FREENESS THEOREMS FOR OPERADS VIA GRÖBNER BASES

VLADIMIR DOTSENKO

ABSTRACT. We show how to use Gröbner bases for operads to prove various freeness theorems: freeness of certain operads as nonsymmetric operads, freeness of an operad $\mathscr{Q}$ as a $\mathscr{P}$-module for an inclusion $\mathscr{P} \hookrightarrow \mathscr{Q}$, freeness of a suboperad. This gives new proofs of many known results of this type and helps to prove some new results.

1. INTRODUCTION

1.1. **Description of results.** Recently, many freeness theorems about operads and free algebras over various operads have been proved. An incomplete list includes the following results:

- free dendriform algebras are free as associative algebras (Loday and Ronco [12]);
- free pre-Lie algebras are free as Lie algebras (Chapoton [4] and Foissy [8]);
- free algebras with two compatible associative products are free as associative algebras (the author's result [6]);
- the nonsymmetric operads Lie and PreLie are free (Salvatore and Tauraso [15], Bergeron and Livernet [2]);
- the suboperad of the operad PreLie generated by the symmetrized pre-Lie product is free (Bergeron and Loday [3]).

In this article, we apply Gröbner bases for operads to derive several freeness theorems that imply all these results and several new ones. Our freeness theorems remind of the Magnus's Freiheitssatz from the group theory [13] and its analogues in other branches of algebra. Many of the results of this paper can be obtained by direct computations that use the Gröbner basis algorithm [7], however, we tried to replace most of computations by ideas coming from the Koszul duality theory [10].

Besides obvious applications of freeness theorems to computations of dimensions and bases for our operads, we hope that in some cases our results have further applications. In particular, results on freeness as a module are interesting from the homological algebra point of view: free modules are used to construct resolutions, so our freeness theorems may be used to ensure freeness of resolutions. We hope to address that elsewhere, together with various questions of shuffle homological algebra for non-free modules.

1

1.2. **Outline of the paper.** The paper is organized as follows. In Section 2, we briefly recall shuffle operads and Gröbner bases. In Section 3, we prove a general criterion for a symmetric operad to be free as a nonsymmetric operad, and show how this criterion applies to the cases of Lie, PreLie, and Lie^2 (first two of these were establised earlier [15, 2], the last one is new). In Section 4, we prove a criterion for a mapping of operads to be an embedding, and apply it to deduce the inclusion Mag $\hookrightarrow$ PreLie (proved by Bergeron and Loday by different methods in a forthcoming paper [3]). In Section 5, we prove a criterion of freeness as a module, and show how this criterion applies to the known cases (Lie, PreLie) [4, 8] and (As, Dend) [12], and, for a certain class of quadratic operads, to the case $(\mathscr{O}, \mathscr{O}^2)$ where $\mathscr{O}^2$ denotes the operad of (weakly) compatible $\mathscr{O}$-structures (generalizing our earlier result on compatible associative products [6]).

1.3. **Acknowledgements.** The author is grateful to Jean-Louis Loday and Bruno Vallette for an invitation to the conference "Operads 2009" where he came up with some of the ideas used in this article. He is also grateful to Paolo Salvatore and Muriel Livernet for some interesting discussions. Special thanks are due to Jean-Louis Loday for sharing an unpublished result on the symmetrized pre-Lie product.

2. Shuffle operads and Gröbner bases

All vector spaces throughout this work are defined over an arbitary field $\Bbbk$ of zero characteristic.

In this section, we give, mostly following [7], a brief outline of definitions and the most important facts. For details on symmetric operads and Koszul duality, see [10] and [14]. For more details on shuffle operads and Gröbner bases, see [7].

2.1. **Shuffle compositions.** We denote by Ord the category of nonempty finite ordered sets (with order-preserving bijections as morphisms), and by Fin — the category of nonempty finite sets (with bijections as morphisms). Also, we denote by Vect the category of vector spaces (with linear operators as morphisms; unlike the first two cases, we do not require a map to be invertible).

Definition 1. (1) A *(nonsymmetric) collection* is a functor from the category Ord to the category Vect.
   (2) A *symmetric collection* (or an $\mathbb{S}$-*module*) is a functor from the category Fin to the category Vect.
For either type of collections, we can consider the category whose objects are collections of this type (and morphisms are morphisms of the corresponding functors). The natural forgetful functor $^f \colon$ Ord $\to$ Fin, $I \mapsto I^f$ leads to a forgetful functor ^{f} from the category of symmetric collections to the category of nonsymmetric ones, $\mathscr{P}^f(I) := \mathscr{P}(I^f)$.

The following monoidal structures on our categories are important for the theory of operads.

Definition 2. Let $\mathscr{P}$ and $\mathscr{Q}$ be two nonsymmetric collections. Define their *(nonsymmetric) composition* $\mathscr{P} \circ \mathscr{Q}$ by the formula

$$(\mathscr{P} \circ \mathscr{Q})(I) := \bigoplus_k \mathscr{P}(k) \otimes \left(\bigoplus_{f \colon I \twoheadrightarrow [k]} \mathscr{Q}(f^{-1}(1)) \otimes \ldots \otimes \mathscr{Q}(f^{-1}(k)) \right),$$

where the sum is taken over all non-decreasing surjections f.

Let $\mathscr{P}$ and $\mathscr{Q}$ be two nonsymmetric collections. Define their *shuffle composition* $\mathscr{P} \circ_{sh} \mathscr{Q}$ by the formula

$$(\mathscr{P} \circ_{sh} \mathscr{Q})(I) := \bigoplus_k \mathscr{P}(k) \otimes \left(\bigoplus_{f \colon I \twoheadrightarrow [k]} \mathscr{Q}(f^{-1}(1)) \otimes \ldots \otimes \mathscr{Q}(f^{-1}(k)) \right),$$

where the sum is taken over all shuffling surjections f, that is surjections for which $\min f^{-1}(i) < \min f^{-1}(j)$ whenever $i < j$.

Let $\mathscr{P}$ and $\mathscr{Q}$ be two symmetric collections. Define their *(symmetric) composition* $\mathscr{P} \circ \mathscr{Q}$ by the formula

$$(\mathscr{P} \circ \mathscr{Q})(I) := \bigoplus_k \mathscr{P}(k) \otimes_{\Bbbk S_k} \left(\bigoplus_{f \colon I \twoheadrightarrow [k]} \mathscr{Q}(f^{-1}(1)) \otimes \ldots \otimes \mathscr{Q}(f^{-1}(k)) \right),$$

where the sum is taken over all surjections f.

