CFTs on Riemann Surfaces of genus $g \geq 1$: Dependence on moduli

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Abstract

The dependence of the Virasoro-N-point function on the moduli of the Riemann surface is investigated. We propose an algebraic geometric approach that applies to any hyperelliptic Riemann surface. Our discussion includes a demonstration of our methods with the case $g = 1$.

1 Introduction

The present article continues the program of determining the N-point functions $\langle\phi_1 \ldots \phi_N\rangle$ of holomorphic fields of rational conformal field theories (RCFTs) on arbitrary Riemann surfaces. In order to actually compute these functions and more specifically the partition function $\langle\mathbf{1}\rangle$ for $N = 0$, one has to study their behaviour under changes of the conformal structure. This is done conveniently by first considering arbitrary changes of the metric. Such a change of $\langle\phi_1 \ldots \phi_N\rangle$ is described by the corresponding $(N + 1)$-point function containing a copy of the Virasosoro field T. For this reason we have previously investigated the N-point functions of T (rather than of more general fields) [11]. In the present paper we study functions on the *moduli space* $\mathcal{M}_g$, which is the space of all possible conformal structures on the genus g surface. For the RCFTs one obtains functions which are meromorphic on a compactification of $\mathcal{M}_g$ or of a finite cover. We shall use that conformal structures occur as equivalence classes of metrics, with equivalent metrics being related by Weyl transformations. The N-point functions of a CFT do depend on the Weyl transformation, but only in a way which can be described by a universal automorphy factor.

For $g = 1$ this can be made explicit as follows. The Riemann surfaces can be described as quotients $\mathbb{C}/\Lambda$, with a lattice Λ generated over $\mathbb{Z}$ by 1 and τ with $\tau \in \mathbb{H}^+$. The upper half plane $\mathbb{H}^+$ is the universal cover of $\mathcal{M}_1$, in other words its Teichmüller space. One has $\mathcal{M}_1 = SL(2, \mathbb{Z}) \setminus \mathbb{H}^+$. Meromorphic functions on finite covers of $\mathcal{M}_1$ are called (weakly) *modular*. They can be described as functions on $\mathbb{H}^+$ which are invariant under a subgroup of $SL(2,\mathbb{Z})$ of finite index. We shall refer to $SL(2,\mathbb{Z})$ as the *full modular group*.

Maps in $SL(2,\mathbb{Z})$ preserve the standard lattice $\mathbb{Z}^2$ together with its orientation and so descend to self-homeomorphisms of the torus. Inversely, every self-homeomorphism of the torus is isotopic to such a map. A modular function is a function on the space $\mathcal{L}$ of all lattices in $\mathbb{C}$ satisfying [16]

$$f(\lambda\Lambda) = f(\Lambda)\,, \quad \forall\Lambda \in \mathcal{L},\ \lambda \in \mathbb{C}^*\,. \tag{1}$$

$\mathcal{L}$ can be viewed as the space of all tori with a flat metric.

Conformal field theories on the torus provide many interesting modular functions, and modular forms. (The latter transform as $f(\lambda\Lambda) = \lambda^{-k} f(\Lambda)$ for some $k \in \mathbb{Z}$ which is specific to f, called the weight of f.)

Little work has been done so far on analogous functions for $g > 1$. Our work develops methods in this direction. The basic idea is that many of the relevant functions are algebraic. In order to proceed step by step, we will restrict our investigations to the locus of hyperelliptic curves, though the methods work in more general context as well.

For an important class of CFTs (the minimal models), the zero-point functions $\langle\mathbf{1}\rangle$ will turn out to solve a linear differential equation so that $\langle\mathbf{1}\rangle$ can be computed for arbitrary hyperelliptic Riemann surfaces. Since $\langle\mathbf{1}\rangle$ is algebraic (namely a meromorphic function on a finite covering of the moduli space), it is clear a priori that the equation can not be solved numerically only, but actually analytically.

The present exposition is a partial and preliminary version of on joint paper with W. Nahm [12]. Sect. 4.3 is based on his ideas.

2 Notations and conventions

Let $\mathbb{H}^+ := \{z \in \mathbb{C}|\ \Im(z) > 0\}$ be the complex upper half plane. $\mathbb{H}^+$ is acted upon by the *full modular group* $\Gamma_1 = SL(2,\mathbb{Z})$ with fundamental domain

$$\mathcal{F} := \left\{ z \in \mathbb{H}^+ \middle|\ |z| > 1,\ \Re(z) < \frac{1}{2} \right\}\,.$$

The operation of Γ_1 on $\mathbb{H}^+$ is not faithful whence we shall also consider the *modular group* $\overline{\Gamma}_1 := \Gamma_1/\{\pm I_2\} = PSL(2,\mathbb{Z})$, (here I_2 is the 2×2 identity matrix). We refer to S, T as the generators of Γ_1 (or of $\overline{\Gamma}_1$) given by the transformations

$$\begin{aligned} S &: z \mapsto -1/z \\ T &: z \mapsto z+1\,. \end{aligned}$$

We shall use the standard notations for G_{2k}, E_{2k} (cf. [16]). The *Dedekind η function* is

$$\eta(\tau) := q^{\frac{1}{24}}(q)_\infty = q^{\frac{1}{24}}\left(1 - q + q^2 + q^5 + q^7 + \ldots\right)\,.$$

$(q)_n := \prod_{k=1}^n (1-q^k)$ is the q-Pochhammer symbol.

