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Three Generations on the Quintic Quotient

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Abstract

A three-generation $SU(5)$ GUT, that is $3 \times (\underline{\mathbf{10}} + \overline{\underline{\mathbf{5}}})$ and a single $\underline{\mathbf{5}}$–$\overline{\underline{\mathbf{5}}}$ pair, is constructed by compactification of the E_8 heterotic string. The base manifold is the $\mathbb{Z}_5 \times \mathbb{Z}_5$-quotient of the quintic, and the vector bundle is the quotient of a positive monad. The group action on the monad and its bundle-valued cohomology is discussed in detail, including topological restrictions on the existence of equivariant structures. This model and a single $\mathbb{Z}_5$ quotient are the complete list of three generation quotients of positive monads on the quintic.

Contents

1 Introduction

Monad bundles [1, 2, 3, 4, 5, 6, 7] are the largest known class of $(0, 2)$-compactifications. However, so far only monad bundles on simply-connected Calabi-Yau manifolds were explicitly constructed. However, just as in the heterotic standard embedding, free quotients are, amongst many other aspects, important in reducing the particle spectrum. For example, the net number of generations was found [5] to peak somewhere around 60. By dividing out the free action of a discrete group G, the number of generations would be divided by $|G|$.

In this paper, I will construct a slope-stable rank 5 vector bundle on the $\mathbb{Z}_5 \times \mathbb{Z}_5$-quotient of the quintic via the monad construction. The best way to work with non-simply connected manifolds is, as usual, to construct everything on the universal cover (the quintic). However, care has to be taken to make everything symmetric under the group action, and as we will see this imposes purely topological restrictions on the Chern classes of the constituents of the monad. Via the so-called "non-standard embedding", this bundle of vanishing first Chern class defines a $(0, 2)$-compactification of the E_8 heterotic string, giving rise to a low-energy $SU(2)$ gauge group. The resulting matter spectrum will be three generations of $\underline{\mathbf{10}} + \underline{\overline{\mathbf{5}}}$ together with an (optional) vector-like pair of $\underline{\mathbf{5}}$–$\underline{\overline{\mathbf{5}}}$. A hopefully useful SINGULAR worksheet demonstrating the spectrum computation is in Appendix B. Although $\mathbb{Z}_5$ Wilson lines cannot be used to break $SU(5)$ to the standard model gauge group, a mechanism like $U(1)_Y$-flux could conceivably be employed. In any case, the technology for constructing equivariant monads will certainly be useful for more thorough searches for heterotic standard models.

2 The Quintic Quotient

The quintic Q is the simplest Calabi-Yau manifold. As the name suggests, it is given by the zero set of a sufficiently generic degree-5 hypersurface $Q(z_0, z_1, z_2, z_3, z_4) = 0$ in projective space $\mathbb{P}^4$. Using this description of a smooth hypersurface, one can show that there are precisely two[1] possible free group actions on the quintic: $\mathbb{Z}_5$ and $\mathbb{Z}_5 \times \mathbb{Z}_5$. However, the complex structure of Q, that is, the quintic polynomial $Q(z)$, has to be chosen suitably to admit this symmetry. Without symmetry, the quintic has $h^{2,1}(Q) = 101$ complex structure moduli. Imposing the symmetries restricts the complex structure moduli to a 25 and 5-dimensional stratum, respectively.

For the purposes of this paper, I will focus solely on the free $\mathbb{Z}_5 \times \mathbb{Z}_5$ group action [8]. It acts projectively linear on the homogeneous coordinates of the ambient projective space; The two group generators are ($\zeta = e^{\frac{2\pi i}{5}}$)

$$g_1(z_i) = z_{i+1}, \qquad g_2(z_i) = \zeta^i z_i. \tag{1}$$

They satisfy

$$g_1^5 = 1 = g_2^5, \quad g_1 g_2 = \zeta^{-1} g_2 g_1 \tag{2}$$

and, therefore, define a $\mathbb{Z}_5 \times \mathbb{Z}_5$ group action on $\mathbb{P}^4$. If the quintic is invariant, then this action defines a group action on the hypersurface, too. The invariant polynomials are best described by a Hironaka decomposition

$$\mathbb{C}[z_0, z_1, z_2, z_3, z_4]^{\mathbb{Z}_5 \times \mathbb{Z}_5} = \bigoplus_{i=1}^{100} \eta_i \, \mathbb{C}[\theta_1, \theta_2, \theta_3, \theta_4, \theta_5]. \tag{3}$$

[1]The $\mathbb{Z}_5$ is a subgroup of $\mathbb{Z}_5 \times \mathbb{Z}_5$, so there is one *maximal* free group action.

Here, the primary invariants are [9]

$$
\begin{array}{lccr}
\theta_1 \stackrel{\text{def}}{=} & z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 & = z_0^5 + (\text{cyc}) \\
\theta_2 \stackrel{\text{def}}{=} & z_0 z_1 z_2 z_3 z_4 & \\
\theta_3 \stackrel{\text{def}}{=} & z_0^3 z_1 z_4 + z_0 z_1^3 z_2 + z_0 z_3 z_4^3 + z_1 z_2^3 z_3 + z_2 z_3^3 z_4 & = z_0^3 z_1 z_4 + (\text{cyc}) \\
\theta_4 \stackrel{\text{def}}{=} & z_0^{10} + z_1^{10} + z_2^{10} + z_3^{10} + z_4^{10} & = z_0^{10} + (\text{cyc}) \\
\theta_5 \stackrel{\text{def}}{=} & z_0^8 z_2 z_3 + z_0 z_1 z_3^8 + z_0 z_2^8 z_4 + z_1^8 z_3 z_4 + z_1 z_2 z_4^8 & = z_0^8 z_2 z_3 + (\text{cyc})
\end{array}
\tag{4}
$$

and the secondary invariants are, in degrees < 10,

$$
\begin{gathered}
\eta_1 \stackrel{\text{def}}{=} 1, \\
\eta_2 \stackrel{\text{def}}{=} z_0^2 z_1 z_2^2 + (\text{cyc}), \quad \eta_3 \stackrel{\text{def}}{=} z_0^2 z_1^2 z_3 + (\text{cyc}), \quad \eta_4 \stackrel{\text{def}}{=} z_0^3 z_2 z_3 + (\text{cyc}).
\end{gathered}
\tag{5}
$$

Hence, an invariant quintic is of the form

$$
Q(z) = c_0 \theta_1 + c_1 \theta_2 + c_2 \theta_3 + c_3 \eta_2 + c_4 \eta_3 + c_5 \eta_4. \tag{6}
$$

The complex structure moduli space of the invariant quintics is an open subset of $\mathbb{P}^5$ parametrized by $[c_0 : \cdots : c_5]$. Finally, one can easily check that a generic member of this family of quintics is smooth and fixed-point free. Therefore, the quotient

$$
X = Q \Big/ \big(\mathbb{Z}_5 \times \mathbb{Z}_5\big) \tag{7}
$$

is a smooth Calabi-Yau threefold with fundamental group $\pi_1(X) = \mathbb{Z}_5 \times \mathbb{Z}_5$.

3 Monadology

3.1 Overview

A monad [10, 1, 2, 3] is a three-step filtration of a vector bundle V. That is, a complex

$$
0 \longrightarrow A \xrightarrow{a} B \xrightarrow{b} C \longrightarrow 0 \tag{8}
$$

whose cohomology is a vector bundle in the middle, at the B entry. In other words, a is injective, b is surjective, $b \circ a = 0$, and $V = \ker(b)/\operatorname{img}(a)$ is the vector bundle. For example, a short exact sequence corresponds to the zero vector bundle.