Definition 3.   (1) A *nonsymmetric operad* is a monoid in the category of nonsymmetric collections with the monoidal structure given by the nonsymmetric composition.
  (2) A *shuffle operad* is a monoid in the category of nonsymmetric collections with the monoidal structure given by the shuffle composition.
  (3) A *symmetric operad* is a monoid in the category of symmetric collections with the monoidal structure given by the (symmetric) composition.

It turns out that the forgetful functor is a monoidal functor between the category of symmetric operads and the category of shuffle operads. Consequently, it turns out that to study various questions of linear algebra for operads, it is sufficient to forget the full symmetric structure because the shuffle structure already captures everything. Further in this section, the word "operad" means a shuffle operad.

2.2. **Tree monomials, divisibility, and Gröbner bases.** We use the usual way to represent operadic elements by decorated rooted trees. A tree has (internal) vertices, directed edges, and inputs (leaves). For a tree whose leaves are labelled by an ordered set, its canonical planar representative is defined as follows. In general, an embedding of a (rooted) tree in the plane is determined by an ordering of inputs for each vertex (in terms of the

planar structure, it is the ordering of inputs from the left to the right). Thus, to define a planar embedding, we should be able to compare two inputs of every vertex v. To do so, we find the minimal leaves that one can reach from v via the corresponding inputs. The input for which the minimal leaf is smaller is considered to be less than the other one. Note that this choice of a representative is essentially the same one as we already made when we identified symmetric compositions with shuffle compositions.

Let us introduce an explicit realisation of the free operad generated by a collection $\mathscr{V}$. The basis of this operad will be indexed by planar representative of trees with decorations of all vertices. First of all, the simplest possible tree is the degenerate tree (without internal vertices); it corresponds to the unit of our operad. The second simplest type of trees is given by corollas, that is trees with one vertex. We shall fix a basis $B^{\mathscr{V}}$ of $\mathscr{V}$ and decorate the vertex of each corolla with a basis element; for a corolla with n inputs, the corresponding element should belong to the basis of $\mathscr{V}(n)$. The basis for whole free operad consists of all planar representatives of trees built from these corollas (explicitly, one starts with this collection of corollas, defines compositions of trees in terms of grafting, and then considers all trees obtained from corollas by iterated shuffle compositions). We shall refer to elements of this basis as *tree monomials*.

An ordering of tree monomials of $\mathscr{F}_{\mathscr{V}}$ is said to be *admissible*, if it is compatible with the operadic structure, that is, replacing the operations in any shuffle compositions with larger operations of the same arities increases the result of the composition. Here we shall describe several admissible orderings which suit our purposes. All results of this section are valid for every admissible ordering of tree monomials.

Recall the following construction crucial for the "path-lexicographic ordering" [7]. Let α be a tree monomial with n inputs. We associate to α a sequence $(a_1, a_2, \ldots, a_n)$ of n words in the alphabet $B^{\mathscr{V}}$ and a permutation $g \in S_n$ as follows. For each leaf i of the underlying tree τ, there exists a unique path from the root to i. The word a_i is the word composed, from left to right, of the labels of the vertices of this path, starting from the root vertex. The permutation g lists the labels of leaves of the underlying tree in the order determined by the planar structure (from left to right).

To compare two tree monomials we always compare their arities first. If the arities are equal, there are several different options of how to proceed. Recall that an ordering of words in the alphabet $B^{\mathscr{V}}$ is said to be admissible, if it is compatible with the semigroup structure on words (increasing factors increases the product). Fix some admissible ordering of words: that can be the lexicographic ordering, or its reverse, or the degree-lexicographic ordering (compare lengths first, and if they are equal, then compare lexicographically, or its reverse, or something else of the same sort). Now, sequences of words can be compared lexicographically: first compare the first words in both sequences, if they are equal, compare the second words

etc. Permutations may be compared in the lexicographic order or its reverse. Also, the result depends on what we compare first, the permutations or the sequences of words. This gives rise to many candidates for an ordering. Generalizing [7, 11], one can prove

Proposition 1. *For each admissible ordering of words, all the orderings described above are admissible.*

For a tree monomial α with the underlying tree T and a subtree T' of T, let us define a tree monomial α' that corresponds to T'. Its vertices are already decorated, so we just need to take care of the leaf labelling. For each leaf l of T', let us consider the smallest leaf of T that can be reached from l. We then number the leaves according to these "smallest descendants": the leaf with the smallest possible descendant gets the label 1, the second smallest — the label 2 etc.

Definition 4. For two tree monomials α, β in the free operad $\mathscr{F}_{\mathscr{V}}$, we say that α *is divisible by* β, if there exists a subtree of the underlying tree of α for which the corresponding tree monomial α' is equal to β.

Definition 5. For an element λ of the free operad, the tree monomial α is said to be its *leading term*, if it is the largest of the terms which occur (with a nonzero coefficient) in the expansion of λ as a linear combination of tree monomials.

Here and below we assume that $\mathscr{M}$ is an operadic ideal of $\mathscr{F}_{\mathscr{V}}$, and $\mathscr{G}$ is a system of generators of $\mathscr{M}$.

Definition 6. $\mathscr{G}$ is called a *Gröbner basis* of $\mathscr{M}$, if for every $f \in \mathscr{M}$ the leading term of f is divisible by the leading term of some element of $\mathscr{G}$. The element $f \in \mathscr{F}_{\mathscr{V}}$ is said to have *the residue $\overline{f}$ modulo $\mathscr{G}$*, if $f - \overline{f} \in \mathscr{M}$, and $\overline{f}$ is a linear combination of tree monomials that are not divisible by leading terms of elements of $\mathscr{G}$ (*normal tree monomials*). Notation: $f \equiv \overline{f} \pmod{\mathscr{G}}$.

For an explicit algorithm computing Gröbner bases, see [7]. Here we only use some simple criteria for Gröbner bases.

