathrm{finite}} \Lambda^{[t,t']}$ is the set of paths taking finitely many values in $\mathbf{Z}^d$. Since we can restrict the union to a sequence of boxes tending to $\mathbf{Z}^d$, the latter is a Borel subset of $(\dot{\mathbf{Z}}^d)^{[t,t']}$. Restricting even further, we define $\mathcal{S}^d_1[t, t'] = \big\{x \in \mathcal{S}^d : x(s^+) - x(s^-) \in \{0\} \cup \{e_1, -e_1, \ldots, e_d, -e_d\}\big\}$. This is easily seen to be a closed subset of $\mathcal{S}^d[t, t']$ and therefore also a Borel subset of $(\dot{\mathbf{Z}}^d)^{[t,t']}$.

Theorem 2.1 *The measure $F_{t',t}$ is concentrated on $\mathcal{S}^d_1[t, t']$. Moreover, the measures $F_{(t',\xi'),(t,\xi)} = \langle \delta_{\xi'} \,|\, F_{t',t} \delta_\xi \rangle$ are concentrated on $\mathcal{S}^d_1[(t', \xi'), (t, \xi)] = \big\{\mathcal{S}^d_1[t, t'] : x(t) = \xi, x(t') = \xi'\big\}$, and all these measures are Radon measures w.r.t. the Skorokhod topology on these spaces.*

Proof. By the fact that $|\langle \psi \,|\, F^\sigma_{t't}(\Phi)\varphi \rangle| \leq \nu^\sigma_{\psi,\varphi}(\Phi)$ it suffices to prove that the projective limit $\nu_{\psi,\varphi}$ of the latter measures is concentrated on $\mathcal{S}^d_1[t, t']$. This follows from a theorem of Doob [17], but is in fact easy to prove directly in this case. Consider the sets

$$\begin{aligned} K_\delta \;=\; \Big\{ x \in (\dot{\mathbf{Z}}^d)^{[t,t']} &: x(t), x(t') \in \mathbf{Z}^d \text{ and } \forall t_1 \in [t, t'] : \\ &\text{either } x(s) = x(t_1) \forall s \in [t_1 - \delta, t_1 + \delta] \\ &\text{or } \exists \xi = \pm e_j, t_2 \in [t_1 - \delta, t_1 + \delta] : x(s_1) - x(s_2) = \xi \\ &\quad \forall s_1 \in [t_1 - \delta, t_2), s_2 \in [t_2, t_1 + \delta] \Big\}. \end{aligned} \tag{32}$$

These are the sets of paths with values in $\mathbf{Z}^d$ such that there is at most one jump in any interval of length 2δ and the jump is of size 1. These sets clearly belong to $\mathcal{S}^d[t,t']$ and they are compact in the Skorokhod topology. The latter follows from the compactness criterion for subsets of $D([t,t'],\mathbf{R}^d)$: see Theorem 6.2 in [16]. In fact (see [18]), for $\eta < 1$,

$$K_\delta = \{x \in \mathcal{S}^d_1[t,t'] : \tilde{\omega}_x(\delta) < \eta\},$$

where the quantity $\tilde{\omega}_x(\delta)$ is given by

$$\begin{aligned} \tilde{\omega}_x(\delta) \;=\; \max\Bigg\{ & \sup_{s-\delta<s'\le s\le s''<s+\delta} \left(|x(s')-x(s)| \wedge |x(s'')-x(s)|\right), \\ & \sup_{t\le s<t+\delta} |x(s)-x(0)|, \sup_{t'-\delta<s\le t'} |x(s)-x(t')| \Bigg\}. \end{aligned} \tag{33}$$

Now, given $\sigma = (t_1,\ldots,t_n)$, it is obvious that $\pi_\sigma^{-1}(x_1,\ldots,x_n) \in K_\delta$ means that whenever $t_{k_2} - t_{k_1} < 2\delta$ with $k_2 \ge k_1 + 2$, then either $x_k = x_{k_1}$ for $k_1 \le k \le k_2$, or there exists k_3 with $k_1 < k_3 < k_2$ such that $x_k = x_{k_3}$ for $k_1 \le k \le k_3$ and $x_k = x_{k_3+1}$ for $k_3 < k \le k_2$. We subdivide the interval $[t,t']$ into $(t'-t)/(2\delta)$ intervals of length 2δ. If $x \notin K_\delta$ then there is a double interval of length 4δ which contains points at distance at most 2δ where x jumps. Consider such a double interval and let t_{k_1-1} be the left-most point of σ and t_{k_2+1} the right-most point of σ contained in this interval. Now, using the bound

$$||A(t)|| \le t||\Gamma||\, \left\|e^{t\Gamma}\right\|, \text{ where } A(t)_{\xi',\xi} = \left(e^{t\Gamma}\right)(\xi',\xi)(1-\delta_{\xi',\xi}),$$

