New Chern-Simons densities in both odd and even dimensions

Eugen Radu[†⋆] and Tigran Tchrakian[‡⋆◇]

[‡]School of Theoretical Physics – DIAS, 10 Burlington Road, Dublin 4, Ireland

[⋆]Department of Computer Science, National University of Ireland Maynooth, Maynooth, Ireland

[◇]Theory Division, Yerevan Physics Institute (YerPhI), AM-375 036 Yerevan 36, Armenia

Abstract

After reviewing briefly the dimensional reduction of Chern–Pontryagin densitie, we define new Chern–Simons densities expressed in terms of Yang-Mills and Higgs fields. These are defined in all dimensions, including in even dimensional spacetimes. They are constructed by subjecting the dimensionally reduced Chern–Pontryagin densites to further descent by two steps.

1 Introduction

The central task of these notes is to explain how to subject $n-$th Chern–Pontryagin (CP) density $\mathcal{C}^{(n)}$,

$$\mathcal{C}^{(n)} = \frac{1}{\omega(\pi)}\varepsilon_{M_1M_2M_3M_4\dots M_{2n-1}M_{2n}}\,\mathrm{Tr}\,F_{M_1M_2}F_{M_3M_4}\dots F_{M_{2n-1}M_{2n}} \tag{1.1}$$

defined on on the $2n-$dimensional space, to dimensional descent to $\mathbb{R}^D$ by considering (1.1) on the direct product space $\mathbb{R}^D \times S^{2n-D}$. The resulting residual density on $\mathbb{R}^D$ will be denoted as $\Omega_D^{(n)}$.

The density $\mathcal{C}^{(n)}$ is by construction, a total divergence

$$\mathcal{C}^{(n)} = \boldsymbol{\nabla}\cdot\boldsymbol{\Omega}^{(n)}\,, \tag{1.2}$$

and it turns out that under certain retrictions, the dimensional descendant of (1.2) is also a total divergence. This result will be applied to the construction of Chern-Simons solitons in all dimensions.

Some special choices, or restrictions, are made for practical reasons. Firstly, we have restricted to the codimension S^{2n-D}, the $(2n-D)-$sphere, as this is the most symmetric compact coset space that is defined both in even and in odd dimensions. It can of course be replaced by any other symmetric and compact coset space.

Secondly, the gauge field of the bulk gauge theory is chosen to be a $2^{n-1}\times 2^{n-1}$ array with complex valued entries. Given our choice of spheres for the codimension, this leads to residual gauge fields on which take their values in the Dirac matrix representation of the residual gauge group $SO(D)$.

As a result of the above two choices, it is possible to make the symmetry imposition (namely the dimensional reduction) such that the residual Higgs field is described by a $D-$component isovector multiplet. With this choice, the asymptotic gauge fields describe a Dirac-Yang [1, 2, 3] monopoles..

The central result exploited here is that the CP density $\mathcal{C}_D^{(n)}$ on $\mathbb{R}^D$ descended from the $2n-$dimensional bulk CP density $\mathcal{C}^{(n)}$ is **also** a *total divergence*

$$\mathcal{C}_D^{(n)} = \boldsymbol{\nabla}\cdot\boldsymbol{\Omega}^{(n,D)} \tag{1.3}$$

like $\mathcal{C}^{(n)}$ formally is on the bulk. From the reduced density $\boldsymbol{\Omega}^{(n,D)}\equiv\Omega_i^{(n,D)}$, where the index i labels the coordinate of the residual space x_i with $i=1,2,\dots,D$, one can identify a Chern–Simons (CS) density as the

$D-$th component of $\mathbf{\Omega}^{(n,D)}$. This quantity can then be interpreted as a CS term on $(D-1)-$ dimensional Monkowski space, *i.e.*, on the spacetime $(t,\mathbb{R}^{D-2})$. The solitons of the corresponding CS–Higgs (CSH) theory can be constructed systematically. Note, that this is not the usual CS term defined in terms of a pure Yang–Mills field on *odd* dimensional spacetime, but rather these new CS terms are defined by both the YM and, the Higgs fields. Most importantly, the definition of these new CS terms is not restricted to *odd* dimensional spacetimes, but covers also *even* dimensional spacetimes. Such CSH solutions have not been studied to date.

2 Dimensional reduction of gauge fields

The calculus of the dimensional reduction of Yang–Mills fields employed here is based on the formalism of A. S. Schwarz [4], which is specially transparent due to the choice of displaying the results only at a fixed point of the compact symmetric codimensional space K^N (the North or South pole for S^N). Our formalism is a straightforward extension of [4, 5, 6].

2.1 Descent over S^N: N odd

For the descent from the bulk dimension $2n=D+N$ down to **odd** D (over odd N), the components of the residual connection evaluated at the Noth pole of S^N are given by

$$\mathcal{A}_i = A_i(\vec{x})\otimes 1\!\!1 \tag{2.1}$$
$$\mathcal{A}_I = \Phi(\vec{x})\otimes \frac{1}{2}\Gamma_I\,. \tag{2.2}$$

The unit matrix in (2.1), like the $N-$dimensional gamma matrix in (2.2), are $2^{\frac{1}{2}(N-1)}\times 2^{\frac{1}{2}(N-1)}$ arrays. Choosing the $2^{n-1}\times 2^{n-1}$ bulk gauge group to be, say, $SU(n-1)$, allows the choice of $SO(D)$ as the gauge group of the residual connection $A_i(x)$. This choice is made such that the asymptotic connections describe a Dirac–Yang monopole.