$\langle\mathbf{1}\rangle$ and $\langle\mathbf{T}\rangle$ (or $\mathbf{A}_1$) are parameters of central importance to this exposition. For better readibility, they appear in bold print throughout.

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3 Differential equations for characters in $(2, \nu)$-minimal models

3.1 Introduction

Characters of conformal field theories are modular functions. A *modular function on* a discrete subgroup Γ of Γ_1 is a Γ-invariant meromorphic function $f : \mathbb{H}^+ \to \mathbb{C}$ with at most exponential growth towards the boundary [16]. For $N \geq 1$, the *principal conguence subgroup* is the group $\Gamma(N)$ such that the short sequence

$$1 \to \Gamma(N) \hookrightarrow \Gamma_1 \xrightarrow{\pi_N} SL(2, \mathbb{Z}/N\mathbb{Z}) \to 1$$

is exact, where π_N is map given by reduction modulo N. A function that is modular on $\Gamma(N)$ is said to be *of level* N. Let $\zeta_N = e^{\frac{2\pi i}{N}}$ be the N-th root of unity with cyclotomic field $\mathbb{Q}(\zeta_N)$. Let F_N be the field of modular functions f of level N which have a Fourier expansion

$$f(\tau) = \sum_{n \geq -n_0} a_n q^{\frac{n}{N}} , \quad q = e^{2\pi i \tau} , \tag{2}$$

with $a_n \in \mathbb{Q}(\zeta_N)$, $\forall n$. The Ramanujan continued fraction

$$r(\tau) := q^{1/5} \frac{1}{1 + \frac{q}{1 + \frac{q^2}{1 + \dots}}} \tag{3}$$

which converges for $\tau \in \mathbb{H}^+$, is an element (and actually a generator) of F_5 [2]. r is algebraic over F_1 which is generated over $\mathbb{Q}$ by the *modular j-function*,

$$j(\tau) = 12^3 \frac{g_2^3}{g_2^3 - 27 g_3^2} .$$

j is associated to the elliptic curve with the affine equation

$$X_{g=1} : \quad y^2 = 4x^3 - g_2 x - g_3 , \quad \text{with} \quad g_2^3 - 27 g_3^2 \neq 0 .$$

Here g_k for $k = 2, 3$ are (specific) modular forms of weight $2k$, so that j is a function of the respective modulus only (the quotient $\tau = \omega_2/\omega_1$ for the lattice $\Lambda = \mathbb{Z}.\omega_1 + \mathbb{Z}.\omega_2$), or rather its orbit under Γ_1 (since we are free to change the basis (ω_1, ω_2) for Λ). In terms of the modulus, a *modular form of weight* $2k$ *on* Γ is a holomorphic function $g : \mathbb{H}^+ \to \mathbb{C}$ with subexponential growth towards the boundary [16] such that $g(\tau)(d\tau)^{2k}$ is Γ-invariant [14]. A modular form on Γ_1 allows a Fourier expansion of the form (2) with $n_0 \geq 0$.

Another way to approach modular functions is in terms of the differential equations they satisfy. The derivative of a modular function is a modular form of weight two, and higher derivatives give rise to *quasi-modular forms*, which we shall also deal with though they are not themselves of primary interest to us.

Geometrically, the conformal structure on the $g = 1$ surface is determined by the tupel (X_1, X_2, X_3, ∞) of its ramification points, and we can change this structure by varying the position of X_1, X_2, X_3 infinitesimally. In this picture, the boundary of the moduli space is approached by letting two ramification points in the quadrupel run together [8].

When changing positions we may keep track of the branch points to obtain a simply connected space [5]. Thus a third way to describe modularity of the characters is by means of a subgroup of the braid group B_3 of 3 strands. The latter is the universal central extension of the quotient group $\overline{\Gamma}_1 = \Gamma_1/\{\pm 1\}$, so that we come full circle.

3.2 Review the differential equation for the characters of the $(2,5)$ minimal model

The character $\langle\mathbf{1}\rangle$ of any CFT on $X_{g=1}$ solves the ODE [6]

$$\frac{d}{d\tau}\langle\mathbf{1}\rangle = \frac{1}{2\pi i}\oint\langle T(z)\rangle\,dz = \frac{1}{2\pi i}\langle\mathbf{T}\rangle\,. \tag{4}$$

Here the contour integral is along the real period, and $\oint dz = 1$. $\langle\mathbf{T}\rangle$, while constant in position, is a modular form of weight two in the modulus. The Virasoro field generates the variation of the conformal structure [6]. In the $(2,5)$ minimal model, we have [6]

$$\langle T(z)T(0)\rangle = \frac{c}{12}\langle\mathbf{1}\rangle\wp''(z|\tau) + 2\langle\mathbf{T}\rangle\wp(z|\tau) + -c\langle\mathbf{1}\rangle G_4 \tag{5}$$

where G_4 is the holomophic Eisenstein series of weight 4. So

$$2\pi i\frac{d}{d\tau}\langle\mathbf{T}\rangle = \oint\langle T(w)T(z)\rangle\,dz = -4\langle\mathbf{T}\rangle G_2 + \frac{22}{5}G_4\langle\mathbf{1}\rangle\,. \tag{6}$$