Proposition 2 ([7]). *The set $\mathscr{G}$ is a Gröbner basis for $\mathscr{M}$ if and only if the (images of) normal tree monomials form a basis of the quotient $\mathscr{F}_{\mathscr{V}}/\mathscr{M}$.*

Definition 7. Let $\mathscr{P}$ be an operad, $\mathscr{P} \simeq \mathscr{F}_{\mathscr{V}}/\mathscr{M}$. A set of tree monomials $B^{\mathscr{P}} \supset \mathscr{V}$ in the free operad $\mathscr{F}_{\mathscr{V}}$ is said to be a *k-triangular basis* of $\mathscr{P}$ if

(1) The image of $B^{\mathscr{P}}$ under the canonical projection

$$\mathscr{F}_{\mathscr{V}} \twoheadrightarrow \mathscr{F}_{\mathscr{V}}/\mathscr{M} \simeq \mathscr{P}$$

is a basis of $\mathscr{P}$.
(2) Every shuffle composition of tree monomials from $B^{\mathscr{P}}$ is either an element of $B^{\mathscr{P}}$ or is congruent modulo $\mathscr{M}$ to a linear combination of smaller elements of $B^{\mathscr{P}}$.

(3) A tree monomial α belongs to $B^{\mathscr{P}}$ if and only if for every its subtree with at most k vertices the corresponding divisor belongs to $B^{\mathscr{P}}$.

Proposition 3 ([7]). *Let $\mathscr{P} \simeq \mathscr{F}_{\mathscr{V}}/\mathscr{M}$ be an operad, $\mathscr{G} \subset \mathscr{M}$ be a system of generators. Then if $\mathscr{G}$ is a Gröbner basis for $\mathscr{M}$, then the set of tree monomials which are not divisible by the leading terms of elements of $\mathscr{G}$ is a k-triangular basis of $\mathscr{P}$, where k is the maximal number of vertices in those leading terms. Conversely, for a k-triangular basis $\mathscr{B}$, there exists a Gröbner basis whose elements are combinations of tree monomials with at most k vertices. In this case, $\mathscr{B}$ coincides with the set of all normal tree monomials.*

Denote by $LT_2(\mathscr{P})$ the set of the leading terms of the quadratic part of the Gröbner basis for the operad $\mathscr{P}$ (that is, the elements of the Gröbner basis which are combinations of tree monomials with two internal vertices). Let us identify, on the level of vector spaces, the free operads whose quotients are, respectively, $\mathscr{P}$ and its Koszul dual $\mathscr{P}^!$ (identifying the dual bases of corollas, and the corresponding tree monomials), taking for the ordering of tree monomials for $\mathscr{P}^!$ the one opposite to the ordering for $\mathscr{P}$. In the case when we have elements with more than 2 inputs among the operad generators, working with the Koszul dual operad brings us to the realm of dg-operads. In that case, the machinery of Gröbner bases exists, but when trying to compute something one should be careful: a tree monomial is defined up to a sign and this should be taken into account when collection terms in a result of a computation.

The following can be easily derived from [11].

Proposition 4. *The set $LT_2(\mathscr{P}^!)$ is the set of quadratic tree monomials complementary to $LT_2(\mathscr{P})$. Also, $\mathscr{P}$ has a quadratic Gröbner basis if and only if $\mathscr{P}^!$ has a quadratic Gröbner basis; in this case both $\mathscr{P}$ and $\mathscr{P}^!$ are Koszul.*

Corollary 1. *Under the conditions of the previous proposition, the set of tree monomials for which every quadratic divisor is a leading monomial of an element of the Gröbner basis of $\mathscr{P}$ spans the Koszul dual $\mathscr{P}^!$; the number of such monomials of arity n gives an upper bound on $\dim \mathscr{P}^!(n)$. If this upper bound is sharp for all n, $\mathscr{P}^!$ has a quadratic Gröbner basis.*

3. Freeness as a nonsymmetric operad

The main idea of Salvatore and Tauraso in [15] was to consider a special basis of the Lie operad, and prove that *prime* elements of this basis (i.e. those which do not admit factorizations as nonsymmetric compositions) generate this operad freely (as a nonsymmetric operad). We adapt their approach to prove a general freeness criterion.

Definition 8. Let α be a tree monomial. For each vertex of α, the (maximal) subtree rooted at this vertex is said to be *connected* if the leaf labels of this subtree form an interval in the ordered set of leaves. A tree monomial is said to be *prime* if its only vertex with a connected subtree is its root.

Example 1. Among the tree monomials of arity 3
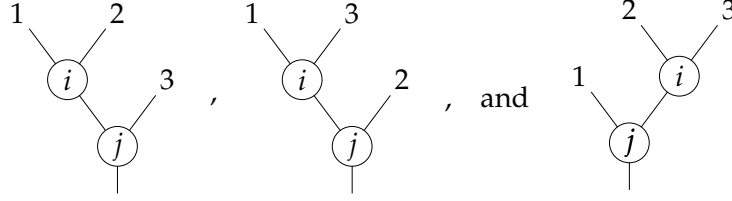
monomials of the second type are prime, and others are not.

Theorem 1. *Let $\mathscr{O} = \mathscr{F}_\mathscr{V}/(\mathscr{R})$ be a shuffle operad for which leading terms of a Gröbner basis consists of prime tree monomials. Then $\mathscr{O}$ is free as a nonsymmetric operad.*

Proof. Under the assumptions of the theorem, the nonsymmetric composition of two normal monomials is again normal, and a divisor of a normal monomial is normal. Thus, our statement follows immediately from

Lemma 1. *Let $\mathscr{O} = \mathscr{F}_\mathscr{V}/(\mathscr{R})$ be a shuffle operad. Assume that there exists a family $\mathscr{B}$ of monomials in $\mathscr{F}_\mathscr{V}$ satisfying the following conditions:*
- *the images of monomials from $\mathscr{B}$ under the natural projection form a basis of $\mathscr{O}$;*
- *$\mathscr{B}$ is closed under nonsymmetric compositions, i.e. it forms a nonsymmetric suboperad of $\mathscr{F}_\mathscr{V}$;*
- *$\mathscr{B}$ is closed under taking nonsymmetric divisors: if $\beta \in \mathscr{B}$ is obtained from α by iterated nonsymmetric compositions, then $\alpha \in \mathscr{B}$.*

Then $\mathscr{O}$ is free as a nonsymmetric operad.