For the same reason, the choice for the multiplet structure of the Higgs field is made to be less restrictive. The (anti-Hermitian) field Φ, which is not necessarily traceless [1], can be and *is* taken to be in the algebra of $SO(D+1)$, in particular, in one or other of the chiral reprentations of $SO(D+1)$, $D+1$ here being even.

$$\Phi=\phi^{ab}\,\Sigma_{ab}\ ,\quad a=i,\,D+1\ ,\quad i=1,2,\dots,D\,. \tag{2.3}$$

(Only in the $D=3$ case does the Higgs field take its values in the algebra of $SO(3)$, since the representations $SO(3)$ coincide with those of chiral $SO(4)$.)

In anticipation of the corresponding situation of even D to be given next, one can specialise (2.3) to the a $D-$component *isovector* expression of the Higgs field

$$\Phi=\phi^{i}\,\Sigma_{i,D+1}\ , \tag{2.4}$$

with the purpose of having a unified notation for both even and odd D, where the Higgs field takes its values in the components $\Sigma_{i,D+1}$ orthogonal to elements Σ_{ij} of the algebra of $SO(D+1)$. This specialisation is not necessary, and is in fact inappropriate should one consider, *e.g.*, axially symmetric fields. It is however adequate for the presentation here and is sufficiently general to describe spherically symmetric monopoles [2].

[1] In practice, when constructing soliton solutions, Φ is taken to be traceless without loss of generality.

[2] While all concrete considerations in the following are restricted to spherically symmetric fields, it should be emphasised that relaxing spherical symmetry results in the Higgs multiplet getting out of the orthogonal complement $\Sigma_{i,D+1}$ to $\Sigma_{i,j}$. Indeed, subject to axial symmetry one has

$$\Phi=f_1(\rho,z)\Sigma_{\alpha\beta}\hat{x}_\beta+f_2(\rho,z)\Sigma_{\beta,D+1}\hat{x}_\beta+f_3(\rho,z)\Sigma_{D,D+1}\,, \tag{2.5}$$

where $x_i=(x_\alpha,z)$, $|x_\alpha|^2=\rho^2$ and with $\hat{x}_\alpha=x_\alpha/\rho$. Clearly, the term in (2.5) multiplying the basis $\Sigma_{\alpha\beta}$ does not occur in (2.4).

In (2.1) and (2.2), and everywhere henceforth, we have denoted the components of the residual coordinates as $x_i = \vec{x}$. The dependence on the codimension coordinate x_I is suppressed since all fields are evaluated at a fixed point (North or South pole) of the codimension space.

The resulting components of the curvature are

$$\begin{aligned}
\mathcal{F}_{ij} &= F_{ij}(\vec{x}) \otimes \mathbb{I} &(2.6)\\
\mathcal{F}_{iI} &= D_i\Phi(\vec{x}) \otimes \frac{1}{2}\Gamma_I &(2.7)\\
\mathcal{F}_{IJ} &= S(\vec{x}) \otimes \Gamma_{IJ}\,, &(2.8)
\end{aligned}$$

where $\Gamma_{IJ} = -\frac{1}{4}[\Gamma_I, \Gamma_J]$ are the Dirac representation matrices of $SO(N)$, the stability group of the symmetry group of the $N-$sphere. In (2.7), $D_i\Phi$ is the covariant derivative of the Higgs field Φ

$$D_i\Phi = \partial_i\Phi + [A_i, \Phi] \tag{2.9}$$

and S is the quantity

$$S = -(\eta^2\, \mathbb{I} + \Phi^2)\,, \tag{2.10}$$

where η is the inverse of the radius of the $N-$sphere.

2.2 Descent over S^N: N even

The formulae corresponding to (2.1)-(2.8) for the case of **even** D are somewhat more complex. The reason is the existence of a chiral matrix Γ_{N+1}, in addition to the Dirac matrices Γ_I, $I = 1, 2, \ldots, N$. Instead of (2.1)-(2.2) we now have

$$\begin{aligned}
\mathcal{A}_i &= A_i(\vec{x}) \otimes \mathbb{I} + B_i(\vec{x}) \otimes \Gamma_{N+1}\\
\mathcal{A}_I &= \phi(\vec{x}) \otimes \frac{1}{2}\Gamma_I + \psi(\vec{x}) \otimes \frac{1}{2}\Gamma_{N+1}\Gamma_I\,,
\end{aligned}$$

where A_i, B_i, ϕ, and ψ are again antihermitian matrices, but with only A_i being traceless. The fact that B_i is not traceless here results in an Abelian gauge field in the reduced system.