Here G_2 is the quasimodular Eisenstein series of weight 2, which enters the equation by means of the identity

$$\int_0^1 \wp(z-w|\tau)\,dz = -2G_2(\tau).$$

In terms of the *Serre derivative*

$$\mathfrak{D}_{2\ell} := \frac{1}{2\pi i}\frac{d}{d\tau} - \frac{\ell}{6}E_2\,, \tag{7}$$

the first order ODEs (4) and (6) combine to give the second order ODE [10]

$$\mathfrak{D}_2\circ\mathfrak{D}_0\langle\mathbf{1}\rangle = \frac{11}{3600}\,E_4\langle\mathbf{1}\rangle\,.$$

This classical example generalises [17]. The two solutions are the well-known Rogers-Ramanujan partition functions [3]

$$\begin{aligned}
\langle\mathbf{1}\rangle_1 &= q^{\frac{11}{60}}\sum_{n\geq 0}\frac{q^{n^2+n}}{(q)_n} = q^{\frac{11}{60}}\left(1+q^2+q^3+q^4+q^5+2q^6+\ldots\right)\,,\\
\langle\mathbf{1}\rangle_2 &= q^{-\frac{1}{60}}\sum_{n\geq 0}\frac{q^{n^2}}{(q)_n} = q^{-\frac{1}{60}}\left(1+q+q^2+q^3+2q^4+2q^5+3q^6+\ldots\right)\,.
\end{aligned}$$

($q = e^{2\pi i\,\tau}$) which are named after the famous Rogers-Ramanujan identities

$$q^{-\frac{11}{60}}\langle\mathbf{1}\rangle_1 = \prod_{n=\pm 2 \bmod 5}(1-q^n)^{-1}\,,\quad q^{\frac{1}{60}}\langle\mathbf{1}\rangle_2 = \prod_{n=\pm 1 \bmod 5}(1-q^n)^{-1}\,.$$

hold. Mnemotechnically, the distribution of indices seems somewhat unfortunate. In general, however, the characters of the $(2,\nu)$ minimal model, of which there are

$$M = \frac{\nu-1}{2} \tag{8}$$

h	0	1	2	3	4	5	6
basis of $F(h)$	1	–	T	∂T	$\partial^2 T$	$\partial^3 T$	$\partial^4 T$ $N_0(T, \partial^2 T)$
$\dim F(h)$	1	0	1	1	1	1	2

Holomorphic fields in the $(2,5)$ minimal model

(ν odd) many, are ordered by their conformal weight, which is the lowest for the respective *vacuum character* $\langle \mathbf{1} \rangle_1$, having weight zero.

The Rogers-Ramanujan identity for $q^{-\frac{11}{60}} \langle \mathbf{1} \rangle_1$ provides the generating function for the partition which to a given holomorphic dimension $h \geq 0$ returns the number of linearly independent holomorphic fields present in the $(2,5)$ minimal model. This number is subject to the constraint $\partial^2 T \propto N_0(T,T)$.

There is a similar combinatorical interpretation for the second Rogers-Ramanujan identity. It involves non-holomorphic fields, however, which we disregard in this thesis.

3.3 Review the algebraic equation for the characters of the $(2,5)$ minimal model

Besides the analytic approach, there is an algebraic approach to the characters. This is due to the fact that $\langle \mathbf{1} \rangle_1, \langle \mathbf{1} \rangle_2$, rather than being modular on the full modular group, are modular on a subgroup of Γ_1: For the generators S, T of Γ_1 we have [2]

$$T \langle \mathbf{1} \rangle_1 = \zeta_{60}{}^{11} \langle \mathbf{1} \rangle_1 \,, \quad T \langle \mathbf{1} \rangle_2 = \zeta_{60}{}^{-1} \langle \mathbf{1} \rangle_2 \,,$$

while under the operation of S, $\langle \mathbf{1} \rangle_1, \langle \mathbf{1} \rangle_2$ transform into linear combinations of one another [2],

$$S \begin{pmatrix} \langle \mathbf{1} \rangle_1 \\ \langle \mathbf{1} \rangle_2 \end{pmatrix} = \frac{2}{\sqrt{5}} \begin{pmatrix} \sin \frac{\pi}{5} & -\sin \frac{2\pi}{5} \\ \sin \frac{2\pi}{5} & \sin \frac{\pi}{5} \end{pmatrix} \begin{pmatrix} \langle \mathbf{1} \rangle_1 \\ \langle \mathbf{1} \rangle_2 \end{pmatrix} .$$

However, $\langle \mathbf{1} \rangle_1, \langle \mathbf{1} \rangle_2$ are modular under a subgroup of Γ_1 of finite index. Its fundamental domain is therefore a finite union of copies of the fundamental domain $\mathcal{F}$ of Γ_1 in $\mathbb{C}$. More specifically, if the subgroup is Γ with index $[\Gamma_1 : \Gamma]$, and if $\gamma_1, \ldots \gamma_{[\Gamma_1:\Gamma]} \in \Gamma_1$ are the coset representatives so that $\Gamma_1 = \Gamma\gamma_1 \cup \ldots \Gamma\gamma_{[\Gamma_1:\Gamma]}$, then we have

$$\mathcal{F} = \gamma_1 \mathcal{F} \cup \ldots \gamma_{[\Gamma_1:\Gamma]} \mathcal{F} \,. \tag{9}$$