Proof. Consider the set $\mathscr{P}$ of all prime monomials in $\mathscr{B}$. We claim that the free nonsymmetric operad $\mathscr{F}$ generated by $\mathscr{P}$ is isomorphic to the nonsymmetric operad $\mathscr{B}$ (which is isomorphic to $\mathscr{O}$ as a nonsymmetric operad due to the first condition). Clearly, there exists a surjection $\phi \colon \mathscr{F} \twoheadrightarrow \mathscr{B}$. We shall describe an inverse mapping from $\mathscr{B}$ to $\mathscr{F}$. For a monomial $\gamma \in \mathscr{B}$, consider the tree whose vertices correspond to vertices of γ with connected subtrees; edges of this tree correspond to the partial order on vertices induced from the tree underlying γ. To such a vertex, we associate a prime monomial; namely, we take the unique prime nonsymmetric divisor rooted at the root of the original monomial (this divisor belongs to $\mathscr{B}$ due to the third condition). Clearly, this mapping to $\mathscr{F}$ is inverse to the surjection ϕ. □

□

Before we formulate and prove the following result, we briefly recall several important definitions. The symmetric operad PreLie controlling pre-Lie algebras is generated by one operation $a, b \mapsto ab$ (without any symmetries) for which the associator $(a, b, c) = (ab)c - a(bc)$ is symmetric in the last two arguments. Its Koszul dual is denoted by Perm, a basis of the space of n-ary operations of Perm is formed by commutative products of

n elements with one element emphasized. The symmetric operad Lie^2 of compatible Lie brackets is generated by two skew-symmetric operations $a, b \mapsto [a, b]$ and $a, b \mapsto \{a, b\}$ which are compatible Lie brackets (every their linear combination is a Lie bracket). Its Koszul dual is denoted by ^2Com, it is generated by two symmetric binary operations which are totally compatible (every composition depends just on the types of the products used, not on how we compose them).

Theorem 2. *The operads* Lie, PreLie, *and* Lie^2 *are free as nonsymmetric operads.*

Proof. Let us consider the ordering for which we first compare the permutations lexicographically, and in case when the permutations are equal — the sequences of words degree-lexicographically. We shall prove that in this case our operads have quadratic Gröbner bases with leading terms of the elements being prime monomials which, by Theorem 1 is sufficient.

For the case of the operad Lie, the leading term of its defining relation $[[a_1, a_2], a_3] - [[a_1, a_3], a_2] - [a_1, [a_2, a_3]]$ is $[[a_1, a_3], a_2]$. Note that this relation alone allows the only normal tree monomial of arity n for the Koszul dual operad, namely

$$[[[[a_1, a_n], a_{n-1}], \ldots, a_3], a_2],$$

so our upper bound on dimensions of components of the Koszul dual operad is sharp. This means that the Gröbner basis for the Koszul dual operad Com is quadratic, hence the Gröbner basis for Lie is quadratic. The leading term is indeed a prime monomial.

For the case of the operad PreLie, we use the operations $\alpha(a_1, a_2) = a_1 a_2$ and $\beta(a_1, a_2) = a_2 a_1$ which generate it as a shuffle operad. The leading terms of its defining relations

$$\alpha(\alpha(a_1, a_2), a_3) - \alpha(a_1, \alpha(a_2, a_3)) - \alpha(\alpha(a_1, a_3), a_2) + \alpha(a_1, \beta(a_2, a_3)),$$

$$\alpha(\beta(a_1, a_2), a_3) - \beta(\alpha(a_1, a_3), a_2) - \beta(a_1, \alpha(a_2, a_3)) + \beta(\beta(a_1, a_3), a_2),$$

$$\alpha(\beta(a_1, a_3), a_2) - \beta(\alpha(a_1, a_2), a_3) - \beta(a_1, \beta(a_2, a_3)) + \beta(\beta(a_1, a_2), a_3)$$

are (under the assumption $\beta > \alpha$)

$$\alpha(\alpha(a_1, a_3), a_2), \beta(\beta(a_1, a_3), a_2), \text{ and } \alpha(\beta(a_1, a_3), a_2).$$

It follows that the normal monomials of arity n for the Koszul dual operad are among

$$\alpha(\ldots \alpha(\beta(\ldots \beta(a_1, a_n) \ldots), a_k), a_{k-1}) \ldots), a_2),$$

with i operations α and j operations β, $i + j = n - 1$. This upper bound for the Koszul dual operad Perm is sharp since $\dim \text{Perm}(n) = n$. This means that there are no more elements in the Gröbner basis and both Perm and PreLie have quadratic Gröbner bases. As we saw above, the leading terms of the relations for PreLie are prime monomials.

For the case of the operad Lie^2, the leading terms of the defining relations

$$[[a_1, a_2], a_3] - [[a_1, a_3], a_2] - [a_1, [a_2, a_3]],$$

$$\{\{a_1, a_2\}, a_3\} - \{\{a_1, a_3\}, a_2\} - \{a_1, \{a_2, a_3\}\},$$

and

$$[\{a_1, a_2\}, a_3] - [\{a_1, a_3\}, a_2] - [a_1, \{a_2, a_3\}] +$$
$$+ \{[a_1, a_2], a_3\} - \{[a_1, a_3], a_2\} - \{a_1, [a_2, a_3]\}$$

are (under the assumption $[\cdot, \cdot] > \{\cdot, \cdot\}$)

$$[[a_1, a_3], a_2], \{\{a_1, a_3\}, a_2\}, \text{ and } [\{a_1, a_3\}, a_2].$$

Similarly to the case of the operad PreLie, this means that the dimension of the arity n component of the Koszul dual operad is at most n. Since the Koszul dual of Lie^2 is $^2\mathrm{Com}$, and $\dim(\,^2\mathrm{Com}(n)) = n$, we deduce that both $^2\mathrm{Com}$ and Lie^2 have quadratic Gröbner bases for our ordering. The leading terms of relations for Lie^2 are prime monomials. $\qquad\square$

4. Embeddings of operads

Proposition 5. *Let $\mathscr{P} = \mathscr{F}_{\mathscr{V}}/(\mathscr{R})$ and $\mathscr{Q} = \mathscr{F}_{\mathscr{V} \oplus \mathscr{W}}/(\mathscr{R} \oplus \mathscr{S})$ be shuffle operads. Assume that the Gröbner basis of $\mathscr{Q}$ is a union of the Gröbner basis of $\mathscr{P}$ and a set of relations whose leading terms are tree monomials not belonging to $\mathscr{F}_{\mathscr{V}} \subset \mathscr{F}_{\mathscr{V} \oplus \mathscr{W}}$. Then the natural mapping $\mathscr{P} \to \mathscr{Q}$ is an embedding.*

Proof. This statement is obvious: because of the condition imposed, the set of normal monomials for $\mathscr{Q}$ contains the set of normal monomials for $\mathscr{P}$. $\qquad\square$

Theorem 3. *The suboperad of the operad* PreLie *generated by the symmetrized pre-Lie product $a \cdot b = ab + ba$ is isomorphic to* Mag, *the free operad on one generator.*

Proof. We consider the operad PreLie with its symmetric generator $\cdot$ and Lie bracket $[\cdot, \cdot]\colon a, b \mapsto [a, b] = ab - ba$. The following lemma is straightforward from the PreLie relations.