Anticipating what follows, it is much more transparent to re-express these formulas in the form

$$\begin{aligned}
\mathcal{A}_i &= A_i^{(+)}(\vec{x}) \otimes\, P_+ + A_i^{(-)}(\vec{x}) \otimes\, P_- + \frac{i}{2}\, a_i(\vec{x})\, \Gamma_{N+1} &(2.11)\\
\mathcal{A}_I &= \varphi(\vec{x}) \otimes \frac{1}{2}\, P_+\, \Gamma_I - \varphi(\vec{x})^\dagger \otimes \frac{1}{2}\, P_-\, \Gamma_I\,, &(2.12)
\end{aligned}$$

where now $P_\pm$ are the $2^{\frac{N}{2}} \times 2^{\frac{N}{2}}$ projection operators

$$P_\pm = \frac{1}{2}\left(\mathbb{I} \pm \Gamma_{N+1}\right)\,. \tag{2.13}$$

In (2.11), the residual gauge connections $A_i^{(\pm)}$ are anti-Hermitian and traceless $2^{\frac{D}{2}} \times 2^{\frac{D}{2}}$ arrays, and the Abelian connection a_i results directly from the trace of the field B_i. The $2^{\frac{D}{2}} \times 2^{\frac{D}{2}}$ "Higgs" field φ in (2.12) is neither Hermitian nor anti-Hermitian. Again, to achieve the desired breaking of the gauge group, to lead eventually to the requisite Higgs *isomultiplet*, we choose the gauge group in the bulk to be $SU(n-1)$, where $2n = D + N$.

The components of the curvaturs are readily calculated to give

$$\begin{aligned}
\mathcal{F}_{ij} &= F_{ij}^{(+)}(\vec{x}) \otimes\, P_+ + F_{ij}^{(-)}(\vec{x}) \otimes\, P_- + \frac{i}{2}\, f_{ij}(\vec{x})\, \Gamma_{N+1} &(2.14)\\
\mathcal{F}_{iI} &= D_i\varphi(\vec{x}) \otimes \frac{1}{2}\, P_+\Gamma_I - D_i\varphi^\dagger(\vec{x}) \otimes \frac{1}{2}\, P_-\Gamma_I &(2.15)\\
\mathcal{F}_{IJ} &= S^{(+)}(\vec{x}) \otimes\, P_+\Gamma_{IJ} + S^{(-)}(\vec{x}) \otimes\, P_-\Gamma_{IJ}\,, &(2.16)
\end{aligned}$$

the curvatures in (2.14) being defined by

$$F_{ij}^{(\pm)} = \partial_i A_j^{(\pm)} - \partial_j A_i^{(\pm)} + [A_i^{(\pm)}, A_j^{(\pm)}] \tag{2.17}$$

$$f_{ij} = \partial_i a_j - \partial_j a_i \,. \tag{2.18}$$

The covariant derivative in (2.15) now is defined as

$$D_i\varphi = \partial_i\varphi + A_i^{(+)}\,\varphi - \varphi\, A_i^{(-)} + i\, a_i\, \varphi \tag{2.19}$$

$$D_i\varphi^\dagger = \partial_i\varphi^\dagger + A_i^{(-)}\,\varphi^\dagger - \varphi^\dagger\, A_i^{(+)} - i\, a_i\, \varphi^\dagger \,, \tag{2.20}$$

and the quantities $S^{(\pm)}$ in (2.16) are

$$S^{(+)} = \varphi\,\varphi^\dagger - \eta^2 \quad , \quad S^{(-)} = \varphi^\dagger\,\varphi - \eta^2 \,. \tag{2.21}$$

In what follows, we will suppress the Abelian field a_i, since only when less stringent symmetry than spherical is imposed is it that it would contribute. In any case, using the formal replacement

$$A_i^{(\pm)} \leftrightarrow A_i^{(\pm)} \pm \frac{i}{2}\, a_i\, \mathrm{1\!I}$$

yields the algebraic results to be derived below, in the general case.

We now refine our calculus of descent over even codimensions further. We see from (2.11) that $A_i^{(\pm)}$ being $2^{\frac{D}{2}} \times 2^{\frac{D}{2}}$ arrays, that they can take their values in the two chiral representations, repectively, of the algebra of $SO(D)$. It is therefore natural to introduce the full $SO(D)$ connection

$$A_i = \begin{bmatrix} A_i^{(+)} & 0 \\ 0 & A_\mu^{(-)} \end{bmatrix} . \tag{2.22}$$

Next, we define the $D-$component *isovector* Higgs field

$$\Phi = \begin{bmatrix} 0 & \varphi \\ -\varphi^\dagger & 0 \end{bmatrix} = \phi^i\, \Gamma_{i,D+1} \tag{2.23}$$

in terms of the Dirac matrix representation of the algebra of $SO(D+1)$, with $\Gamma_{i,D+1} = -\frac{1}{2}\Gamma_{D+1}\Gamma_i$.