[9]. Thus $\langle \mathbf{1} \rangle_1$ and $\langle \mathbf{1} \rangle_2$ define functions on a finite covering of the moduli space $\Gamma_1 \setminus \mathbb{H}^+$ and are algebraic. We can write [2]

$$\langle \mathbf{1} \rangle_1 = \frac{\theta_{5,2}}{\eta} \,, \quad \langle \mathbf{1} \rangle_2 = \frac{\theta_{5,1}}{\eta} \,,$$

where the functions $\eta, \theta_{5,1}, \theta_{5,2}$ on the r.h.s. are specific theta functions,

$$\theta(\tau) = \sum_{n \in \mathbb{Z}} f(n) \,, \quad f(n) \sim q^{n^2} \,, \quad q = e^{2\pi i \tau} \,.$$

The characters' common denominator is the Dedekind η function. Using the Poisson transformation formula, one finds that $\eta, \theta_{5,1}, \theta_{5,2}$ are all modular forms of weight $\frac{1}{2}$ ([16], Propos. 9, p. 25). For the quotient $\langle \mathbf{1} \rangle_1 / \langle \mathbf{1} \rangle_2$ and $\tau \in \mathbb{H}^+$, we find [2],

$$\frac{\langle \mathbf{1} \rangle_1}{\langle \mathbf{1} \rangle_2} = \frac{\theta_{5,2}}{\theta_{5,1}} = q^{\frac{1}{5}} \prod_{n=1}^{\infty} (1 - q^n)^{\left(\frac{n}{5}\right)} = r(\tau) \,,$$

where $r(\tau)$ is the Ramanujan continued fraction introduced in eq. (3). (Here $(n/5) = 1, -1, 0$ for $n = \pm 1, \pm 2, 0 \pmod 5$, respectively, is the Legendre symbol.)

$r(\tau)$ is modular on $\Gamma(5)$ with index $[\Gamma_1 : \Gamma(5)] = 120$ [9]. The quotient $\Gamma(5) \setminus \mathbb{H}^+$ can be compactified and made into a Riemann surface, which is referred to as the modular curve

$$X(5) = \Gamma(5) \setminus \mathbb{H}^* .$$

Here $\mathbb{H}^* := \mathbb{H}^+ \cup \mathbb{Q} \cup \{\infty\}$ is the extended complex upper half plane. $X(5)$ has genus zero and the symmetry of an icosahedron. The rotation group of the sphere leaving an inscribed icosahedron invariant is A_5, the alternating group of order 60. By means of a stereographic projection, the notion of edge center, face center and vertex are induced on the extended complex plane [4]. They are acted upon by the icosahedral group $G_{60} \subset PSL(2, \mathbb{C})$. The face centers and finite vertices define the simple roots of two monic polynomials $F(z)$ and $V(z)$ of degree 20 and 11, respectively, which transform in such a way under G_{60} that

$$J(z) := \frac{F^3(z)}{V^5(z)}$$

is invariant. It turns out that $J(r(\tau))$ for $\tau \in \mathbb{H}^+$ is $\Gamma(1)$-invariant, and in fact that $J(r(\tau)) = j(\tau)$. Thus $r(\tau)$ satisfies

$$F^3(z) - j(\tau)V^5(z) = 0$$

(for the same value of τ), which is equivalent to $r^5(\tau)$ solving the icosahedral equation

$$(X^4 - 228X^3 + 494X^2 + 228X + 1)^3 + j(\tau)X(X^2 + 11X - 1)^5 = 0 .$$

This is actually the minimal polynomial of r^5 over $\mathbb{Q}(j)$, so that $\mathbb{Q}(r)$ defines a function field extension of degree 60 over $\mathbb{Q}(j)$.

This construction which goes back to F. Klein, doesn't make use of a metric. In order to determine the centroid of a face (or of the image of its projection onto the sphere) only the conformal structure on S^2 is required. Indeed, the centroid of a regular polygone is its center of rotations, thus a fixed point under an operation of $Aut(S^2) = SL(2, \mathbb{C})$.

3.4 Higher order modular ODEs

To the $(2, \nu)$ minimal model, where $\nu \geq 3$ is odd, we associate [3]

- the number $M = \frac{\nu-1}{2}$ introduced in eq. (8), which counts the characters,
- the sequence

$$\kappa_s = \frac{(\nu - 2s)^2}{8\nu} - \frac{1}{24} , \quad s = 1, \ldots, M , \tag{10}$$

  which parametrises the characters of the $(2, \nu)$ minimal model,
- the rank $r = \frac{\nu-3}{2}$.

The character corresponding to κ_s is

$$\langle \mathbf{1} \rangle_s = f_{A,B,s} := q^{\kappa_s} \sum_{\mathbf{n} \in (\mathbb{N}_0)^r} \frac{q^{\mathbf{n}^t A \mathbf{n} + \mathbf{B}^t \mathbf{n}}}{(q)_{\mathbf{n}}} . \tag{11}$$

where

$$A = C(T_r)^{-1} \in \mathbb{Q}^{r\times r}, \quad \mathbf{B} \in \mathbb{Q}^r \,,$$

C being a Cartan matrix. The diagram of T_r is the diagram of A_{2r} folded by its $\mathbb{Z}_2$ symmetry.