we have

$$\begin{aligned} & \nu^\sigma_{\psi,\varphi}(\{\text{At least 2 jumps between } t_{k_1} \text{ and } t_{k_2}\}) \\ & \le \sum_{k=k_1-1}^{k_2-1} \sum_{k'=k+1}^{k_2} \sum_{\xi',\xi_n,\ldots,\xi_{k'+1}} \sum_{\xi_{k'}\neq\xi_{k'+1}} \sum_{\xi_{k'-1},\ldots,\xi_{k+1}} \\ & \qquad \times \sum_{\xi_k\neq\xi_{k+1}} \sum_{\xi_{k-1},\ldots,\xi_1,\xi} |\psi(\xi')| \left(e^{(t'-t_n)\Gamma}\right)(\xi',\xi_n)\ldots\left(e^{(t_1-t)\Gamma}\right)(\xi_1,\xi)|\varphi(\xi)| \\ & \le \sum_{k=k_1-1}^{k_2-1} \sum_{k'=k+1}^{k_2} (t_{k+1}-t_k)(t_{k'+1}-t_{k'})||\Gamma||^2 e^{2d(t'-t)}||\varphi||\,||\psi|| \\ & \le \frac{t_{k_2+1}-t_{k_1-1}}{k_2-k_1+2}||\Gamma||^2 e^{2d(t'-t)}||\varphi||\,||\psi|| \\ & \qquad \times \sum_{k=k_1-1}^{k_2-1} \sum_{k'=k+1}^{k_2} \left(t_{k_2+1}-t_{k_1-1} - \frac{t_{k_2+1}-t_{k_1-1}}{k_2-k_1+2}(k-k_1+3)\right) \\ & \le 32\delta^2 d^2\, e^{2d(t'-t)}||\varphi||\,||\psi||. \end{aligned} \tag{34}$$

Since there are $(t'-t)/(2\delta)$ such intervals,

$$\nu^{\sigma}_{\psi,\varphi}(K^c_\delta) \le 16(t'-t)\delta d^2 e^{2d(t'-t)} ||\varphi||\, ||\psi|| \to 0.$$

Finally, we notice that, on a metric space, every bounded Borel measure is outer regular, and inner-regular with respect to closed sets (see [16], Theorem 1.2 of Chapter 2). Since we have already shown that the measure $\nu_{\psi,\phi}$ is concentrated on a compact set in $\mathcal{S}^d_1$ up to any $\epsilon > 0$, it follows that it is a Radon measure. $\square$

3. THE FEYNMAN INTEGRAL FORMULA

To derive the Feynman integral formula for the solution of the Schrödinger equation, let, for simplicity, V be a bounded potential, $V : \mathbf{Z}^d \to \mathbf{R}$. Then the integral

$$\int_t^{t'} V(x(s))\, ds = \lim_{n\to\infty} \sum_{k=1}^n V(x(t_k))(t_k - t_{k-1}) = \lim_{n\to\infty} \Sigma^{\sigma_n}_V(x) \tag{35}$$

is well-defined and continuous as a function of $x \in \mathcal{S}^d_1[t,t']$. Indeed, $s \mapsto V(x(s))$ is a step function, hence integrable, and the set of points where $x \in \mathcal{S}^d_1[t,t']$ has a jump has measure 0. Therefore, if Δ_x is the set of points of discontinuity of x, $|\{s \in [t,t'] :\, d(s,\Delta_x) < \epsilon\}| \to 0$ as $\epsilon \to 0$. Now, if $x_n \to x$ in $\mathcal{S}^d_1[t,t']$, let n be so large that $\rho(x,x_n) < \epsilon$, where ρ denotes the Skorokhod metric:

$$\rho(x,x') = \inf_{\lambda in H[t,t']} ||x - x' \circ \lambda||_\infty + ||\lambda - \mathrm{id}||_\infty. \tag{36}$$

(Here $H[t,t']$ denotes the continuous increasing functions from $[t,t']$ onto itself.) Then there exists $\lambda \in H$ such that $||x - x_n \circ \lambda||_\infty < \epsilon$ and $||\lambda - \mathrm{id}||_\infty < \epsilon$. Assuming $\epsilon < 1$ we have: $x_n(s) = x(s)$ unless $d(s,\Delta_x) < \epsilon$. If $M = ||V||_\infty$, and taking $t_k = t + k(t'-t)/n$ with $\frac{1}{n} < \epsilon$, we get

$$\begin{aligned} &\left| \sum_{k=1}^n V(x_n(t_k))(t_k - t_{k-1}) - \sum_{k=1}^n V(x(t_k))(t_k - t_{k-1}) \right| \\ &\le\ 2M \sum_{k=1}^n 1_{\{k:\, d(t_k,\Delta_x)<\epsilon\}}(t_k - t_{k-1}) \\ &\le\ 2M |\{s :\, d(s,\Delta_x) < \epsilon\}| \to 0. \end{aligned} \tag{37}$$