Note here the formal equivalence between the Higgs multiplet (2.23) in even D, to the corresponding one (2.4) in odd D. This formal equivalence turns out to be very useful in the calulus employed in following Sections. In contrast with the former case of odd D however, the form (2.23) for even D is much more restrictive. This is because in this case the Higgs multiplet is restricted to take its values in the components $\Gamma_{i,D+1}$ orthogonal to the elements Γ_{ij} of $SO(D)$ by definition, irrespective of what symmetry is imposed. It is clear that relaxing the spherical symmetry here, does not result in Φ getting out of the orthogonal complement of Γ_{ij}, when D is even.

From (2.22), follows the $SO(D)$ curvature

$$F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j] = \begin{bmatrix} F_{ij}^{(+)} & 0 \\ 0 & F_{ij}^{(-)} \end{bmatrix} \tag{2.24}$$

and from (2.22) and (2.23) follows the covariant derivative

$$D_\mu\Phi = \partial_i\Phi + [A_i, \Phi] = \begin{bmatrix} 0 & D_i\varphi \\ -D_i\varphi^\dagger & 0 \end{bmatrix} . \tag{2.25}$$

From (2.23) there simply follows the definition of S for even D

$$S = -(\eta^2\, \mathrm{1\!I} + \Phi^2) = \begin{bmatrix} S^{(+)} & 0 \\ 0 & S^{(-)} \end{bmatrix} . \tag{2.26}$$

3 New Chern–Simons terms

First, we recall the usual dynamical Chern–Simons in odd dimensions defined in terms of the non-Abelian gauge connection. Topologically massive gauge field theories in $2+1$ dimensional spacetimes were first introduced in [7, 8]. The salient feature of these theories is the presence of a Chern-Simons (CS) dynamical term. To define a CS density one needs to have a gauge connection, and hence also a curvature. Thus, CS densities can be defined both for Abelian (Maxwell) and non-Abelian (Yang–Mills) fields. They can also be defined for the gravitational [9] field since in that system too one has a (Levi-Civita or otherwise) connection, akin to the Yang-Mills connection in that it carries frame indices analogous to the isotopic indices of ther YM connection. Here we are interested exclusively in the (non-Abelian) YM case, in the presence of an *isovector* valued Higgs field.

The definition of a Chern-Simons (CS) density follows from the definition of the corresponding Chern-Pontryagin (CP) density (1.1). As stated by (1.2), this quantity is a total divergence and the density $\mathbf{\Omega}^{(n)} = \Omega^{(n)}_M$ ($M = 1, 2, \dots, 2n$) in that case has $(2n)-$components. The Chern-Simons density is then defined as one fixed component of $\mathbf{\Omega}^{(n)}$, say the $2n-$th component,

$$\Omega^{(n)}_{\rm CS} = \Omega^{(n)}_{2n} \tag{3.1}$$

which now is given in one dimension less, where $M = \mu, 2n$ and $\mu = 1, 2, \dots (2n-1)$.

This definition of a (dynamical) CS term holds in all odd dimensional spacetimes $(t, \mathbb{R}^D)$, with $x_\mu = (x_0, x_i)$, $i = 1, 2, \dots, D$, with D being an even integer. That D must be even is clear since $D + 2 = 2n$, the $2n$ dimensions in which the CP density (1.1) is defined, is itself even.

The properties of CS densities are reviewed in [10]. Most remarkably, CS densities are defined in odd (space or spacetime) dimensions and are *gauge variant*. The context here is that of a $(2n-1)-$dimensional Minkowskian space. It is important to realise that dynamical Chern-Simons theories are defined on spacetimes with Minkowskian signature. The reason is that the usual CS densities appearing in the Lagrangian are by construction *gauge variant*, but in the definition of the energy densities the CS term itself does not feature, resulting in a Hamiltonian (and hence energy) being *gauge invariant* as it should be [3].

Of course, the CP densities and the resulting CS densities, can be defined in terms of both Abelian and non-Abelian gauge connections and curvatures. The context of the present notes is the construction of soliton solutions [4], unlike in [7, 8]. Thus in any given dimension, our choice of gauge group must be made with due regard to regularity, and the models chosen must be consistent with the Derrick scaling requirement for the finiteness of energy. Accordingly, in all but $2+1$ dimensions, our considerations are restricted to non-Abelian gauge fields.

Clearly, such constructions can be extended to all odd dimensional spacetimes systematically. We list $\Omega_{\rm CS}$, defind by (3.1), for $D = 2, 4, 6$, familiar densities

$$\Omega^{(2)}_{\rm CS} = \varepsilon_{\lambda\mu\nu} {\rm Tr}\, A_\lambda \left[F_{\mu\nu} - \frac{2}{3} A_\mu A_\nu\right] \tag{3.2}$$

$$\Omega^{(3)}_{\rm CS} = \varepsilon_{\lambda\mu\nu\rho\sigma} {\rm Tr}\, A_\lambda \left[F_{\mu\nu}F_{\rho\sigma} - F_{\mu\nu}A_\rho A_\sigma + \frac{2}{5} A_\mu A_\nu A_\rho A_\sigma\right] \tag{3.3}$$

$$\begin{aligned}\Omega^{(4)}_{\rm CS} = \varepsilon_{\lambda\mu\nu\rho\sigma\tau\kappa} {\rm Tr}\, A_\lambda \bigg[&F_{\mu\nu}F_{\rho\sigma}F_{\tau\kappa} - \frac{4}{5}F_{\mu\nu}F_{\rho\sigma}A_\tau A_\kappa - \frac{2}{5}F_{\mu\nu}A_\rho F_{\sigma\tau}A_\kappa \\ &+ \frac{4}{5}F_{\mu\nu}A_\rho A_\sigma A_\tau A_\kappa - \frac{8}{35}A_\mu A_\nu A_\rho A_\sigma A_\tau A_\kappa\bigg] .\end{aligned} \tag{3.4}$$