It turns out that $\langle \mathbf{1} \rangle_s$ satisfies an Mth order ODE [13]. Given M differentiable functions $f_1, \ldots, f_M$ there always exists an ODE having these as solutions. Consider the determinant

$$\det\begin{pmatrix} f & \mathfrak{D}^1 f & \ldots & \mathfrak{D}^M f \\ f_1 & \mathfrak{D}^1 f_1 & \ldots & \mathfrak{D}^M f_1 \\ \ldots & \ldots & \ldots & \ldots \\ f_M & \mathfrak{D}^1 f_M & \ldots & \mathfrak{D}^M f_M \end{pmatrix} =: \sum_{i=0}^{M} w_i \, \mathfrak{D}^i f \,.$$

Here for $m \geq 1$,

$$\mathfrak{D}^m := \mathfrak{D}_{2(m-1)} \circ \cdots \circ \mathfrak{D}_2 \circ \mathfrak{D}_0$$

is the order m differential operator which maps a modular function into a modular form of weight $2m$. ($\mathfrak{D}_k$ is the first order Serre differential operator introduced in eq. (7).) For $m = 0$ we set $\mathfrak{D}^0 = 1$.

Whenever f equals one of the f_i, $1 \leq i \leq M$, the determinant is zero, so we obtain an ODE in f whose coefficients are Wronskian minors containing $f_1, \ldots, f_M$ and their derivatives only. These are modular when the $f_1, \ldots, f_M$ and their derivatives are or when under modular transformation, they transform into linear combinations of one another (as the characters do).

Lemma 1. *Let $3 \leq \nu \leq 13$, ν odd. The characters of the $(2, \nu)$ minimal model satisfy*

$$D^{(2,\nu)} \langle \mathbf{1} \rangle = 0 \,, \tag{12}$$

where $D^{(2,\nu)}$ is the differential operator

$$\begin{aligned} D^{(2,\nu)} &:= \mathfrak{D}^M + \sum_{m=0}^{M-2} \sum_{\Omega_{2(M-m)}} \Omega_{2(M-m)} \mathfrak{D}^m \qquad (13) \\ \Omega_{2(M-m)} &:= \alpha_m E_{2(M-m)} \,, \qquad 2 \leq M - m \leq 5 \,, \\ \Omega_{12} &:= \alpha_0 E_{12} + \alpha_0^{(cusp)} \Delta \,. \end{aligned}$$

Here $\Delta = \eta^{24}$ is the modular discriminant function, E_{2k} is the holomorphic Eisenstein series of weight $2k$, and the nonzero numbers α_m and $\alpha_0^{(cusp)}$ are given by the table below:

$(2,\nu)$	$(2,3)$	$(2,5)$	$(2,7)$	$(2,9)$	$(2,11)$	$(2,13)$
M	1	2	3	4	5	6
κ_M	0	$-\frac{1}{60}$	$-\frac{1}{42}$	$-\frac{1}{36}$	$-\frac{1}{33}$	$-\frac{5}{156}$
α_M	1	1	1	1	1	1
α_{M-2}		$-\frac{11}{60^2}$	$-\frac{5\cdot7}{42^2}$	$-\frac{2\cdot3\cdot13}{36^2}$	$-\frac{11\cdot53}{2^2\cdot33^2}$	$-\frac{7\cdot13\cdot67}{156^2}$
α_{M-3}			$\frac{5\cdot17}{42^3}$	$\frac{2^3\cdot53}{36^3}$	$\frac{3\cdot5\cdot11\cdot59}{2^3\cdot33^3}$	$\frac{2^3\cdot13\cdot17\cdot193}{156^3}$
α_{M-4}				$-\frac{3\cdot11\cdot23}{36^4}$	$-\frac{11\cdot6151}{2^4\cdot33^4}$	$-\frac{5\cdot11\cdot13\cdot89\cdot127}{156^4}$
α_{M-5}					$\frac{2^4\cdot17\cdot29}{33^5}$	$\frac{2^3\cdot3\cdot5\cdot13\cdot31\cdot2437}{156^5}$
α_{M-6}						$-\frac{5^4\cdot7^2\cdot23\cdot31\cdot67}{156^6}$
$\alpha^{\text{(cusp)}}_{M-6}$						$\frac{5^2\cdot7\cdot11\cdot23^2\cdot167}{2^5\cdot3^2\cdot13^4\cdot691}$

The nonzero coefficients in the order M differential operator in the $(2,\nu)$ minimal model. κ_M is displayed to explain the standard denominators of the α_m (and mark deviations from them).

Remark 2. *The prime* 691 *displayed in the denominator of* $\alpha^{(cusp)}_{M-6}$ *suggests that Bernoulli numbers are involved in the computations. This is an artefact of the choice of basis, however. Using the identity [16]*

$$E_{12} = \frac{1}{691}(441E_4^3 + 250E_6^2)\,,$$

we can write

$$\alpha_0 E_{12} + \alpha_0^{(cusp)}\Delta = -\frac{5^2\cdot7\cdot23}{2^7\cdot3^5\cdot13^6}\left(\frac{53\cdot1069}{2^5}E_4^3 + \frac{6047}{3}E_6^2\right)\,.$$

Only the specific values of the coefficients in eq. (12) seem to be new. Rather than setting up a closed formula for α_m, we shall outline the algorithm to determine these numbers, and leave the actual computation as an easy numerical exercise.