For the purposes of this paper, I will only consider *positive monads* where $A = 0$ and B, C are very ample bundles. In this case, V *is* defined by a short exact sequence

$$
0 \longrightarrow V \longrightarrow B \xrightarrow{f} C \longrightarrow 0 \tag{9}
$$

In particular, I will always take the base space to be $Q \subset \mathbb{P}^4$ and

$$B = \bigoplus_{i=1}^{n} \mathcal{O}(b_i), \qquad C = \bigoplus_{j=1}^{m} \mathcal{O}(c_j) \tag{10}$$

with b_i, $c_j > 0$. The positive monads for rank 3, 4, and 5 bundles satisfying heterotic anomaly cancellation without anti-branes were classified in [4, 5, 6]. There are 43 such monads, none of which gives rise to 3 generations[2]. Adding the additional constraint that the number of generations is at least a multiple of 3, the authors found 15

rank V	$\{b_i\}$	$\{c_i\}$	Ind V	$G_{\text{3-gen}}$
3	$\{2, 2, 1, 1, 1\}$	$\{4, 3\}$	-60	–
3	$\{2, 2, 2, 1, 1\}$	$\{5, 3\}$	-105	–
3	$\{3, 2, 1, 1, 1\}$	$\{4, 4\}$	-75	–
3	$\{1, 1, 1, 1, 1, 1\}$	$\{2, 2, 2\}$	-15	$\mathbb{Z}_5$
3	$\{2, 2, 2, 1, 1, 1\}$	$\{3, 3, 3\}$	-45	–
3	$\{3, 3, 3, 1, 1, 1\}$	$\{4, 4, 4\}$	-90	–
3	$\{2, 2, 2, 2, 2, 2, 2, 2\}$	$\{4, 3, 3, 3, 3\}$	-90	–
3	$\{2, 2, 2, 2, 2, 2, 2, 2, 2\}$	$\{3, 3, 3, 3, 3, 3\}$	-75	–
4	$\{2, 2, 1, 1, 1, 1\}$	$\{4, 4\}$	-90	–
4	$\{1, 1, 1, 1, 1, 1, 1\}$	$\{3, 2, 2\}$	-30	–
4	$\{2, 2, 2, 1, 1, 1, 1\}$	$\{4, 3, 3\}$	-75	–
4	$\{2, 2, 2, 2, 1, 1, 1, 1\}$	$\{3, 3, 3, 3\}$	-60	–
5	$\{1, 1, 1, 1, 1, 1, 1, 1\}$	$\{3, 3, 2\}$	-45	–
5	$\{1, 1, 1, 1, 1, 1, 1, 1\}$	$\{4, 2, 2\}$	-60	–
5	$\{2, 2, 2, 2, 2, 1, 1, 1, 1, 1\}$	$\{3, 3, 3, 3, 3\}$	-75	$\mathbb{Z}_5 \times \mathbb{Z}_5$

Table 1: *Positive monad bundles on the quintic. The entries marked in red are obstructions to a $\mathbb{Z}_5 \times \mathbb{Z}_5$-action, see Subsection 3.4. The last column $G_{\text{3-gen}}$ are free group actions such that the quotient has three generations.*

monads on the quintic, which are reproduced in Table 1 for convenience. As I will discuss shortly, most of these monad bundles do not have a suitable symmetry for the quotient to yield a three generation model. However, two of them do. The first one is a rank 3 bundle which we can divide by $\mathbb{Z}_5$. This yields a low-energy E_6 gauge group with three generations of $\underline{\mathbf{27}}$ and no anti-generations. The $\mathbb{Z}_5$ fundamental group can be used to break the E_6 to the standard model gauge group plus two extra $U(1)$ [11], though it would be difficult to remove the exotic matter coming from the

[2]The net number of generations, that is the difference between generations and anti-generations, equals the index $\mathrm{Ind}(V)$ of the vector bundle.

decomposition of the $\underline{\mathbf{27}}$. The other model, and subject of this paper, is in the last row of Table 1. As we will see in the remainder of this section, being able to divide out the non-cyclic group $\mathbb{Z}_5 \times \mathbb{Z}_5$ will pose many more restrictions on the monad bundle than just $\mathbb{Z}_5$. At the same time, the larger group makes it much easier to project out unwanted matter states.

3.2 Symmetry Considerations

For the reasons just mentioned, in the following I will only consider the monad bundle[3]

$$0 \longrightarrow V \longrightarrow 5\mathcal{O}_Q(1) \oplus 5\mathcal{O}_Q(2) \xrightarrow{f} 5\mathcal{O}_Q(3) \longrightarrow 0. \tag{11}$$

However, eq. (11) only defines a vector bundle on $Q \subset \mathbb{P}^4$. In order to divide out the group action and obtain a bundle on X, we need to first *specify* a group action on V.

The choice of a group action on the vector bundle is called an equivariant structure. This is a choice bundle map $\gamma(g) : V \to V$ covering the group action $g : Q \to Q$ on the base of the bundle for all group elements $g \in G$. In other words, we need to pick a linear map $\gamma_p(g) : V_p \to V_{gp}$ holomorphically varying over each point $p \in Q$. Similarly, an equivariant map $f : (V, \gamma^V) \to (W, \gamma^W)$ between equivariant bundles is a map of the vector bundles that intertwines the equivariant structure in the obvious way,

$$\begin{tikzcd} V_p \arrow[r, "f_p(g)"] \arrow[d, "\gamma_p^V(g)"'] & W_p \arrow[d, "\gamma_p^W(g)"] \\ V_{gp} \arrow[r, "f_{gp}(g)"] & W_{gp} \end{tikzcd} \quad \Leftrightarrow \quad \gamma_p^W(g) \circ f_p(g) = f_{gp}(g) \circ \gamma_p^V(g). \tag{12}$$

Direct sums and tensor products of equivariant bundles are equivariant in the obvious way, turning the category Vect_G of G-equivariant vector bundles into a ring. Finally, a monad is equivariant if the objects in the complex and the maps are equivariant; The cohomology of an equivariant monad is an equivariant vector bundle.

For example, consider the trivial line bundle $\mathcal{O}$ on a compact complex manifold. Up to an overall factor, there is a unique section s which is nowhere vanishing. Hence, the choice of a G-equivariant structure on $\mathcal{O}$ is equivalent to the choice of a G-representation on $\Gamma(\mathcal{O}) = \mathbb{C}s$, that is, a multiplicative character of G. By abuse of notation [12], we denote by $\chi \in \mathrm{Hom}(G, \mathbb{C}^\times)$ also the corresponding line bundle. Moreover, we write χV for the tensor product of the line bundle χ and the vector bundle V.

Since we are particularly interested in slope-stable bundles, let us note that the equivariant structure on such a bundle is unique up to multiplication with a multiplicative character. The proof is as follows, assume that you have two different

[3] I will always write $r\mathcal{O}(n)$ for the rank-r bundle $\oplus_{i=1}^r \mathcal{O}(n) = \mathcal{O}(n)^{\oplus r} = \underline{\mathbf{r}} \otimes \mathcal{O}(n)$.

equivariant structures $V_1 = (V, \gamma_{(1)})$ and $V_2 = (V, \gamma_{(2)})$. Then $\gamma_{(2)}^{-1} \circ \gamma_{(1)}$ is a nontrivial automorphism of V. But the only automorphisms of stable bundles are multiplication by a constant [13]. The constant can depend on $g \in G$, but must be a 1-dimensional representation χ. Therefore, $V_1 = \chi V_2$.