Lemma 2. *The defining relations of* PreLie *for this choice of generators are*

$$[[a_1, a_2], a_3] - [[a_1, a_3], a_2] - [a_1, [a_2, a_3]],$$

$$(a_1 \cdot a_2) \cdot a_3 - a_1 \cdot (a_2 \cdot a_3) - a_1 \cdot [a_2, a_3] - [a_1, a_2] \cdot a_3 - 2[a_1, a_3] \cdot a_2 +$$
$$+ [a_1, a_2 \cdot a_3] + [a_1 \cdot a_2, a_3] + [[a_1, a_3], a_2],$$

and

$$(a_1 \cdot a_3) \cdot a_2 - a_1 \cdot (a_2 \cdot a_3) + a_1 \cdot [a_2, a_3] - [a_1, a_3] \cdot a_2 - 2[a_1, a_2] \cdot a_3 +$$
$$+ [a_1, a_2 \cdot a_3] + [a_1 \cdot a_3, a_2] + [[a_1, a_2], a_3].$$

Let us choose an ordering of tree monomials as follows. To compare two tree monomials, we first compare their arities, then the number of the corollas corresponding to the product $\cdot$, and then compare the tree monomials using the original path-lexicographic ordering from [7]. It is easy to see that this ordering is admissible. The leading monomials of the

relations of PreLie are (assuming that the bracket corolla is greater than the product corolla)

$$[[a_1, a_2], a_3], [a_1 \cdot a_2, a_3], \text{ and } [a_1 \cdot a_3, a_2].$$

It follows that the normal monomials of arity n for the Koszul dual operad are among

$$[[[[a_1 \cdot a_k, a_n], a_{n-1}], \dots, \hat{a}_k, \dots, a_3], a_2], k \geq 2, \text{ and } [[[[a_1, a_n], a_{n-1}], \dots, a_3], a_2],$$

which gives the upper bound n on the dimensions of the arity n component of the Koszul dual operad, and, as we have seen before, this implies that our relations form a Gröbner basis of PreLie for this ordering. Clearly, there are no leading terms of our relations that are made entirely of the product corollas, so by Proposition 5 the product corolla defines an embedding $\mathrm{Mag} \hookrightarrow \mathrm{PreLie}$.   □

5. Freeness as a module

Theorem 4. *Let $\mathscr{P} \hookrightarrow \mathscr{Q}$ be an embedding of shuffle operads, $\mathscr{P} = \mathscr{F}_\mathscr{V}/(\mathscr{R})$, $\mathscr{Q} = \mathscr{F}_{\mathscr{V} \oplus \mathscr{W}}/(\mathscr{R} \oplus \mathscr{S})$. Fix a basis of tree monomials in $\mathscr{F}_{\mathscr{V} \oplus \mathscr{W}}$ for which the basis corollas is the union of the bases for $\mathscr{V}$ and $\mathscr{W}$.*

(1) *Assume that the Gröbner basis of $\mathscr{Q}$ is a union of the Gröbner basis of $\mathscr{P}$ and a set of relations whose leading terms are tree monomials with the root from the basis of $\mathscr{W}$. Then $\mathscr{Q}$ is free as a left $\mathscr{P}$-module, $\mathscr{Q} \simeq \mathscr{P} \circ \mathscr{K}$ for some collection $\mathscr{K}$.*

(2) *Assume that the Gröbner basis of $\mathscr{Q}$ is a union of the Gröbner basis of $\mathscr{P}$ and a set of relations whose leading terms are tree monomials where the parent vertex of each leaf belongs to basis of $\mathscr{W}$. Then $\mathscr{Q}$ is free as a right $\mathscr{P}$-module, $\mathscr{Q} \simeq \mathscr{K} \circ \mathscr{P}$ for some collection $\mathscr{K}$.*

Proof. These two statements are absolutely analogous to each other; let us prove the first one. Define $\mathscr{K}$ to be the set of all normal monomials whose roots belong to the complementary set of generators $\mathscr{W}$. There exists a natural mapping $\mathscr{P} \circ \mathscr{K} \to \mathscr{Q}$. This mapping is surjective for obvious reasons, moreover, it is injective because a composition of a normal tree monomial from $\mathscr{P}$ and a normal tree monomial from $\mathscr{K}$ cannot be reduced due to our condition on the Gröbner basis.   □

Corollary 2. *The operad* PreLie *is free as a* Lie*-module. Consequently, free pre-Lie algebras are free as Lie algebras.*

Proof. Let us choose an ordering of tree monomials as follows. To compare two tree monomials, we first compare their arities, then corresponding permutations lexicographically, then the corresponding sequences of words degree-lexicographically. The leading monomials of the relations of PreLie from Lemma 2 are (assuming that the bracket corolla is greater than the product corolla)

$$[[a_1, a_3], a_2], [a_1, a_3] \cdot a_2, \text{ and } (a_1 \cdot a_3) \cdot a_2.$$

Similarly to the proof of Theorem 2, this means that the dimension of the arity n component of the Koszul dual operad is at most n. Since the dimension of that component is equal to n, we conclude that both PreLie and its dual have quadratic Gröbner bases for this ordering as well. For the last two relations which are complementary to the Lie relation, the roots of the leading terms belong to the complementary set of generators, which proves our statement.   □

Remark 1. Theorem 4, applied to the Gröbner basis for the operad PreLie that we computed in Theorem 3, implies that PreLie is free as a left Mag-module as well, PreLie $\simeq$ Mag $\circ \mathscr{K}$ for some collection $\mathscr{K}$. A direct computation shows that dim $\mathscr{K}(n) = n^{n-2}$, the number of labelled trees on $[n]$. It would be interesting to describe $\mathscr{K}$ combinatorially in the spirit of the description of PreLie via labelled rooted trees from [5].