Proof. (Sketch) We first show that the highest order coefficient α_M of the ODE can be normalised to one. For every κ_s in the list (10) and for $0 \le m \le M-1$, we have

$$\mathfrak{D}^m\langle\mathbf{1}\rangle_s \propto q^{\kappa_s}(1+O(q))\,. \tag{14}$$

Since the κ_s are all different, we know that

$$w_M \sim \prod_s q^{\kappa_s}\,, \quad q \text{ close to zero}\,.$$

By construction, w_M has no pole at finite τ. The number of zeros can be calculated using Cauchy's Theorem [16]: Since $\mathfrak{D}^m\langle\mathbf{1}\rangle$ has weight $2m$, we find

$$\text{weight}\, w_M = 2\sum_{\ell=0}^{M-1}\ell = M(M-1)\,.$$

The order of vanishing $\mathrm{ord}_P(w_M)$ of w_M at a point $P \in \Gamma\setminus\mathbb{H}^+$ depends only on the orbit ΓP [16]. Denote by $\mathrm{ord}_\infty(w_M)$ the order of vanishing of w_M at ∞ (i.e. the smallest integer $n \geq 0$ such that $a_n \neq 0$ in the Fourier expansion for w_M). By eq. (9) for the fundamental domain of the finite index subgroup Γ of Γ_1, all orders of vanishing for Γ differ from those for Γ_1 by the same factor. Thus ([16], Propos. 2 on p. 9) generalises to subgroups $\Gamma \subset \Gamma_1$ and to

$$\mathrm{ord}_\infty(w_M) + \sum_{P\in\Gamma\setminus\mathbb{H}^+}\frac{1}{n_P}\mathrm{ord}_P(w_M) = \frac{M(M-1)}{12}\,, \tag{15}$$

where n_P is the order of the stabiliser. Since

$$\mathrm{ord}_\infty(w_M) = \sum_{s=1}^{M}\kappa_s = \frac{M(M-1)}{12}\,,$$

we have $\mathrm{ord}_P(w_M) = 0$ for $P \in \Gamma\setminus\mathbb{H}^+$. Thus we can divide by w_M to yield

$$\sum\tilde{\alpha}_i\mathfrak{D}^i\langle\mathbf{1}\rangle_j = 0$$

for $j = 1,\ldots,M$ and the modular forms $\tilde{\alpha}_i = \frac{w_i}{w_M}$.

By (14), $D^{(2,\nu)}\langle\mathbf{1}\rangle_s$ is a power series of order $\geq \kappa_s$ in q. The coefficient of q^{κ_s} is a monic degree M polynomial in κ_s, and we have

$$[D^{(2,\nu)}]_0 q^\kappa = q^\kappa\prod_{s=1}^{M}(\kappa-\kappa_s)\,, \tag{16}$$

since by assumption $\langle\mathbf{1}\rangle_{\kappa_s} \in \ker D^{(2,\nu)}$ for $s = 1,\ldots M$. (Here $[D^{(2,\nu)}]_0$ denotes the cut-off of the differential operator $D^{(2,\nu)}$ at power zero in q.) For $2 \leq k \leq 5$, the space of modular forms of weight $2k$ is spanned by the Eisenstein series E_{2k}, while for $k = 6$, the space is two dimensional and spanned by E_{12} and Δ. However, only the Eisenstein series have a constant term, so that actually all coefficients α_m are determined by eq. (16). Note that vanishing of α_{M-1} (the coefficient of $\mathfrak{D}^{M-1}$ in $D^{(2,\nu)}$) implies the equality

$$-\sum_{s=1}^{M}\kappa_s = \sum_{\ell=1}^{M}\frac{1-\ell}{6}\,. \tag{17}$$

Indeed, the l.h.s. of eq. (17) equals the coefficient of κ^{M-1} in the polynomial $q^{-\kappa}[D^{(2,\nu)}]_0 q^\kappa$ in eq. (16), while the r.h.s. equals the coefficient of κ^{M-1} in $q^{-\kappa}[\mathfrak{D}^M]_0 q^\kappa$, where

$$q^{-\kappa}[\mathfrak{D}^{M-i}]_0 q^\kappa = \prod_{\ell=0}^{M-i-1}(\kappa-\frac{\ell}{6})\,,\quad 0 \leq i \leq M-1\,.$$

Equality (17) thus states that $q^{-\kappa}[\mathfrak{D}^{M-1}]_0 q^\kappa$ (with leading term κ^{M-1}) does not contribute, and so is equivalent to $\alpha_{M-1} = 0$.

$\alpha_0^{(\text{cusp})}$ is determined by considering the next highest order $[D^{(2,\nu)}\langle\mathbf{1}\rangle]_{\kappa+1}$ for some character. (Since modular transformations permute the characters only and have no effect on $D^{(2,\nu)}$, it is sufficient to do the computation for the vacuum character $\langle\mathbf{1}\rangle_1 = q^{\kappa_1}(1+O(q^2))$). □

3.5 Outlook: Generalisation to other minimal models

For $(\mu, \nu) \in \mathbb{Z}^2$, the (μ, ν)-minimal model has

$$M = \frac{(\nu - 1)(\mu - 1)}{2}$$

different characters. The set of all characters is parametrised by

$$\kappa_{r,s} = \frac{(\nu r - \mu s)^2}{4\mu\nu} - \frac{1}{24}\,, \quad 1 \le r \le \mu - 1\,, \quad 1 \le s \le \nu - 1\,.$$

Due to periodicity of the conformal weights $\kappa_{r,s} + \frac{c}{24}$ (which we shall not go into here) this listing makes us count every character twice. The characters are modular functions on some finite index subgroup Γ of Γ_1 satisfying an order M differential equation, and it remains to verify that the latter has highest order coefficient $\alpha_M = 1$. We have

$$\mathrm{ord}_\infty(w_M) = \frac{1}{2} \sum_{1 \le r \le \mu-1; 1 \le s \le \nu-1} \kappa_{r,s} = \frac{M(M-1)}{12}\,,$$

where the factor of $1/2$ in front of the sum has been inserted to prevent the double counting mentioned above. As before, we conclude that w_M has no zeros in $\mathbb{H}^+$ and with the