3.3 Schur Covers

Not every vector bundle can carry an equivariant structure. The simplest example would be the line bundle $\mathcal{O}_Q(1)$. Let us look at the problem in some detail. First, let us identify the sections with the homogeneous coordinates on $\mathbb{P}^4$ in the usual way,

$$\Gamma\mathcal{O}(1) = \langle z_0, z_1, z_2, z_3, z_4 \rangle. \tag{13}$$

The naive guess for an equivariant structure would be the tautological action

$$\gamma(g_1)(z) = g_1(z), \quad \gamma(g_2)(z) = g_2(z). \tag{14}$$

This fails to be a $\mathbb{Z}_5 \times \mathbb{Z}_5$-equivariant structure because the two actions do not commute,

$$\gamma(g_2)\gamma(g_1)\gamma(g_2)^{-1}\gamma(g_1)^{-1} = \zeta \neq 1, \tag{15}$$

see eq. (2). We can define our way out of this problem by introducing another generator g_3 such that $\gamma(g_3)(z) = \zeta z$ and $g_2 g_1 g_2^{-1} g_1^{-1} = g_3$. This enlarged group is the Heisenberg group

$$H_5 = \langle g_1, g_2, g_3 \rangle = \big(\mathbb{Z}_5 \times \mathbb{Z}_5\big) \rtimes \mathbb{Z}_5, \tag{16}$$

and we just defined a H_5-equivariant structure on $\mathcal{O}_Q(1)$. We note that the Heisenberg group is a Schur cover of $\mathbb{Z}_5 \times \mathbb{Z}_5$,

$$0 \longrightarrow \mathbb{Z}_5 \longrightarrow H_5 \longrightarrow \mathbb{Z}_5 \times \mathbb{Z}_5 \longrightarrow 0\,, \tag{17}$$

and that this construction can be generalized: Given a projective action on the homogeneous coordinates of a projective space, $\mathcal{O}(1)$ is equivariant with respect to a Schur cover.

But we wanted a $\mathbb{Z}_5 \times \mathbb{Z}_5$-equivariant structure, and not a H_5-equivariant structure! Note that a H_5-equivariant structure γ is, in fact, a $\mathbb{Z}_5 \times \mathbb{Z}_5$-equivariant structure if and only if the kernel $\langle g_3 \rangle = \mathbb{Z}_5$ of the cover is represented trivially. Therefore, we arrive at the following characterization of $\mathbb{Z}_5 \times \mathbb{Z}_5$-equivariant bundles:

- Every vector bundle (V, γ) is H_5-equivariant.
- The vector bundle (V, γ) is $\mathbb{Z}_5 \times \mathbb{Z}_5$-equivariant if and only if[4] $\gamma(g_3) = \text{id}$.

[4] Note that g_3 acts trivially on the base space, so its action on the bundle is just a linear map of the fiber to itself.

This shows that eq. (14) does not define an $\mathbb{Z}_5 \times \mathbb{Z}_5$-equivariant structure on $\mathcal{O}_Q(1)$. Could there be another H_5-equivariant structure that *does* descend to a $\mathbb{Z}_5 \times \mathbb{Z}_5$-equivariant structure? To rule out this possibility first note that, like all line bundles, $\mathcal{O}_Q(1)$ is a slope-stable vector bundle. By the argument in Subsection 3.2, any other equivariant structure must differ from (14) by a multiplicative character of H_5. As we will see in more detail in Subsection 3.6, the multiplicative characters of H_5 are just the 1-d representations of $\mathbb{Z}_5 \times \mathbb{Z}_5$. Therefore, any character χ of H_5 satisfies $\chi(g_3) = 1$ and we clearly cannot use this freedom to turn eq. (14) into a $\mathbb{Z}_5 \times \mathbb{Z}_5$-equivariant structure. Hence, there cannot be any $\mathbb{Z}_5 \times \mathbb{Z}_5$-equivariant structure on $\mathcal{O}_Q(1)$.

3.4 Topological Restrictions

Although we have used the holomorphic structure in the above argument, one can exclude the existence of a $\mathbb{Z}_5 \times \mathbb{Z}_5$-equivariant structure on $\mathcal{O}_Q(1)$ on grounds of topology alone. Recall that the topological isomorphism class of a line bundle is classified by its first Chern class. In the case at hand, it is

$$c_1\big(\mathcal{O}_Q(1)\big) = 1 \in \mathbb{Z} = H^2\big(Q, \mathbb{Z}\big). \tag{18}$$

The cohomology of the cover Q and the quotient $X = Q/(\mathbb{Z}_5 \times \mathbb{Z}_5)$ is related via the Leray-Serre spectral sequence,

$$E_2^{p,q} = H^q\Big(Q, H^p(\mathbb{Z}_5 \times \mathbb{Z}_5, Z)\Big) \quad \Rightarrow \quad H^{p+q}_{\mathbb{Z}_5 \times \mathbb{Z}_5}(Q, \mathbb{Z}) = H^{p+q}(X, \mathbb{Z}). \tag{19}$$

For our purposes, the important part is the nonvanishing [14, 15, 16, 17] differential d_3 in the tableau

$$E_2^{p,q} = \begin{array}{c|ccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
q{=}2 & \mathbb{Z} & 0 & \mathbb{Z}_5^2 & \mathbb{Z}_5 & \mathbb{Z}_5^3 & \mathbb{Z}_5^2 & \cdots \\
q{=}1 & 0 & 0 \;\; d_3 & 0 & 0 & 0 & 0 & \cdots \\
q{=}0 & \mathbb{Z} & 0 & \mathbb{Z}_5^2 & \mathbb{Z}_5 & \mathbb{Z}_5^3 & \mathbb{Z}_5^2 & \cdots \\
\hline
 & p{=}0 & p{=}1 & p{=}2 & p{=}3 & p{=}4 & p{=}5 & \cdots
\end{array} . \tag{20}$$

Therefore, only the multiples of $5 \in H^2(Q, \mathbb{Z})$ survive to the E_3 tableau[5]. In particular, we obtain

$$H^2\big(X, \mathbb{Z}\big) = \ker(d_3) \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5 \;\simeq\; \mathbb{Z} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5 \tag{21}$$

[5] And, since the Leray-Serre spectral sequence is a first quadrant spectral sequence, $E_3^{0,2} = E_\infty^{0,2}$.

on the quotient[6]. Another way of expressing this factor of 5 is the following. Consider the quotient map $q : Q \to X$, and pull back the cohomology groups. Then

$$q^* : H^2\big(X, \mathbb{Z}\big) \longrightarrow H^2\big(Q, \mathbb{Z}\big), \quad (f, t_1, t_2) \mapsto 5f \tag{22}$$

is multiplication by 5.

Now we apply the usual tautology

$$\begin{tikzcd} \boxed{\text{Equivariant bundles on } Q} \arrow[rr, bend left, "\text{quotient}"] & & \boxed{\text{Bundles on } X} \arrow[ll, bend left, "\text{pullback } q^*"] \end{tikzcd}$$

that equivariant bundles on Q are the same as bundles on the quotient X. The Chern classes are natural, that is, any bundle W on X satisfies $c_i\big(q^*W\big) = q^*\big(c_i(W)\big)$. Therefore, a necessary condition for a bundle V on Q to have a $\mathbb{Z}_5 \times \mathbb{Z}_5$-equivariant structure is that its Chern classes are in the image of q, that is,

$$V \in \mathrm{Vect}_{\mathbb{Z}_5\times\mathbb{Z}_5}(Q) \quad \Rightarrow \quad c_i(V) \in \mathrm{img}(q^*). \tag{23}$$

In particular, $c_1(V) \in 5\mathbb{Z} \subset H^2(Q, \mathbb{Z})$ is a necessary condition for a bundle on Q to be $\mathbb{Z}_5 \times \mathbb{Z}_5$-equivariant. This restriction on the first Chern class already forbids equivariant structures on most of the B, C bundles in Table 1.