To formulate the main new result of this section, we recall the following general definitions that were given in [16], as a generalisation of the notion of compatible Lie brackets (and the algebraic structure controlled by its Koszul dual operad) to general algebraic structures.

Let $\mathscr{O} = \mathscr{F}_{\mathscr{V}}/(\mathscr{R})$ be a binary quadratic operad generated by binary operations $\alpha_1, \dots, \alpha_s$ with t relations

$$\left\{ \sum_{1 \le i,j \le s} \gamma_{i,j}^{k,1} \alpha_j(\alpha_i(a_1, a_2), a_3) + \gamma_{i,j}^{k,2} \alpha_j(a_1, \alpha_i(a_2, a_3)) + \gamma_{i,j}^{k,3} \alpha_j(\alpha_i(a_1, a_3), a_2) \right\}_{1 \le k \le t} .$$

Consider two operads $\mathscr{O}_\circ = \mathscr{F}_{\mathscr{V}_\circ}/(\mathscr{R}_\circ)$ and $\mathscr{O}_\bullet = \mathscr{F}_{\mathscr{V}_\bullet}/(\mathscr{R}_\bullet)$ both isomorphic to $\mathscr{O}$. We choose $\Bbbk$-bases $\alpha_1^1, \dots, \alpha_s^1$ of $\mathscr{V}_\circ$ and $\alpha_1^2, \dots, \alpha_s^2$ of $\mathscr{V}_\bullet$. The relations $\mathscr{R}_\circ$ and $\mathscr{R}_\bullet$ can then be given by the same $\gamma_{i,j}^{k,l}$. The suboperads $\mathscr{O}_\circ$ and $\mathscr{O}_\bullet$ of $\mathscr{F}_{\mathscr{V}_\circ \oplus \mathscr{V}_\bullet}/(\mathscr{R}_\circ \cup \mathscr{R}_\bullet)$ control a pair of $\mathscr{O}$-structures which are not related in any way. Now we are going to define two important ways of imposing additional compatibility relations.

Definition 9. Relations

$$\left\{ \sum_{1 \le i,j \le s} \gamma_{i,j}^{k,1} \alpha_j^1(\alpha_i^2(a_1, a_2), a_3) + \gamma_{i,j}^{k,2} \alpha_j^1(a_1, \alpha_i^2(a_2, a_3)) + \gamma_{i,j}^{k,3} \alpha_j^1(\alpha_i^2(a_1, a_3), a_2) + \right.$$

$$\left. + \gamma_{i,j}^{k,1} \alpha_j^2(\alpha_i^1(a_1, a_2), a_3) + \gamma_{i,j}^{k,2} \alpha_j^2(a_1, \alpha_i^1(a_2, a_3)) + \gamma_{i,j}^{k,3} \alpha_j^2(\alpha_i^1(a_1, a_3), a_2) \right\}_{1 \le k \le t}$$

are called the linear compatibility (weak compatibility) relations.

Definition 10. Relations

$$\alpha_j^1(\alpha_i^2(a_1, a_2), a_3) = \alpha_j^2(\alpha_i^1(a_1, a_2), a_3) \quad (1 \le i, j \le s),$$

$$\alpha_j^1(\alpha_i^2(a_1, a_3), a_2) = \alpha_j^2(\alpha_i^1(a_1, a_3), a_2) \quad (1 \le i, j \le s),$$

$$\alpha_j^1(a_1, \alpha_i^2(a_2, a_3)) = \alpha_j^2(a_1, \alpha_i^1(a_2, a_3)) \quad (1 \le i, j \le s),$$

$$\left\{ \sum_{1 \le i, j \le s} \gamma_{i,j}^{k,1} \alpha_j^1(\alpha_i^2(a_1, a_2), a_3) + \gamma_{i,j}^{k,2} \alpha_j^1(a_1, \alpha_i^2(a_2, a_3)) + \gamma_{i,j}^{k,3} \alpha_j^1(\alpha_i^2(a_1, a_3), a_2) \right\}_{1 \le k \le t},$$

$$\left\{ \sum_{1 \le i, j \le s} \gamma_{i,j}^{k,1} \alpha_j^2(\alpha_i^1(a_1, a_2), a_3) + \gamma_{i,j}^{k,2} \alpha_j^2(a_1, \alpha_i^1(a_2, a_3)) + \gamma_{i,j}^{k,3} \alpha_j^2(\alpha_i^1(a_1, a_3), a_2) \right\}_{1 \le k \le t}$$

are called the total compatibility (strong compatibility) relations (the last one clearly follows from the first four and so is not necessary to include).

Definition 11.   (1) The operad $\mathscr{O}^2$ is the quotient of $\mathscr{F}_{\mathscr{V}_\circ \oplus \mathscr{V}_\bullet}$ modulo the ideal generated by $\mathscr{R}_\circ$, $\mathscr{R}_\bullet$, and the linear compatibility relations. It is called the operad of linearly compatible $\mathscr{O}$-stuctures.
    (2) The operad $^2\mathscr{O}$ is the quotient of $\mathscr{F}_{\mathscr{V}_\circ \oplus \mathscr{V}_\bullet}$ modulo the ideal generated by $\mathscr{R}_\circ$, $\mathscr{R}_\bullet$ and the total compatibility relations. It is called the operad of totally compatible $\mathscr{O}$-structures.

This notation agrees with the notation introduced earlier for the particular cases Lie^2 and $^2\mathrm{Com}$.