[3]Should one employ a CS density on a space with Euclidean signature, with the CS density appearing in the static Hamiltonian itself, then the energy would not be *gauge invariant*. Hamiltonians of this type have been considered in the literature, *e.g.*, in [11]. Chern-Simons densities on Euclidean spaces, defined in terms of the composite connection of a sigma model, find application as the topological charge densities of Hopf solitons.

[4]The term soliton solutions here is used rather loosely, implying only the construction of regular and finite energy solutions, without insisting on topological stability in general.

Concerning the choice of gauge groups, one notes that the CS term in $D+1$ dimensions features the product of D powers of the (algebra valued) gauge field/connection infront of the Trace, which would vanish if the gauge group *is not larger than* $SO(D)$. In that case, the YM connection would describe only a 'magnetic' component, with the 'electric' component necessary for the the nonvanishing of the CS density would be absent. As in [12], the most convenient choice is $SO(D+2)$. Since D is always even, the representation of $SO(D+2)$ are the *chiral* representation in terms of (Dirac) spin matrices. This completes the definition of the usual non-Abelian Chern-Simons densities in $D+1$ spacetimes.

From (3.2)-(3.4), it is clear that the CS density is *gauge variant*. The Euler"=Lagrange equations of the CS density is nonetheless *gauge invariant*, such that for the exaples (3.2)-(3.4) the corresponding arbitarry variations are

$$\delta_{A_\lambda}\Omega_{\rm CS}^{(2)} = \varepsilon_{\lambda\mu\nu}F_{\mu\nu} \tag{3.5}$$

$$\delta_{A_\lambda}\Omega_{\rm CS}^{(3)} = \varepsilon_{\lambda\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \tag{3.6}$$

$$\delta_{A_\lambda}\Omega_{\rm CS}^{(4)} = \varepsilon_{\lambda\mu\nu\rho\sigma\kappa\eta}F_{\mu\nu}F_{\rho\sigma}F_{\kappa\eta}\,. \tag{3.7}$$

This, and other interesting properties of CS densities are given in [10]. A remarkable property of a CS density is its transformation under the action of an element, g, of the (non-Abelian) gauge group. We list these for the two examples (3.2)-(3.3),

$$\Omega_{\rm CS}^{(2)}\to\tilde\Omega_{\rm CS}^{(2)} = \Omega_{\rm CS}^{(2)}-\frac{2}{3}\varepsilon_{\lambda\mu\nu}{\rm Tr}\,\alpha_\lambda\alpha_\mu\alpha_\nu-2\varepsilon_{\lambda\mu\nu}\,\partial_\lambda{\rm Tr}\,\alpha_\mu\,A_\nu \tag{3.8}$$

$$\begin{aligned}\Omega_{\rm CS}^{(3)}\to\tilde\Omega_{\rm CS}^{(3)} &= \Omega_{\rm CS}^{(3)}-\frac{2}{5}\,\varepsilon_{\lambda\mu\nu\rho\sigma}{\rm Tr}\,\alpha_\lambda\alpha_\mu\alpha_\nu\alpha_\rho\alpha_\sigma \\ &+2\,\varepsilon_{\lambda\mu\nu\rho\sigma}\,\partial_\lambda{\rm Tr}\,\alpha_\mu\Big[A_\nu\left(F_{\rho\sigma}-\frac{1}{2}A_\rho A_\sigma\right)+\left(F_{\rho\sigma}-\frac{1}{2}A_\rho A_\sigma\right)A_\nu \\ &\qquad -\frac{1}{2}\,A_\nu\,\alpha_\rho\,A_\sigma-\alpha_\nu\,\alpha_\rho\,A_\sigma\Big]\,,\end{aligned} \tag{3.9}$$

where $\alpha_\mu=\partial_\mu g\,g^{-1}$, as distinct from the algebra valued quantity $\beta_\mu=g^{-1}\,\partial_\mu g$ that appears as the inhomogeneous term in the gauge transformation of the non-Abelian curvature (in our convention).

As seen from (3.8)-(3.9), the gauge variation of $\Omega_{\rm CS}$ consists of a term which is explicitly a total divergence, and, another term

$$\omega^{(n)}\simeq\varepsilon_{\mu_1\mu_2\dots\mu_{2n-1}}{\rm Tr}\,\alpha_{\mu_1}\alpha_{\mu_2}\dots\alpha_{\mu_{2n-1}}\,, \tag{3.10}$$

which is *effectively total divergence*, and in a concrete group representation parametrisation becomes *explicitly total divergence*. This can be seen by subjecting (3.10) to variations with respect to the function g, and taking into account the Lagrange multiplier term resulting from the (unitarity) constraint $g^\dagger\,g=g\,g^\dagger=1\!\!{\rm I}$.