In fact, next to the monad bundle that I am investigating in this paper, only the monad with $B = 3\mathcal{O}(2) \oplus 4\mathcal{O}(1)$ and $C = \mathcal{O}(4) \oplus 2\mathcal{O}(3)$ seems to be allowed. However, the latter is excluded by its second Chern classes. For completeness, the expressions for the integrally normalized Chern classes and the pullback $q^* : H^{\mathrm{ev}}(X) \to H^{\mathrm{ev}}(Q)$ are in Appendix A.

3.5 Heterotic Anomaly Cancellation

Of course, the heterotic anomaly cancellation condition

$$c_2(X) - c_2(W) - c_2(W_{\mathrm{hidden}}) = PD(C) \quad \in H^4\big(X, \mathbb{Z}\big) \tag{24}$$

has to be satisfied on the quotient manifold $X = Q/(\mathbb{Z}_5 \times \mathbb{Z}_5)$ and bundle $W = V/\gamma$, where $PD(C)$ is the Poincaré-dual of the curve wrapped by five-branes.

A necessary but not sufficient criterion for the anomaly cancellation is that the image of both sides under the pull-back is the same,

$$c_2(Q) - c_2(V) - c_2(V_{\mathrm{hidden}}) = q^* PD(C) \quad \in H^4\big(Q, \mathbb{Z}\big). \tag{25}$$

[6]That is, the line bundles on the quotient are classified by the de Rham part of the first Chern class $\frac{1}{2\pi}[F]$ together with two $\mathbb{Z}_5$ phases for the discrete Wilson lines.

However, on the quintic we are in the favorable circumstance that

$$q^* : \underbrace{H^4(X,\mathbb{Z})}_{\simeq\mathbb{Z}} \to \underbrace{H^4(Q,\mathbb{Z})}_{\simeq\mathbb{Z}}, \quad n \mapsto 25n \tag{26}$$

is injective[7]. Hence, the anomaly cancellation condition on the covering space and on the quotient are equivalent. In particular, the monad bundle eq. (11) satisfies $c_2(V) = c_2(Q)$, and therefore cancels the heterotic anomaly on the cover Q as well as on the quotient X without hidden bundle or branes.

3.6 Representation Theory

In order to better understand the Schur cover

$$H_5 = \left\langle g_1, g_2, g_3 \middle| g_1^5 = g_2^5 = g_3^5 = 1,\ g_2 g_1 g_2^{-1} g_1^{-1} = g_3,\ g_1 g_3 = g_3 g_1,\ g_2 g_3 = g_3 g_2 \right\rangle \tag{27}$$

let us quickly discuss its representation theory [18]. First of all, its Abelianisation is $\mathbb{Z}_5 \times \mathbb{Z}_5$. Therefore, each 1-dimensional representation has to factor through $\mathbb{Z}_5 \times \mathbb{Z}_5$, see eq. (17), and H_5 has 25 one-dimensional representations $r_1^i r_2^j$, $0 \geq i, j < 5$. In addition, there are four[8] useful to take irreducible representations $R_1, \ldots, R_4$ of dimension 5, distinguished by the weight

$$R_i(g_3) = \zeta^i \operatorname{diag}(1,1,1,1,1). \tag{28}$$

For example, we defined the representation carried by the 5 homogeneous variables of $\mathbb{P}^4$ to be R_1, see eq. (1). Together, these are all irreducible representations:

$$\sum_{i,j=0}^{4} \dim\left(r_1^i r_2^j\right)^2 + \sum_{i=1}^{4} \dim\left(R_i\right)^2 = 125 = |H_5|. \tag{29}$$

The tensor products of the irreducible representations can be summarized in the representation ring

$$\begin{split} R(H_5) = \mathbb{Z}\big[r_1, r_2, R_1, R_2, R_3, R_4\big] \Big/ \Big\langle r_i^5 = 1, \quad r_i R_j = R_j, \\ R_a R_{5-a} = \sum r_1^i r_2^j, \quad R_a R_b = 5R_{a+b} \text{ if } a+b \neq 0 \bmod 5 \Big\rangle. \end{split} \tag{30}$$

One observes that

[7] Modulo torsion (the finite part in H^4), the pull-back is always injective. However, it is important to cancel it in integral cohomology on the quotient manifold as there is a danger of a discrete anomaly. Moreover, the existence of holomorphic curves can depend on the torsion part of their homology class [15, 16, 17].

[8] In the following, we will use the notation where these four representations are indexed by $\mathbb{Z}$ mod 5, with 0 being disallowed. In other words, $R_{i+5} \overset{\text{def}}{=} R_i$.

- Take one of the genuinely H_5 representations R_i.
- Tensor with R_{5-i}.
- The result is a $\mathbb{Z}_5 \times \mathbb{Z}_5$ representation, that is, $R_i R_{5-i}(g_3) = \mathrm{id}$.

This will be important in the following to construct $\mathbb{Z}_5 \times \mathbb{Z}_5$-equivariant rank-5 vector bundles.

Finally, since we will be interested in polynomials, we will need the symmetric powers of R_1. They are

$$\mathrm{Sym}^k(R_1) = \begin{cases} \frac{1}{5}\binom{4+k}{k} R_k & k \neq 0 \bmod 5, \\ 1 + \frac{1}{25}\left[\binom{4+k}{k} - 1\right] \sum_{i,j} r_1^i r_2^j & k = 0 \bmod 5. \end{cases} \tag{31}$$

3.7 Equivariance

Every representation $\rho \in R(H_5)$ of dimension $r = \dim(\rho)$ defines an H_5-equivariant vector bundle on Q by taking the trivial vector bundle $r\mathcal{O}_Q$ with its global sections $\vec{s} = (s_\alpha)$, $\alpha \in \{0, \dots, r-1\}$, and defining an equivariant structure

$$\gamma(g)(\vec{s}) = \rho(g)\vec{s}. \tag{32}$$

By abuse of notation, I will denote the corresponding vector bundle by ρ as well. In particular, taking $\rho = R_{5-n}$ and tensoring with a line bundle defines the rank 5 vector bundles

$$\Phi(n) \stackrel{\text{def}}{=} R_{5-n} \otimes \mathcal{O}_Q(n), \qquad n \neq 0 \bmod 5. \tag{33}$$

Previously, in Subsection 3.3, I defined a H_5-equivariant structure on $\mathcal{O}_Q(1)$ and, hence, on $\mathcal{O}_Q(n) = \mathcal{O}_Q(1)^{\otimes n}$. Therefore, $\Phi(n)$ is H_5-equivariant as the tensor product of equivariant bundles. Moreover, g_3 acts trivially, and $\Phi(n)$ is actually a $\mathbb{Z}_5 \times \mathbb{Z}_5$-equivariant vector bundle. Topologically $\Phi(n) = 5\mathcal{O}_Q(n)$ is decomposable, but as an equivariant bundle it is not.