Corollary 3. *The characters of the* (μ, ν) *minimal model satisfy an order* M *differential equation*

$$D^{(\nu,\mu)}\langle \boldsymbol{1} \rangle = 0\,,$$

where $D^{(\nu,\mu)}$ *is a differential operator of the form*

$$D^{(\nu,\mu)} = \mathfrak{D}^M + \sum_{m=0}^{M-2} \sum_{\Omega_{2(M-m)}} \Omega_{2(M-m)} \mathfrak{D}^m$$

where summation is over modular forms $\Omega_{2(M-m)}$ *of weight* $2(M - m)$*.*

4 A new variation formula

4.1 Introduction

Suppose $X = \mathbb{C}/\Lambda$ where $\Lambda = (\mathbb{Z}.1 + \mathbb{Z}.i\beta)$ with $\beta \in \mathbb{R}$. Thus the fundamental domain is a rectangle in the (x^0, x^1) plane with length $\Delta x^0 = 1$ and width $\Delta x^1 = \beta$. The dependence of $\langle \mathbf{1} \rangle$ from the modulus $i\beta$ follows from the identity

$$\langle \mathbf{1} \rangle = \mathrm{tr}\, e^{-H\beta}\,, \quad H = \int T^{00} dx^0,$$

where T^{00} is a real component of the Virasoro field.[1] Stretching $\beta \mapsto (1+\varepsilon)\beta$ changes the Euclidean metric according to

$$(ds)^2 \ \mapsto\ (ds)^2 + 2\varepsilon (dx^1)^2 + O(\varepsilon^2)\,.$$

Thus $dG_{11} = 2\frac{d\beta}{\beta}$, and

$$\begin{aligned} d\langle \mathbf{1} \rangle = -\mathrm{tr}(H d\beta\, e^{-H\beta}) &= -\frac{dG_{11}}{2} \left(\int \langle T^{00} \rangle dx^0 \right) \beta \\ &= -\frac{dG_{11}}{2} \iint \langle T^{00} \rangle dx^0 dx^1\,. \end{aligned} \tag{18}$$

The fact that $\int \langle T^{00} \rangle dx^0$ does not depend on x^1 follows from the conservation law $\partial_\mu T^{\mu\nu} = 0$:

$$\frac{d}{dx^1} \oint \langle T^{00} \rangle\, dx^0 = \oint \partial_1 \langle T^{00} \rangle\, dx^0 = -\oint \partial_0 \langle T^{10} \rangle\, dx^0 = 0\,,$$

using Stokes' Theorem.

When $g > 1$, equation (18) generalises to

$$d\langle \mathbf{1} \rangle = -\frac{1}{2} \iint dG_{\mu\nu}\, \langle T^{\mu\nu} \rangle\ \sqrt{G}\, dx^0 \wedge dx^1\,. \tag{19}$$

Here $G := |\det G_{\mu\nu}|$, and $dvol_2 = \sqrt{G}\, dx^0 \wedge dx^1$ is the volume form which is invariant under base change.[2] The normalisation is in agreement with eq. (18) (see also [3], eq. (5.140) on p. 139).

Methods that make use of the flat metric do not carry over to surfaces of higher genus. We may choose a specific metric of prescribed constant curvature to obtain mathematically correct but cumbersome formulas. Alternatively, we consider quotients of N-point functions over $\langle \mathbf{1} \rangle$ only (as done in [6]) so that the dependence on the specific metric drops out. Yet we suggest to use a singular metric that is adapted to the specific problem.

[1] We have [1]

$$T_{zz} = \frac{1}{4}(T_{00} - 2iT_{10} - T_{11})\,.$$

For the relation with the Virasoro field $T(z)$ discussed in [11], cf. Subsection 4.2 below.

[2] The change to complex coordinates is a more intricate, however: We have $dx^0 \wedge dx^1 = iG_{z\bar{z}}\, dz \wedge d\bar{z}$ with $G_{z\bar{z}} = \frac{1}{2}$, as can be seen by setting $z = x^0 + ix^1$.

4.2 The variation formula in the literature

Formula (19) describes the effect on $\langle \mathbf{1} \rangle$ of a change $dG_{\mu\nu}$ in the metric. It generalises to the variation of N-point functions $\langle \varphi \ldots \rangle$ as follows: Suppose the metric is changed on an open subset $N \subseteq X$. Then

$$d\langle \varphi \ldots \rangle = -\frac{1}{2} \iint_X (dG_{\mu\nu}) \, \langle T^{\mu\nu} \varphi \ldots \rangle \, dvol_2 \; . \tag{20}$$

([15], eq. (12.2.2) on p. 360; see also eq. (11) in [6].)[3], provided that

$$\text{the positions of the fields } \varphi, \ldots \text{ do not lie in } N. \tag{21}$$

Note that in order for the formula to be well-defined, $T_{\mu\nu} dx^\mu dx^\nu$ must be quadratic differential on X, i.e. one which transforms homogeneously under coordinate changes.