The volume integaral of the CS density then transforms under a gauge transformation as follows. Given the appropriate asymptotic decay of the connection (and hence also the curvature), the surface integrals in (3.8)-(3.9) vanish. The only contribution to the gauge variation of the CS action/energy then comes from the integral of the density (3.10), which (in the case of Euclidean signature) for the appropriate choice of gauge group yields an integer, up to the angular volume as a multiplicative factor.

All above stated properties of the Chern-Simons (CS) density hold irrespective of the signature of the space. Here, the signature is taken to be Minkowskian, such that the CS density in the Lagrangian does not contribute to the energy density directly. As a consequence the energy of the soliton is gauge invariant and does not suffer the gauge transformation (3.8)-(3.9). Should a CS density be part of a static Hamiltonian (on a space of Euclidean signature), then the energy of the soliton would change by a multiple of an integer.

3.1 New Chern–Simons terms in all dimensions

The plan to introduce a completely new type of Chern-Simons term. The usual CS densities $\mathbf{\Omega}_{\rm CS}^{(n)}$, (3.1), are defined with reference to the total divergence expression (1.2) of the $n-$th Chern-Pontryagin density (1.1), as the $2n-$th component $\Omega_{2n}^{(n)}$ of the density $\mathbf{\Omega}^{(n)}$. Likewise, the new CS terms are defined with reference to the total divergence expression (1.3) of the dimensionally reduced $n-$th CP density, with the dimension D of the residual space replaced formally by $\bar{D}$

$$\mathcal{C}_{\bar{D}}^{(n)} = \nabla \cdot \mathbf{\Omega}^{(n,\bar{D})} \,. \tag{3.11}$$

The densities $\mathbf{\Omega}^{(n,\bar{D})}$ can be read off from $\mathbf{\Omega}^{(n,D)}$ given in Section **5**, with the formal replacement $D \to \bar{D}$. The new CS term is now identified as the $\bar{D}-$th component of $\mathbf{\Omega}^{(n,\bar{D})}$. The final step in this identification is to assign the value $\bar{D} = D + 2$, where D is the spacelike dimension of the $D+1$ dimensional Minkowski space, with the new Chern-Simons term defined as

$$\tilde{\Omega}_{\rm CS}^{(n,D+1)} \stackrel{def}{=} \Omega_{D+2}^{(n,D+2)} \tag{3.12}$$

The departure of the new CS densitiess from the usual CS densities is stark, and these differ in several essential respects from the usual ones described in the previous subsection. The most important new features in question are

- The field content of the new CS systems includes Higgs fields in addition to the Yang-Mills fields, as a consequence of the dimensional reduction of gauge fields described in Section **4**. It should be emphasised that the appearance of the Higgs field here is due to the imposition of symmetries in the descent mechanism, in contrast with its presence in the models [13, 14, 15] supporting $2+1$ dimensional CS vortices, where the Higgs field was introduced by hand with the expedient of satisfying the Derrick scaling requirement.

- The usual dynamical CS densities defined with reference to the $n-$th CP density live in $2n-1$ dimensional Minkowski space, *i.e.*, only in odd dimensional spacetime. By contrast, the new CS densities defined with reference to the $n-$th CP densities live in $D+1$ dimensional Minkowski space, for all D subject to
$$2n - 2 \ge D \ge 2 \,, \tag{3.13}$$
i.e., in both odd, as well as even dimensions. Indeed, in any given D there is an infinite tower of new CS densities characterised by the integer n subject to (3.13). This is perhaps the most important feature of the new CS densities.

- The smallest simple group consistent with the nonvanishing of the *usual* CS density in $2n-1$ dimensional spacetime is $SO(2n)$, with the gauge connection taking its values in the *chiral* Dirac representation. By contrast, the gauge groups of the new CS densities in $D+1$ dimensional spacetime are fixed by the prescription of the dimensional descent from which they result. As *per* the prescription of descent described in Section **4**, the gauge group now will be $SO(D+2)$, independently of the integer n, while the Higgs field takes its values in the orthogonal complement of $SO(D+2)$ in $SO(D+3)$. As such, it forms an iso-$(D+2)-$vector multiplet.

- Certain properties of the new CS densities are remarkably different for D even and D odd.
  - Odd D: Unlike in the usual case (3.2)-(3.3), the new CS terms are *gauge invariant*. The gauge fields are $SO(D+2)$ and the Higgs are in $SO(D+3)$. D being odd, $D+3$ is even and hence the fields can be parametrised with respect to the *chiral* (Dirac) representations of $SO(D+3)$. An important consequence of this is the fact that now, both (electric) A_0 and (magnetic) A_i fields lie in the same isotopic multiplets, in contrast to the *pseudo*$-$dyons described in the previous section.
  - Even D: The new CS terms now consist of a *gauge variant* part expressed only in terms of the gauge field, and a *gauge invariant* part expressed in terms of both gauge and Higgs fields. The leading, *gauge variant*, term differs from the corresponding usual CS terms (3.2)-(3.3) only due to the presence of a (chiral) Γ_{D+3} matrix infront of the Trace. The gauge and Higgs fields are

again in $SO(D+2)$ and in $SO(D+3)$ respectively, but now, D being even $D+3$ is odd and hence the fields are parametrised with respect to the (chirally doubled up) full Dirac representations of $SO(D+3)$. Hence the appearance of the chiral matrix infront of the Trace.