Let us take a closer look at this definition. A basis for $H^0\big(Q, \Phi(n)\big)$ is the same as for $5\mathcal{O}_Q(n)$, namely the $5 \cdot \binom{4+n}{n}$ sections

$$s_{\alpha,(i_1,\dots,i_n)} \stackrel{\text{def}}{=} \begin{array}{cl} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ z_{i_1} z_{i_2} \cdots z_{i_n} \\ 0 \\ \vdots \\ 0 \end{pmatrix} & \begin{array}{l} \\ \\ \\ \longleftarrow \text{row } \alpha \\ \\ \\ \\ \end{array} \end{array}, \qquad 0 \leq \alpha, i_1, \dots, i_n < 5. \tag{34}$$

The generators g_1, g_2 act in an obvious way on the homogeneous coordinates $z_0, \ldots, z_4$, this is the H_5 action on $\mathcal{O}_Q(1)$. The action on the sections of $\Phi(n)$ is slightly different, and a convenient basis choice is

$$\begin{aligned} g_1\big(s_{\alpha,(i_1,\ \dots,\ i_n)}\big) &\stackrel{\text{def}}{=} s_{\alpha-n,(i_1+1,\ \dots,\ i_n+1)}, \\ g_2\big(s_{\alpha,(i_1,\ \dots,\ i_n)}\big) &\stackrel{\text{def}}{=} \zeta^{\alpha+i_1+\cdots+i_n} s_{\alpha,(i_1,\ \dots,\ i_n)}. \end{aligned} \tag{35}$$

One can easily check that

$$g_1 \circ g_2\big(s_{\alpha,(i_1,\ \dots,\ i_n)}\big) = \zeta^{\alpha+i_1+\cdots+i_n} s_{\alpha-n,(i_1+1,\ \dots,\ i_n+1)} = g_2 \circ g_1\big(s_{\alpha,(i_1,\ \dots,\ i_n)}\big). \tag{36}$$

Therefore, eq. (35) defines a $\mathbb{Z}_5 \times \mathbb{Z}_5$-equivariant structure on $\Phi(n)$.

To summarize, we have now defined equivariant structures on the entries of the monad under consideration. They are

$$B = \Phi(1) \oplus \Phi(2), \quad C = \Phi(3). \tag{37}$$

If we can find a equivariant morphism from B to C, then we have defined an equivariant monad.

3.8 Morphisms

A morphism $B \to C$ is simply given by a $\text{rank}(C) \times \text{rank}(B)$ matrix of polynomials such that the (j,i) entry is of degree $c_j - b_i$. Keeping track of the equivariant structure on the bundles $\Phi(n)$ and assuming[9] $m - n \bmod 5 \neq 0$, one finds

$$\begin{aligned} \operatorname{Hom}\big(\Phi(n), \Phi(m)\big) &= \operatorname{Hom}\Big(R_{5-n}\mathcal{O}_Q(n), R_{5-m}\mathcal{O}_Q(m)\Big) \\ &= R^\vee_{5-n} R_{5-m} \operatorname{Sym}^{m-n}(R_1) = \binom{4+m-n}{m-n} \sum_{i,j=0}^{4} r_1^i r_2^j. \end{aligned} \tag{38}$$

Let me describe the invariant homomorphisms in more detail. First, note the obvious basis

$$\operatorname{Hom}\big(\Phi(n), \Phi(n+k)\big) = \big\langle f^\alpha_{\beta,(i_1,\dots,i_k)} \big\rangle,$$

$$f^\alpha_{\beta,(i_1,\dots,i_k)} \stackrel{\text{def}}{=} (\delta^\alpha_\beta)\, z_{i_1}\cdots z_{i_k} = \begin{array}{cc} \begin{matrix} \text{column } \alpha \\ \downarrow \end{matrix} & \\ \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & & z_{i_1}\cdots z_{i_k} & & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} & \begin{matrix} \\ \\ \longleftarrow \text{ row } \beta, \\ \\ \\ \end{matrix} \end{array} \tag{39}$$

[9] I will allways assume this in the following. For the purposes of this paper, only the case where $1 \geq n, m \geq 3$ is relevant.

where (δ^α_β) is the 5×5-matrix with a single non-zero entry $= 1$ and the z_i are again the homogeneous coordinates of $\mathbb{P}^4$. The action on a section of $\Phi(n)$ is simply matrix-vector multiplication, that is,

$$f^\alpha_{\beta,(i_{n+1},\dots,i_{n+k})}\big(s_{\epsilon,(i_1,\dots,i_n)}\big) = \delta^\alpha_\epsilon s_{\beta,(i_1,\dots,i_{n+k})}. \tag{40}$$

Finally, the g_1, g_2 action on the space of maps is the usual action on the homogeneous coordinates combined with a matrix action to correctly intertwine between $\Phi(n)$ and $\Phi(n+k)$, see eq. (12). Explicitly, the action is

$$\begin{aligned} g_1\big(f^\alpha_{\beta,(i_1,\dots,i_k)}\big) &= f^{\alpha-n}_{\beta-n-k,(i_1+1,\dots,i_k+1)}, \\ g_2\big(f^\alpha_{\beta,(i_1,\dots,i_k)}\big) &= \zeta^{\beta-\alpha+i_1+\cdots+i_k} f^\alpha_{\beta,(i_1,\dots,i_k)}. \end{aligned} \tag{41}$$

The g_1 and g_2 actions commute,

$$g_2 g_1\big(f^\alpha_{\beta,(i_1,\dots,i_k)}\big) = \zeta^{\beta-\alpha+i_1+\cdots+i_k} f^{\alpha-n}_{\beta-n-k,(i_1+1,\dots,i_k+1)} = g_1 g_2\big(f^\alpha_{\beta,(i_1,\dots,i_k)}\big), \tag{42}$$

and therefore decompose into $\mathbb{Z}_5 \times \mathbb{Z}_5$-representations as we argued above. For example, the 5-dimensional space of invariant homomorphisms from $\Phi(2)$ to $\Phi(3)$ is spanned by

$$\begin{pmatrix} z_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & z_3 \\ 0 & 0 & 0 & z_1 & 0 \\ 0 & 0 & z_4 & 0 & 0 \\ 0 & z_2 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & z_1 & 0 & 0 & 0 \\ z_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & z_2 \\ 0 & 0 & 0 & z_0 & 0 \\ 0 & 0 & z_3 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & z_2 & 0 & 0 \\ 0 & z_0 & 0 & 0 & 0 \\ z_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & z_1 \\ 0 & 0 & 0 & z_4 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & z_3 & 0 \\ 0 & 0 & z_1 & 0 & 0 \\ 0 & z_4 & 0 & 0 & 0 \\ z_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & z_0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & z_4 \\ 0 & 0 & 0 & z_2 & 0 \\ 0 & 0 & z_0 & 0 & 0 \\ 0 & z_3 & 0 & 0 & 0 \\ z_1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{43}$$

We now have all ingredients to define a $\mathbb{Z}_5 \times \mathbb{Z}_5$-equivariant monad bundle on the quintic. For explicitness, I will from now on take the quintic to be the Fermat quintic

$$Q = z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5. \tag{44}$$

Let the $\mathbb{Z}_5 \times \mathbb{Z}_5$-equivariant rank-5 vector bundle be the kernel of the positive monad

$$0 \longrightarrow V \longrightarrow \Phi(1) \oplus \Phi(2) \xrightarrow{f} \Phi(3) \longrightarrow 0 \tag{45}$$

with an invariant map given by the polynomial matrix

$$f \stackrel{\text{def}}{=} \begin{pmatrix} 0 & z_3^2 & 0 & 0 & z_2^2 & 0 & z_1 & 0 & z_3 & z_4 \\ 0 & z_0^2 & 0 & z_1^2 & 0 & z_4 & 0 & z_1 & z_2 & 0 \\ z_4^2 & 0 & 0 & z_3^2 & 0 & 0 & z_4 & z_0 & 0 & z_2 \\ z_1^2 & 0 & z_2^2 & 0 & 0 & z_2 & z_3 & 0 & z_0 & 0 \\ 0 & 0 & z_4^2 & 0 & z_0^2 & z_1 & 0 & z_3 & 0 & z_0 \end{pmatrix}. \tag{46}$$

As we will see soon, the map f has been chosen generic enough so that the monad is, indeed, a vector bundle. Yet it is special enough so that one $\underline{\mathbf{5}}$–$\underline{\overline{\mathbf{5}}}$ pair survives[10].