Proposition 6 ([16]). *We have $(\mathscr{O}^2)^! = {}^2(\mathscr{O}^!)$ and $^2\mathscr{O} \simeq {}^2\mathrm{Com} \otimes \mathscr{O}$.*

Theorem 5. *Let $\mathscr{O}$ be an operad which has a quadratic Gröbner basis (for some admissible ordering). Furthermore, assume that all the monomials $\alpha_j(a_1, \alpha_i(a_2, a_3))$ are normal monomials relative to that Gröbner basis. Then there exists an admissible ordering for which the operad $\mathscr{O}^2$ has a quadratic Gröbner basis as well, and $\mathscr{O}^2$ is a free $\mathscr{O}$-module. In particular, free $\mathscr{O}^2$-algebras are free as $\mathscr{O}$-algebras.*

Proof. Let us choose an ordering of tree monomials in $\mathscr{F}_{\mathscr{V}_\circ \oplus \mathscr{V}_\bullet}$ as follows. To compare two tree monomials, we first compare their arities, then the number of corollas from the set of generators $\mathscr{V}_\circ$, then compare the images of these monomials under the natural homomorphism $\mathscr{F}_{\mathscr{V}_\circ \oplus \mathscr{V}_\bullet} \twoheadrightarrow \mathscr{F}_\mathscr{V}$ (defined on generators by $\alpha_j^i \mapsto \alpha_j$), and if all the above procedures give the same result, compare these monomials using the path-lexicographic order, assuming that $\alpha_u^1 > \alpha_v^2$ for all u, v. It is clear that this ordering is admissible.

Note that from our normal monomial condition it follows that for the operad $\mathscr{O}^!$ all the operations $\alpha_j(a_1, \alpha_i(a_2, a_3))$ are among the leading terms of the Gröbner basis. It follows that all underlying trees of normal tree monomials for this operad are "combs": all their vertices form a chain between the root and the leaf labelled 1.

Let us consider the operad $^2(\mathcal{O}^!)$. Since this operad is isomorphic to $^2\mathrm{Com} \otimes \mathcal{O}^!$, we can easily construct a monomial basis for $^2(\mathcal{O}^!)$ from a monomial basis of $\mathcal{O}^!$. Let us take the set of all normal tree monomials for $\mathcal{O}^!$ and replicate every normal monomial of arity n exactly n times; for the kth replica ($1 \le k \le n$), we look at the path from the root to the leaf labelled 1, and replace the last $k-1$ vertices of this path α_v by the corresponding generator α_v^1 and the remaining $n-k$ vertices by the corresponding generator α_v^2. As a result, the corresponding tree monomial is the least (with respect to our ordering) among all the monomials that can be obtained this way (note that in $^2(\mathcal{O}^!)$ all these monomials are equal to the least one because of the strong compatibility relations).

From Proposition 3, we see that normal tree monomials for $\mathcal{O}^!$ form a 2-triangular basis. This implies that our monomial basis for $^2(\mathcal{O}^!)$ is 2-triangular as well (the fact that all normal monomials are combs guarantees "2" in "2-triangular"), so the total compatibility relations form a Gröbner basis. Consequently, the operad $\mathcal{O}^2$ has a Gröbner basis as well, as the Koszul dual to $^2(\mathcal{O}^!)$.

Finally, let us notice that for the leading terms of relations of $\mathcal{O}^2$ which are not the relations of the first copy of $\mathcal{O}_\circ$, the roots belong to $\mathcal{V}_\bullet$ (consistent orderings of monomials for an operad and its dual are opposite to each other), which proves the desired result.   □

Corollary 3. *Under the assumption of Theorem 5, the operads $\mathcal{O}^2$ and $^2(\mathcal{O}^!)$ are Koszul.*

Our result, despite of the additional assumption on the operad $\mathcal{O}$, are applicable to some new (not covered by [16]) cases, for example, the operad Com (its Koszul dual Lie is not set-theoretic, and so Strohmayer's results do not work, but our assumption applies). It turns out that some restrictions on the operad $\mathcal{O}$ should be imposed anyway (both for the Koszulness and for freeness of $\mathcal{O}^2$ as an $\mathcal{O}$-module). Indeed, let us take the nilpotent free operad $\mathcal{N}$ with one symmetric generator. For this operad, it is easy to see that $\dim \mathcal{N}^2(2) = 2$, $\dim \mathcal{N}^2(3) = 3$, $\dim \mathcal{N}^2(k) = 0$ for $k > 3$. Thus the Hilbert series [10] of this operad is $f(t) = t + t^2 + \frac{t^3}{2}$, which immediately implies that this operad is not Koszul (since the inverse of $f(-t)$ has negative coefficients). Also, we immediately see that $\mathcal{N}^2$ is not free as a left $\mathcal{N}$-module (if it was, it would have an additional generator in arity 2 which, in turn, would lead to non-zero operations of arity 4).

It turns out that the result of Corollary 3 can be substantially generalized. For a binary quadratic operad $\mathcal{O} = \mathcal{F}_\mathcal{V}/(\mathcal{R})$ let $\pi_\mathcal{O}\colon \mathcal{F}_\mathcal{V} \to \mathcal{O}$ be the natural projection. Let λ be a tree monomial with $n-1$ vertices from the free operad on one generator. Fix an ordering of vertices of λ, and let $\mathcal{L}_\lambda^\mathcal{V}$ denote the induced decoration morphism from $\mathcal{V}^{\otimes(n-1)}$ to $\mathcal{F}_\mathcal{V}$; it decorates vertices of λ with elements of V. An operad $\mathcal{O}$ is said to be *small* if for every $n \ge 3$ and every tree monomial γ with $n-1$ vertices the composite map

$\pi_{\mathcal{O}} \circ \mathcal{L}_\gamma^{\mathcal{V}} : \mathcal{V}^{\otimes(n-1)} \to \mathcal{O}(n)$ is surjective (the class of small operads was studied in [17] in relation to Manin products for operads; by [17, Prop. 15], smallness of a binary quadratic operad $\mathcal{A}$ guarantees that for every binary quadratic operad $\mathcal{B}$ the Hadamard product $\mathcal{A} \otimes \mathcal{B}$ and the white product $\mathcal{A} \circ \mathcal{B}$ are isomorphic). Modelling the proof of Theorem 5, one can easily derive

Proposition 7. *Suppose that the operad $\mathcal{A}$ is small. Under the assumption of Theorem 5, the operads $\mathcal{O} \bullet \mathcal{A}^{!}$ and $\mathcal{O}^{!} \circ \mathcal{A}$ have quadratic Gröbner bases and hence are Koszul.*

Remark 2. In fact, in most of the cases of Koszul operads, we know an admissible ordering of tree monomials for which the Gröbner basis is quadratic. Moreover, we do not know any example of a Koszul operad generated by binary operations which has no quadratic Gröbner basis. Such examples should exist; in the case of quadratic algebras (that is, quadratic operads generated by unary operations), there are some known examples of that sort, see [1].