As in the usual CS models, the regular finite energy solutions of the new CS models are not topologically stable. These solutions can be constructed numerically.

Before proceeding to display some typcal examples in the Subsection following, it is in order to make a small diversion at this point to make a clarification. The new CS densities proposed are functionals of both the Yang–Mills, and, the "isovector" Higgs field. Thus, the systems to be described below are Chern-Simons–Yang-Mills-Higgs models in a very specific sense, namely that the Higgs field is an intrinsic part of the new CS density. This is in contrast with Yang-Mills–Higgs-Chern-Simons or Maxwell–Higgs-Chern-Simons models in $2+1$ dimensional spacetimes that have appeared ubiquitously in the literature. It is important to emphasise that the latter are entirely different from the systems introduced here, simply because the CS densities they employ are the *usual* ones, namely (3.2) or more often its Abelian [5] version

$$\Omega^{(2)}_{\rm U(1)} = \varepsilon_{\lambda\mu\nu}\, A_\lambda F_{\mu\nu}\ ,$$

while the CS densities employed here are *not* simply functionals of the gauge field, but also of the (specific) Higgs field. To put this in perspective, let us comment on the well known *Abelian* CS-Higgs solitons in $2+1$ dimensions constructed in [13, 14] support self-dual vortices, which happen to be unique inasfar as they are also topologically stable. (Their non-Abelian counterparts [15] are not endowed with topological stability.) The presence of the Higgs field in [13, 14, 15] enables the Derrick scaling requirement to be satisfied by virtue of the presence of the Higgs self-interaction potential. In the Abelian case in addition, it results in the topological stability of the vortices. If it were not for the topological stability, it would not be necessary to have a Higgs field merely to satisfy the Derrick scaling requirement. That can be achieved instead, *e.g.*, by introducing a negative cosmological constant and/or gravity, as was done in the $4+1$ dimensional case studied in [12]. Thus, the involvement of the Higgs field in conventional (*usual*) Chern-Simons theories is not the only option. The reason for emphasising the optional status of the Higgs field in the usual $2+1$ dimensional Chern-Simons–Higgs models is, that in the new models proposed here the Higgs field is intrinsic to the definition of the (new) Chern-Simons density itself.

3.2 Examples

As discussed above, the new dynamical Chern-Simons densities

$$\tilde\Omega^{(n,D+1)}_{\rm CS}[A_\mu,\Phi]$$

are characterised by the dimensionality of the space D and the integer n specifying the dimension $2n$ of the bulk space from which the relevant residual system is arrived at.

The case $n=2$ is empty, since according to (3.13) the largest spacetime in which a new CS density can be constructed is $2n-2$, *i.e.*, in $1+1$ dimensional Minkowsky space which we ignore.

The case $n=3$ is not empty, and affords two nontrivial examples. The largest spacetime $2n-2$, in which a new CS density can be constructed in this case is $3+1$ and the next in $2+1$ Minkowski space. These, are, repectively,

$$\tilde\Omega^{(3,3+1)}_{\rm CS} = \varepsilon_{\mu\nu\rho\sigma}\,{\rm Tr}\,F_{\mu\nu}\,F_{\rho\sigma}\,\Phi \qquad (3.14)$$

$$\tilde\Omega^{(3,2+1)}_{\rm CS} = \varepsilon_{\mu\nu\lambda}\,{\rm Tr}\,\gamma_5\left[-2\eta^2 A_\lambda\left(F_{\mu\nu}-\frac{2}{3}A_\mu A_\nu\right)+(\Phi\,D_\lambda\Phi-D_\lambda\Phi\,\Phi)\,F_{\mu\nu}\right]. \qquad (3.15)$$

[5]There are, of course, Abelian CS densities in all odd spacetime dimensions but these do not concern us here since in all $D+1$ dimensions with $D=2n\geq 4$, no regular solitons can be constructed.