[10] The observation of [4, 5, 6] that a completely generic map leads to no vector-like pairs whatsoever remains true.

4 Particle Spectrum

4.1 Machinery

Massless fields in 10 dimensions give rise to massless fields in the 4-dimensional effective action if their dependence on the Calabi-Yau coordinates is a zero mode of the Dirac operator. This boils down to a question about harmonic forms on the vector bundle V and its exterior powers $\wedge^i V$, $1 \geq i \geq 4$. By changing our model for the cohomology to sheaf cohomology, this becomes a tractable computation. In particular, we can use the defining monad to convert $\wedge^i V$ into something that depends only on B and C and the map in-between. But since B and C are only sums of line bundles, all cohomology groups are now given explicitly as vector spaces spanned by polynomials.

Explicitly, we first replace $\wedge^i V$ by equivalent objects in the derived category,[11]

$$\begin{aligned} V &= \Big[0 \longrightarrow \underline{B} \xrightarrow{f} C \longrightarrow 0\Big], \\ \wedge^2 V &= \Big[0 \longrightarrow \underline{\wedge^2 B} \xrightarrow{b_1\wedge b_2 \mapsto [b_1\otimes f(b_2)]} B\otimes C \xrightarrow{(b,c)\mapsto \{f(b)\otimes c\}} \operatorname{Sym}^2 C \longrightarrow 0\Big]. \end{aligned} \tag{47}$$

The remaining exterior powers are just duals, namely

$$\wedge^3 V = \wedge^2 V^\vee, \quad \wedge^4 V = V^\vee, \quad \wedge^5 V = \det V = \mathcal{O}_Q, \tag{48}$$

and their cohomology can be determined via Serre duality. As a necessary evil we have to deal with cohomology for complexes, the so-called hypercohomology. Note that there are two ways to "take cohomology" here: There is the cohomology of a complex $\mathcal{K}^\bullet$, and the cohomology of the objects in the complex. Here, we only need the case where the cohomology of the complex is a vector bundle (or sheaf) V located at a single position,

$$V = \ker(\mathcal{K}^0 \to \mathcal{K}^1)\Big/ \operatorname{img}(\mathcal{K}^{-1}), \qquad \ker(\mathcal{K}^p \to \mathcal{K}^{p+1}) = \operatorname{img}(\mathcal{K}^{p-1}) \text{ if } p \neq 0. \tag{49}$$

In this case, the hypercohomology $H(\mathcal{K}^\bullet)$ is simply the cohomology of V. The other way of taking cohomologies gives rise to the hypercohomology spectral sequence

$$E_1^{p,q} = H^q\big(Q, \mathcal{K}^p\big) \quad \Rightarrow \quad H^{p+q}(V) \tag{50}$$

Thanks to Kodaira vanishing, only the $q=0$ row will be nonzero. Moreover, the first and only non-vanishing differential $d_1 : E_1^{p,0} \to E_1^{p+1,0}$ is just multiplication by $H(f)$ induced from $f : B \to C$ with suitable (anti-)symmetrization.

To summarize, using the defining map f, see eq. (46), defines maps between spaces of polynomials

$$\begin{aligned} H^0\big(Q, B\big) &\xrightarrow{H(f)} H^0\big(Q, C\big), \\ H^0\big(Q, \wedge^2 B\big) \xrightarrow{F_B} H^0\big(Q, B\otimes C\big) &\xrightarrow{F_C} H^0\big(Q, \operatorname{Sym}^2 C\big). \end{aligned} \tag{51}$$

[11] The underlined entry marks the zero position of the complex.

Here, $H(f)$ is tautologically the same matrix as in eq. (46). The polynomial matrices[12] F_B and F_C are of dimension 50×45 and 15×50, respectively, and are explicitly constructed by the script in Appendix B. Taking the cohomology, one finds

$$H^i\big(Q, V\big) = \begin{cases} 0 \\ 0 \\ \text{coker}\, H(f) \\ \ker H(f) \end{cases} \quad , \quad H^i\big(Q, \wedge^2 V\big) = \begin{cases} 0 & i = 3 \\ \text{coker}\, F_C & i = 2 \\ \ker F_C \big/ \operatorname{img} F_B & i = 1 \\ \ker F_B & i = 0 \end{cases} . \tag{52}$$

Finally, using Serre duality,

$$H^i\big(Q, \wedge^k V\big) = H^{3-i}\big(Q, \wedge^{5-k} V\big), \tag{53}$$

we determined all cohomology groups of exterior powers of V. The relevant computations with polynomials can be easily done with SINGULAR [19] and are recorded in Appendix B.

4.2 Slope Stability

Since the quintic has only a one-dimensional $H^2(Q, \mathbb{Z})$ we are in the lucky case where we can apply Hoppe's criterion [20]. Specifically, we have

- The Fermat quintic Q is a smooth manifold with dim $H^2(Q, \mathbb{Z}) = 1$.
- The monad defines a *vector bundle* V; This requires a short computation that Q and the 5×5 minors of f do not vanish simultaneously.
- The first Chern class of V vanishes by construction.
- Finally, $H^0(Q, \wedge^i V)$ must vanish for $1 \geq i \geq 4$. The potential contributions are $\ker H(f)$ and $\ker F_B$, and a short computation shows that they indeed vanish, see again Appendix B. This can also be argued more generally using the Koszul resolution [5, 6].

Hoppe's criterion then guarantees that the bundle V is slope-stable and, therefore, admits a Hermitian Yang-Mills connection.