Theorem 3.1 *Let $H = H_0 + V$, where $V : \mathbf{Z}^d \to \mathbf{R}$ is a bounded potential. Then*

$$e^{-i(t'-t)H} = \int_{\mathcal{S}^d_1[t,t']} \exp\left[-i \int_t^{t'} V(x(s))ds\right] F_{t',t}(dx). \tag{38}$$

Moreover, the matrix elements of $e^{-i(t'-t)H}$ *are given by*

$$e^{-i(t'-t)H}(\xi',\xi) = \int_{\mathcal{S}_1^d[(t,\xi),(t',\xi')]} \exp\left[-i\int_t^{t'} V(x(s))ds\right] F_{t',t}(dx). \tag{39}$$

Proof. We only have to prove that

$$\lim_{n\to\infty} \int e^{-i\Sigma_V^{\sigma_n}(x)} F_{t',t}(dx) = e^{-i(t'-t)H},$$

where the integral is over $(\dot{\mathbf{Z}}^d)^{[t,t']}$. This follows from the definition and the Trotter product formula: Writing $\varphi_k(\xi) = e^{-i(t_k-t_{k-1})V(\xi)}$,

$$e^{-i\Sigma_V^{\sigma_n}(x)} = (\varphi_n \otimes \cdots \otimes \varphi_1) \circ \pi_{\sigma_n}(x)$$

and hence, with $t_k = t + k(t'-t)/n$ as above,

$$\begin{aligned}\int e^{-i\Sigma_V^{\sigma_n}(x)} F_{t',t}(dx) &= \int (\varphi_n \otimes \cdots \otimes \varphi_1) dF_{t',t}^{\sigma_n} \\ &= \mathcal{M}_{\varphi_n} U^0_{t_n-t_{n-1}} \mathcal{M}_{\varphi_{n-1}} \ldots \mathcal{M}_{\varphi_1} U^0_{t_1-t} \\ &= \left(e^{-i(t'-t)V/n} U^0_{(t'-t)/n}\right)^n.\end{aligned}$$

By Trotter's product formula (in fact, the simple form of Theorem XIII.30 of [19] suffices), the right-hand side tends to $e^{-i(t'-t)(H_0+V)}$. The formula for the matrix elements follows from the fact that $F_{(t',\xi'),(t,\xi)}$ is concentrated on $\mathcal{S}_1^d[(t,\xi),(t',\xi')]$. □

In fact, the Feynman integral formula can be extended to time-dependent potentials: Assume that $V : \mathbf{Z}^d \times [t,t'] \to \mathbf{R}$ is a uniformly bounded potential which depends continuously on the time (i.e. the second variable). A minor modification of the above argument shows that $\int_t^{t'} V(x(s),s)ds$ is still well-defined and continuous as a function of $x \in \mathcal{S}_1^d[t,t']$. We now approximate V by a step-function, as follows. We subdivide $[t,t']$ into subintervals $[t_k,t_{k+1}]$ as before and put $V^{(n)}(x(s),s) = V(x(t_k),t_k)$ if $t_k \le s < t_{k+1}$. The solution of the Schrödinger equation

$$i\frac{\partial}{\partial s}\psi_s = (H_0 + V^{(n)})\psi_s \tag{40}$$

with initial condition ψ_t^0 is obviously given by

$$\psi_{t'}^{(n)} = U^{(n)}_{t_n-t_{n-1}} \ldots U^{(n)}_{t_1-t}\psi_t^0$$

where

$$U_s^{(n)} = e^{-is(H_0+V^{(n)}(\cdot,t_k))} \text{ if } t_k \le s < t_{k+1}. \tag{41}$$

Now, since $V^{(n)} \to V$ in L^1-norm,

$$\int_t^{t'} V^{(n)}(x(s), s)ds \to \int_t^{t'} V(x(s), s)ds$$

for every $x \in \mathcal{S}_1^d[t, t']$, and by the bounded convergence theorem,

$$\int \exp\left[-i\int_t^{t'} V^{(n)}(x(s), s)ds\right] F_{t',t}(dx)\psi_t^0 \to \exp\left[-i\int_t^{t'} V(x(s), s)ds\right]\psi_t^0.$$

On the other hand, the solution of (40) converges to that of

$$i\frac{\partial}{\partial s}\psi_s = (H_0 + V)\psi_s. \tag{42}$$

This follows from Picard's method (the method of successive approximations) applied to the corresponding integral equations. We thus have:

Theorem 3.2 *Let $H = H_0 + V$, where $V : \mathbf{Z}^d \times [t, t'] \to \mathbf{R}$ is a uniformly bounded potential depending continuously on the time. Then, for any initial condition ψ^0, the solution of the Schrödinger equation (42) is given by*

$$\psi_{t'} = \int_{\mathcal{S}_1^d[t,t']} \exp\left[-i\int_t^{t'} V(x(s), s)ds\right] F_{t',t}(dx)\psi^0. \tag{43}$$

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