In the last example we wish to consider, the operads are regular, so everything is essentially reduced to the level of nonsymmetric operads. The symmetric operad Dend of dendriform algebras is generated by two binary operations $\prec$ and $\succ$ which satisfy the relations

$$(a_1 \prec a_2) \prec a_3 = a_1 \prec (a_2 \prec a_3) + a_1 \prec (a_2 \succ a_3),$$
$$(a_1 \succ a_2) \prec a_3 = a_1 \succ (a_2 \prec a_3),$$
$$(a_1 \prec a_2) \succ a_3 + (a_1 \succ a_2) \succ a_3 = a_1 \succ (a_2 \succ a_3).$$

Its Koszul dual operad Dias of diassociative algebras is generated by two binary operations $\dashv$ and $\vdash$ which satisfy the relations

$$(a_1 \dashv a_2) \dashv a_3 = a_1 \dashv (a_2 \dashv a_3), \quad (a_1 \vdash a_2) \vdash a_3 = a_1 \vdash (a_2 \vdash a_3),$$
$$(a_1 \dashv a_2) \dashv a_3 = a_1 \dashv (a_2 \vdash a_3), \quad (a_1 \dashv a_2) \vdash a_3 = a_1 \vdash (a_2 \vdash a_3),$$
$$(a_1 \vdash a_2) \dashv a_3 = a_1 \vdash (a_2 \dashv a_3).$$

It is well known that the operad Dias is a symmetrization of a nonsymmetric operad whose n^{th} component has the dimension n. We shall see that this allows for proofs similar to those for the pair of Koszul dual operads PreLie and Perm.

If we introduce the new generator $\star$, $a_1 \star a_2 = a_1 \prec a_2 + a_1 \succ a_2$, it is easy to see that this operation is an associative product. In [12], Loday and Ronco proved that the free dendriform algebra on one generator is free as an associative algebra with respect to this product. Since the dendriform operad is a symmetrization of a nonsymmetric operad, this essentially means that Dend is a free left module over As and hence the free dendriform algebra on any number of generators is free as an associative algebra. We shall prove this statement using Gröbner bases for nonsymmetric operads.

Proposition 8. *The nonsymmetric operad Dend is a free left module over the nonsymmetric operad As.*

Proof. In terms of the generators $\star$ and $>$, the defining relations of Dend are

$$(a_1 \star a_2) \star a_3 = a_1 \star (a_2 \star a_3),$$
$$(a_1 > a_2) > a_3 - 2a_1 > (a_2 > a_3) + a_1 > (a_2 \star a_3) = 0,$$
$$(a_1 \star a_2) > a_3 = a_1 > (a_2 > a_3).$$

If we assume that the operation $\star$ is greater than the operation $>$, and use the path-lexicographic ordering of trees, the leading terms of relations are $(a_1 \star a_2) \star a_3$, $(a_1 > a_2) > a_3$, and $(a_1 \star a_2) > a_3$. We already know that this means an upper bound n on the dimension of the nth component of the Koszul dual operad. Thus, the Gröbner basis is quadratic since this bound is sharp. The leading terms of the relations (except for the associativity relation) have their roots labelled by $>$ which proves that Dend is a free left As-module. $\qquad\square$

Remark 3. A freeness theorem which is not covered by this text is the result of Foissy [9] stating that free brace algebras are free as pre-Lie algebras. It remains an open question to compute the Gröbner basis for the brace operad and derive the result of Foissy from Theorem 4.

References

[1] Roland Berger, Weakly Confluent Quadratic Algebras. Algebras and Representation Theory, **1** (1998), no. 3, 189–213.

[2] Nantel Bergeron and Muriel Livernet, The non-symmetric operad pre-Lie is free. Preprint `arXiv:0811.0888`.

[3] Nantel Bergeron and Jean-Louis Loday, in preparation.

[4] Frédéric Chapoton, Free pre-Lie algebras are free as Lie algebras. Preprint `arXiv:0704.2153`, to appear in Bulletin canadien de mathématiques.

[5] Frederic Chapoton and Muriel Livernet, Pre-Lie algebras and the rooted trees operad. Internat. Math. Res. Notices 2001, no. 8, 395–408.

[6] Vladimir Dotsenko, Compatible associative products and trees. Algebra and Number Theory, **3** (2009), no. 5, 567–586.

[7] Vladimir Dotsenko and Anton Khoroshkin, Gröbner bases for operads. Duke Math. Journal, to appear.

[8] Loïc Foissy, Finite-dimensional comodules over the Hopf algebra of rooted trees. J. Algebra **255** (2002), no. 1, 89–120.

[9] Loïc Foissy, Free brace algebras are free pre-Lie algebras. Preprint `arXiv:0809.2866`.

[10] Victor Ginzburg and Mikhail Kapranov, Koszul duality for operads. Duke Math. J., 76:1 (1994), 203–272.

[11] Eric Hoffbeck, A Poincaré–Birkhoff–Witt criterion for Koszul operads. Manuscripta Math., to appear.

[12] Jean-Louis Loday and Maria Ronco, Hopf algebra of the planar binary trees. Adv. Math. **139** (1998), no. 2, 293–309.

[13] Wilhelm Magnus, Über discontinuierliche Gruppen mit einer definierenden Relation (Der Freiheitssatz). J. Reine Angew. Math., **163** (1930), 141–165.

[14] Martin Markl, Steve Shnider, Jim Stasheff, Operads in Algebra, Topology and Physics, Mathematical Surveys and Monographs, vol. 96, AMS, Providence, Rhode Island, 2002.

[15] Paolo Salvatore and Roberto Tauraso, The operad Lie is free. J. Pure Appl. Algebra **213** (2009), 224–230.

[16] Henrik Strohmayer, Operads of compatible structures and weighted partitions, J. Pure Appl. Algebra **212** (2008), no. 11, 2522–2534.

[17] Bruno Vallette, Manin products, Koszul duality, Loday algebras and Deligne conjecture. J. Reine Angew. Math., **620** (2008), 105–164.

DUBLIN INSTITUTE FOR ADVANCED STUDIES, 10 BURLINGTON ROAD, DUBLIN 4, IRELAND AND SCHOOL OF MATHEMATICS, TRINITY COLLEGE, DUBLIN 2, IRELAND

E-mail address: `vdots@maths.tcd.ie`