4.3 Light Matter

Using the well-known embedding of $SU(5) \times SU(5) \subset E_8$, the matter spectrum of the $E_8 \times E_8$ heterotic string compactified on a slope-stable vector bundle over a Calabi-Yau

[12] Of course they satisfy $F_C F_B = 0$, as required for the d_1 differentials in the Hypercohomology spectral sequence.

manifold is determined by the multiplicities[13]

$$\begin{gathered} n_{\underline{\mathbf{10}}} = h^1(V), \quad n_{\overline{\underline{\mathbf{10}}}} = h^1(V^\vee) = h^2(V), \\ n_{\underline{\mathbf{5}}} = h^1(\wedge^2 V^\vee) = h^2(\wedge^2 V), \quad n_{\overline{\underline{\mathbf{5}}}} = h^1(\wedge^2 V) = n_{\underline{\mathbf{10}}} - n_{\overline{\underline{\mathbf{10}}}} + n_{\underline{\mathbf{5}}}. \end{gathered} \tag{54}$$

We already noted that $h^0(V)$ had to vanish for slope-stability. Therefore, index theory determines the only non-vanishing cohomology group $h^1(V) = 75$. Moreover, the $\mathbb{Z}_5 \times \mathbb{Z}_5$-action is uniquely determined by the corresponding character-valued index, and one obtains

$$H^1\big(Q, V\big) = 3 \sum_{i,j=0}^{4} r_1^i r_2^j. \tag{55}$$

Next, consider $H^2(Q, \wedge^2 V) = \operatorname{coker} F_C$. We can think of it as symmetric tensors in $H^0(Q, C \otimes C)$ with the basis

$$\Big\{ \mathbb{C}\big[z_0, z_1, z_2, z_3, z_4\big]_6 \Big/ \langle Q = 0 \rangle \Big\} \otimes \big\{ \vec{e}_{(\alpha,\beta)} \big|\; 0 \geq \alpha \geq \beta \geq 4 \big\} \tag{56}$$

The $\mathbb{Z}_5 \times \mathbb{Z}_5$-group action can easily be identified as the one coming from

$$\Gamma\Phi(3) \times \Gamma\Phi(3) \longrightarrow H^0\big(Q, \operatorname{Sym}^2 C\big), \quad \big(s_{\alpha,(i_1,i_2,i_3)}, s_{\beta,(i_4,i_5,i_6)}\big) \mapsto e_{(\alpha,\beta)} \prod_{k=1}^{6} z_{i_k}. \tag{57}$$

When asked nicely, Singular can compute a basis for the cokernel, see Appendix B. One obtains

$$\operatorname{coker} F_C = \big\langle z_4^6 \vec{e}_{(1,4)},\; z_0^3 z_2 z_4^2 \vec{e}_{(0,0)},\; z_3^6 \vec{e}_{(1,1)},\; z_1 z_3^5 \vec{e}_{(2,2)},\; z_4^6 \vec{e}_{(3,3)},\; z_2 z_4^5 \vec{e}_{(4,4)} \big\rangle. \tag{58}$$

In general, the $\mathbb{Z}_5 \times \mathbb{Z}_5$-action will map the representatives to different representatives of the same quotient space. These must be projected back onto the chosen representatives to read off the group action. Decomposing this 6-dimensional $\mathbb{Z}_5 \times \mathbb{Z}_5$-representation into irreducible representations yields

$$H^1\big(Q, \wedge^2 V\big) = H^2\big(Q, \wedge^2 V\big) = r_2^4 + 1 + r_1 + r_1^2 + r_1^3 + r_1^4. \tag{59}$$

To summarize, we defined an equivariant bundle (V, γ) on Q and computed the $\mathbb{Z}_5 \times \mathbb{Z}_5$-action on its cohomology. Now, finally, we extract the invariants to obtain the cohomology of the quotient bundle $W = V/\gamma$ on the quotient space $X = Q/(\mathbb{Z}_5 \times \mathbb{Z}_5)$,

$$H^i\big(X, W\big) = \begin{cases} 0 \\ 0 \\ 3 \\ 0 \end{cases}, \quad H^i\big(X, \wedge^2 W\big) = \begin{cases} 0 & i = 3 \\ 1 & i = 2 \\ 1 + 3 & i = 1 \\ 0 & i = 0 \end{cases}. \tag{60}$$

[13]The last equation follows from $\operatorname{Ind}(V) = \operatorname{Ind}(\wedge^2 V)$ for a rank-5 bundle or anomaly cancellation in the low-energy action.

Therefore, the ensuing $SU(5)$ GUT has a matter spectrum of three $\underline{\mathbf{10}} + \underline{\overline{\mathbf{5}}}$ and a single vector-like pair $\underline{\mathbf{5}}$–$\underline{\overline{\mathbf{5}}}$. Perturbing the bundle moduli ϕ by changing the map eq. (46) removes this vector-like pair, so there must be a μ-term $\sim \phi\psi_{\underline{\mathbf{5}}}\psi_{\underline{\overline{\mathbf{5}}}}$ in the superpotential.

A Chern classes on the Quintic and Quotient

The even-degree cohomology of the quintic is one-dimensional in degrees 0 to 6, but the integral normalization of the cup product still provides some interesting structure. If we denote the (positive) generator of $H^2(Q,\mathbb{Z})$ by J, then

$$H^{\mathrm{ev}}(Q,\mathbb{Z}) = \mathbb{Z}\big[J, \tfrac{1}{5}J^2\big] = \mathbb{Z} \ \oplus \ \mathbb{Z}\cdot J \ \oplus \ \mathbb{Z}\cdot\big(\tfrac{1}{5}J^2\big) \ \oplus \ \mathbb{Z}\cdot\big(\tfrac{1}{5}J^3\big)\,. \tag{61}$$

In other words, the square $J^2 \in H^4(J,\mathbb{Z})$ can be divided by 5 in the integral cohomology. Therefore, the integrally normalized Chern classes of a positive monad, see eq. (9), with $c_1(V) = 0$ are

$$\begin{aligned}
\mathrm{rank}(B) &= n, & \mathrm{rank}(V) &= n-m, \\
c_1(B) &= \sum_{i=1}^{n} b_i, & c_1(V) &= \sum_{i=1}^{n} b_i - \sum_{j=1}^{m} c_j \stackrel{!}{=} 0, \\
c_2(B) &= 5\sum_{i<j} b_i b_j, & c_2(V) &= -\frac{5}{2}\left(\sum_{i=1}^{n} b_i^2 - \sum_{j=1}^{m} c_j^2\right), \\
c_3(B) &= 5\sum_{i<j<k} b_i b_j b_k, & c_3(V) &= \frac{5}{3}\left(\sum_{i=1}^{n} b_i^3 - \sum_{j=1}^{m} c_j^3\right).
\end{aligned} \tag{62}$$

The pull-back by the $\mathbb{Z}_5 \times \mathbb{Z}_5$ quotient map $q : Q \to X$ is [16]

$$\begin{array}{llr}
q^* : & \mathbb{Z} = H^0(X,\mathbb{Z}) \to H^0(Q,\mathbb{Z}) = \mathbb{Z}, & n \mapsto n \\
q^* : & \mathbb{Z}\oplus\mathbb{Z}_5\oplus\mathbb{Z}_5 = H^2(X,\mathbb{Z}) \to H^2(Q,\mathbb{Z}) = \mathbb{Z}, & (n,\psi_1,\psi_2) \mapsto 5n \\
q^* : & \mathbb{Z} = H^4(X,\mathbb{Z}) \to H^4(Q,\mathbb{Z}) = \mathbb{Z}, & n \mapsto 25n \\
q^* : & \mathbb{Z} = H^6(X,\mathbb{Z}) \to H^6(Q,\mathbb{Z}) = \mathbb{Z}, & n \mapsto 25n
\end{array} \tag{63}$$

Therefore, a necessary condition for B, C to posses a $\mathbb{Z}_5 \times \mathbb{Z}_5$-equivariant structure is that their first Chern class is divisible by 5 and that their second and third Chern classes are divisible by 25.