The case $n=4$ affords four nontrivial examples, those in $5+1$, $4+1$, $3+1$ and $2+1$ Minkowski space. These are, repectively,

$$\tilde{\Omega}_{\rm CS}^{(4,5+1)} = \varepsilon_{\mu\nu\rho\sigma\tau\lambda}\,\mathrm{Tr}\;F_{\mu\nu}\,F_{\rho\sigma}\,F_{\tau\lambda}\,\Phi \tag{3.16}$$

$$\tilde{\Omega}_{\rm CS}^{(4,4+1)} = \varepsilon_{\mu\nu\rho\sigma\lambda}\,\mathrm{Tr}\,\Gamma_7\bigg[A_\lambda\left(F_{\mu\nu}F_{\rho\sigma}-F_{\mu\nu}A_\rho A_\sigma+\frac{2}{5}A_\mu A_\nu A_\rho A_\sigma\right) +D_\lambda\Phi\left(\Phi F_{\mu\nu}F_{\rho\sigma}+F_{\mu\nu}\Phi F_{mn}+F_{\mu\nu}F_{mn}\Phi\right)\bigg] \tag{3.17}$$

$$\tilde{\Omega}_{\rm CS}^{(4,3+1)} = \varepsilon_{\mu\nu\rho\sigma}\,\mathrm{Tr}\bigg[\Phi\left(\eta^2\,F_{\mu\nu}F_{\rho\sigma}+\frac{2}{9}\,\Phi^2\,F_{\mu\nu}F_{\rho\sigma}+\frac{1}{9}\,F_{\mu\nu}\Phi^2F_{\rho\sigma}\right) -\frac{2}{9}\left(\Phi D_\mu\Phi D_\nu\Phi-D_\mu\Phi\Phi D_\nu\Phi+D_\mu\Phi D_\nu\Phi\Phi\right)F_{\rho\sigma}\bigg] \tag{3.18}$$

$$\tilde{\Omega}_{\rm CS}^{(4,2+1)} = \varepsilon_{\mu\nu\lambda}\,\mathrm{Tr}\,\Gamma_5\bigg\{6\eta^4\,A_\lambda\left(F_{\mu\nu}-\frac{2}{3}\,A_\mu\,A_\nu\right) -6\,\eta^2\left(\Phi\,D_\lambda\Phi-D_\lambda\Phi\,\Phi\right)\,F_{\mu\nu} +\left[\left(\Phi^2\,D_\lambda\Phi\,\Phi-\Phi\,D_\lambda\Phi\,\Phi^2\right)-2\left(\Phi^3\,D_\lambda\Phi-D_\lambda\Phi\,\Phi^3\right)\right]F_{\mu\nu}\,.\bigg\} \tag{3.19}$$

It is clear that in any $D+1$ dimensional spacetime an infinite tower of CS densities $\tilde{\Omega}_{\rm CS}^{(n,D+1)}$ can be defined, for all positive integers n. Of these, those in even dimensional spacetimes are gauge invariant, *e.g.*, (3.14), (3.16) and (3.18), while those in odd dimensional spacetimes are gauge variant, *e.g.*, (3.15), (3.17) and (3.19), the gauge variations in these cases being given formally by (3.8) and (3.9), with g replaced by the appropriate gauge group here.

Static soliton solutions to models whose Lagrangians consist of the above introduced types of CS terms together with Yang-Mills–Higgs (YMH) terms are currently under construction. The only constraint in the choice of the detailed models employed is the requirement that the Derrick scaling requirement be satisfied. Such solutions are constructed numerically. In contrast to the monopole solutions, they are not endowed with topological stability because the gauge group must be larger than $SO(D)$, for which the solutions to the constituent YMH model is a stable monopole. Otherwise the CS term would vanish.

Acknowledgement

This work is carried out in the framework of Science Foundation Ireland (SFI) project RFP07-330PHY.

References

[1] P.A.M. Dirac, Proc. Roy. Soc. A **133** (1931) 60.

[2] C. N. Yang, J. Math. Phys. **19** (1978) 320.

[3] T. Tchrakian, Phys. Atom. Nucl. **71** (2008) 1116.

[4] A. S. Schwarz, Commun. Math. Phys. **56** (1977) 79.

[5] V. N. Romanov, A. S. Schwarz and Yu. S. Tyupkin, Nucl. Phys. B **130** (1977) 209.

[6] A. S. Schwarz and Yu. S. Tyupkin, Nucl. Phys. B **187** (1981) 321.

[7] S. Deser, R. Jackiw and S. Templeton, Phys. Rev. Lett. **48** (1982) 975.

[8] S. Deser, R. Jackiw and S. Templeton, Annals Phys. **140** (1982) 372 [Erratum-ibid. **185** (1988) 406] [Annals Phys. **185** (1988) 406] [Annals Phys. **281** (2000) 409].

[9] R. Jackiw and S. Y. Pi, Phys. Rev. D **68** (2003) 104012 [arXiv:gr-qc/0308071].

[10] R. Jackiw, "Chern-Simons terms and cocycles in physics and mathematics", in E.S. Fradkin *Festschrift*, Adam Hilger, Bristol (1985)

[11] V. A. Rubakov and A. N. Tavkhelidze, Phys. Lett. B **165** (1985) 109.

[12] Y. Brihaye, E. Radu and D. H. Tchrakian, Phys. Rev. D **81** (2010) 064005 [arXiv:0911.0153 [hep-th]].

[13] J. Hong, Y. Kim and P. Y. Pac, Phys. Rev. Lett. **64** (1990) 2230.

[14] R. Jackiw and E. J. Weinberg, Phys. Rev. Lett. **64** (1990) 2234.

[15] F. Navarro-Lerida and D. H. Tchrakian, Phys. Rev. D **81** (2010) 127702 [arXiv:0909.4220 [hep-th]].