B Singular

```
LIB "random.lib";
LIB "matrix.lib";
```

```
LIB "solve.lib";

ring r = 0,(u,v,x,y,z),dp;
poly Q = u^5+v^5+x^5+y^5+z^5;

matrix f0[5][5]=u,0,0,0,0 ,0,0,0,0,y ,0,0,0,v,0 ,0,0,z,0,0 ,0,x,0,0,0;
matrix f1[5][5]=0,v,0,0,0 ,z,0,0,0,0 ,0,0,0,0,x ,0,0,0,u,0 ,0,0,y,0,0;
matrix f2[5][5]=0,0,x,0,0 ,0,u,0,0,0 ,y,0,0,0,0 ,0,0,0,0,v ,0,0,0,z,0;
matrix f3[5][5]=0,0,0,y,0 ,0,0,v,0,0 ,0,z,0,0,0 ,x,0,0,0,0 ,0,0,0,0,u;
matrix f4[5][5]=0,0,0,0,z ,0,0,0,x,0 ,0,0,u,0,0 ,0,y,0,0,0 ,v,0,0,0,0;

matrix f00[5][5]=u^2,0,0,0,0,0,0,y^2,0,0,0,0,0,0,v^2,0,z^2,0,0,0,0,0,0,x^2,0;
matrix f01[5][5]=0,u*v,0,0,0,0,0,0,y*z,0,v*x,0,0,0,0,0,0,z*u,0,0,0,0,0,0,x*y;
matrix f02[5][5]=0,0,u*x,0,0,0,0,0,0,y*u,0,v*y,0,0,0,0,0,0,z*v,0,x*z,0,0,0,0;
matrix f03[5][5]=0,0,0,y*u,0,v*y,0,0,0,0,0,0,z*v,0,0,0,0,0,0,x*z,0,u*x,0,0,0;
matrix f04[5][5]=0,0,0,0,z*u,0,x*y,0,0,0,0,0,0,u*v,0,y*z,0,0,0,0,0,0,v*x,0,0;
matrix f11[5][5]=0,0,v^2,0,0,0,0,0,0,z^2,0,x^2,0,0,0,0,0,0,u^2,0,y^2,0,0,0,0;
matrix f12[5][5]=0,0,0,v*x,0,z*u,0,0,0,0,0,0,x*y,0,0,0,0,0,0,u*v,0,y*z,0,0,0;
matrix f13[5][5]=0,0,0,0,v*y,0,z*v,0,0,0,0,0,0,x*z,0,u*x,0,0,0,0,0,0,y*u,0,0;
matrix f14[5][5]=z*v,0,0,0,0,0,0,x*z,0,0,0,0,0,0,u*x,0,y*u,0,0,0,0,0,0,v*y,0;
matrix f22[5][5]=0,0,0,0,x^2,0,u^2,0,0,0,0,0,0,y^2,0,v^2,0,0,0,0,0,0,z^2,0,0;
matrix f23[5][5]=x*y,0,0,0,0,0,0,u*v,0,0,0,0,0,0,y*z,0,v*x,0,0,0,0,0,0,z*u,0;
matrix f24[5][5]=0,x*z,0,0,0,0,0,0,u*x,0,y*u,0,0,0,0,0,0,v*y,0,0,0,0,0,0,z*v;
matrix f33[5][5]=0,y^2,0,0,0,0,0,0,v^2,0,z^2,0,0,0,0,0,0,x^2,0,0,0,0,0,0,u^2;
matrix f34[5][5]=0,0,y*z,0,0,0,0,0,0,v*x,0,z*u,0,0,0,0,0,0,x*y,0,u*v,0,0,0,0;
matrix f44[5][5]=0,0,0,z^2,0,x^2,0,0,0,0,0,0,u^2,0,0,0,0,0,0,y^2,0,v^2,0,0,0;

matrix BtoC = concat( f22+f33, f4+f1+f3 );

matrix Alt2BtoBB[10*10][10*(10-1)/2]=0;
int pos=1;
for (int i=0; i<10; i++) {
    for (int j=i+1; j<10; j++) {
       Alt2BtoBB[10*i+j+1, pos] =  1;
       Alt2BtoBB[10*j+i+1, pos] = -1;
       pos++;
    }
}
matrix CCtoSym2C[5*(5+1)/2][5*5]=0;
int pos=1;
for (int i=0; i<5; i++) {
    for (int j=i; j<5; j++) {
       CCtoSym2C[pos, 5*i+j+1] = 1;
       CCtoSym2C[pos, 5*j+i+1] = 1;
       pos++;
    }
}
// BB = B tensor B is indexed by (b1,b2)
// CB = C tensor B is indexed by  (c,b2)
matrix BBtoCB[5*10][10*10]=0;
for (int b1=0; b1<10; b1++) {
  for (int b2=0; b2<10; b2++) {
    for (int c=0; c<5; c++) {
        BBtoCB[10*c+b2+1, 10*b1+b2+1] = BtoC[c+1, b1+1];
} } }
// CB = C tensor B is indexed by (c1,b)
// CC = C tensor C is indexed by (c1,c2)
matrix CBtoCC[5*5][5*10]=0;
for (int c1=0; c1<5; c1++) {
  for (int b=0; b<10; b++) {
      for (int c2=0; c2<5; c2++) {
        CBtoCC[5*c1+c2+1, 10*c1+b+1] = BtoC[c2+1, b+1];
} } }
matrix Alt2BtoCB = BBtoCB * Alt2BtoBB;
matrix CBtoSym2C = CCtoSym2C * CBtoCC;

// So far we defined polynomial matrices representing
// H^0(B) ----BtoC----> H^0(C)
// H^0(wedge^2 B) ----Alt2BtoBC----> H^0(C tensor B) ----CBtoSym2C----> H^0(Sym^2 C)

// composition must be zero
compress( CBtoSym2C * Alt2BtoCB );

// The image of BtoC lives in 5 O(3) and the quintic constraint does not matter
// The image of CBtoSym2C lives in 25 O(6), so modding out the quintic Q is important
module higgs = std(CBtoSym2C + freemodule(15)*Q);

// This computes that dim coker BtoC = 5+15+25+30 = 75
hilb(std(BtoC));

// This computes that dim coker CBtoSym2C = 6 = number of Higgs
hilb(higgs);
// Lets find the 6 representatives
kbase(higgs,6);
// Figure out the g_1 action on H^2(wedge^2 V); g_2 action is easy
```

```
reduce( u6*gen(8),     higgs);   // = g_1( z6*gen(9)    )
reduce( z6*gen(13),    higgs);   // = g_1( y6*gen(6)    )
reduce( v3yu2*gen(10), higgs);   // = g_1( u3xz2*gen(1) )
reduce( xz5*gen(15),   higgs);   // = g_1( vy5*gen(10)  )
reduce( u6*gen(1),     higgs);   // = g_1( z6*gen(13)   )
reduce( yu5*gen(6),    higgs);   // = g_1( xz5*gen(15)  )

// As a 5 x 10 matrix, BtoC has a kernel. But it requires polynomials of degree >=6
// so BtoC: H^0( Phi(1)+Phi(2) ) ---> H^0( Phi(3) ) has no kernel
module ker = std(modulo(BtoC,0*BtoC));
intvec ker_deg = 0:ncols(ker);
for (int i=1; i<=ncols(ker); i++) { ker_deg[i] = maxdeg1(ker[i]); }; ker_deg;

// similarly, Alt2BtoCB is injective
module ker = std(modulo(Alt2BtoCB,0*Alt2BtoCB));
intvec ker_deg = 0:ncols(ker);
for (int i=1; i<=ncols(ker); i++) { ker_deg[i] = maxdeg1(ker[i]); }; ker_deg;

// lets compute where the bundle is singular
ideal cym = ideal(Q);
ideal sing = cym+ideal(minor(BtoC,5,cym));
dim(std(sing)); // by homogeneity, a discrete solution set must be { u=v=x=y=z=0 }
solve(sing);    // indeed. Hence, no singularity in P^4 and V is a bundle
```

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