Due to invariance of N-point functions under diffeomorphisms, $T_{\mu\nu}$ satisfies the conservation law

$$\begin{aligned} 0 = \nabla_\mu T^\mu{}_z &= \nabla_z T^z{}_z + \nabla_{\bar{z}} T^{\bar{z}}{}_z \\ &= \partial_z T^z{}_z + G^{z\bar{z}} \partial_{\bar{z}} T_{zz} \; . \end{aligned} \tag{22}$$

Here we have used that $T^z{}_z$ transforms like a scalar, whence $\nabla_z T^z{}_z = \partial_z T^z{}_z$. Moreover, $\nabla_\mu G^{\mu\nu} = 0$, and $\nabla_{\bar{z}} T_{zz} = \partial_{\bar{z}} T_{zz}$, which is true since T_{zz} takes values in a holomorphic line bundle.

A Weyl transformation $G_{\mu\nu} \mapsto \mathcal{W} G_{\mu\nu}$ changes the metric only within the respective conformal class. Its effect on N-point functions is described by the trace of T (eq. (3) on p. 310 in [6]), which equals

$$T_\mu{}^\mu = T_z{}^z + T_{\bar{z}}{}^{\bar{z}} = 2T_z{}^z = \frac{c}{24\pi} \mathcal{R}.1 \; , \tag{23}$$

([3], eq. (5.144) on page 140, which is actually true for the underlying fields), where $\mathcal{R}$ is the scalar curvature of the of the Levi-Cività connection ∇ on X. The non-vanishing of the trace (23) is referred to as the *trace* or *conformal anomaly*.

Since $T_\mu{}^\mu$ is a multiple of the unit field, the restriction (21) is unnecessary. Thus under a Weyl transformation $G_{\mu\nu} \mapsto \mathcal{W} G_{\mu\nu}$, all N-point functions change by the same factor Z (equal to $\langle \mathbf{1} \rangle$), given by

$$d \log Z = -\frac{c}{24\pi} \iint \mathcal{R} \, d\mathcal{W} \, dvol_2$$

Lemma 4. *[6] Suppose X has scalar curvature $\mathcal{R} = const.$ Let*

$$\frac{1}{2\pi} T(z) := T_{zz} - \frac{c}{24\pi} t_{zz} \; , \tag{24}$$

where

$$t_{zz} := \partial_z \Gamma^z{}_{zz} - \frac{1}{2} (\Gamma^z{}_{zz})^2 \; .$$

with $\Gamma^z{}_{zz} = \partial_z \log G_{z\bar{z}}$ being the Christoffel symbol. We have

$$\partial_{\bar{z}} T(z) = 0 \; .$$

[3] Note that both references introduce the Virasoro field with the opposite sign. Our sign convention follows e.g. [3], cf. eq. (5.148) on p. 140.

Proof. Direct computation shows that

$$\partial_{\bar{z}} t_{zz} = -\frac{1}{2} G_{z\bar{z}}\, \partial_z(\mathcal{R}.1)\ .$$

From the conservation law eq. (22) follows

$$\begin{aligned}\partial_{\bar{z}} T_{zz} &= -\, G_{z\bar{z}}\, \partial_z T^z{}_z \\ &= -\, \frac{c}{48\pi} G_{z\bar{z}}\, \partial_z(\sqrt{G}\, \mathcal{R}.1) = \frac{c}{24\pi}\, \partial_{\bar{z}} t_{zz}\ .\end{aligned}$$

□

Remark 5. *t_{zz} defines a projective connection: Under a holomorphic coordinate change, $z \mapsto w$ such that $w \in \mathcal{D}(S)$,*

$$t_{ww}\ (dw)^2 = t_{zz}\ (dz)^2 - S\,(w)(z)\ (dz)^2\ ,$$

where $S\,(w)$ is the Schwarzian derivative. t_{zz} is known as the ***Miura transform*** *of the affine connection given by the differentials $\Gamma^z{}_{zz} dz$.*

$T(z)$ is the holomorphic field considered in [11].[4]

4.3 A new variation formula

Let X be a Riemann surface. We introduce

- γ : one-dimensional smooth submanifold of X , topologically an S^1,
- N : a tubular neighbourhood of γ in X ,
- A : a vector field which conserves the metric on X and is holomorphic on N .

We think of $A = \alpha(z)\frac{\partial}{\partial z} \in TN$ as an infinitesimal coordinate transformation

$$\begin{aligned} z \quad \mapsto \quad w(z) &= \left(1 + \varepsilon\alpha(z)\frac{\partial}{\partial z}\right) z \\ &= z + \varepsilon\alpha(z)\ , \end{aligned} \tag{25}$$

where $|\varepsilon| \ll 1$. We suppose $\alpha = 1$.

Theorem 1. *Suppose X has scalar curvature $\mathcal{R} = 0$. Let φ be a holomorphic field on X. The effect of the transformation (25) with $\alpha = 1$ on $\langle\varphi(w)\rangle$ is*

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \langle\varphi(w)\rangle = -i \oint_\gamma \langle T_{zz}\ \varphi(w)\rangle\ dz\ ,$$

provided that

$$w \text{ does not lie on the curve } \gamma\ . \tag{26}$$

In particular, as w is not enclosed by γ, $\langle\varphi(w)\rangle$ doesn't change.

[4]Our notations differ from those used in [6]. Thus the standard field $T(z)$ in [6] equals $-T_{zz}$ in our exposition, and the field $\tilde{T}(z)$ in [6] equals $-\frac{1}{2\pi} T(z)$ here.

Proof. By property (26), the position of φ is not contained in a small tubular neighbourhood N of γ. Let

$$N \setminus \gamma = N_L \sqcup N_R$$

be the decomposition in connected parts left and right of γ (we assume γ has positive orientation). Let $W \subset X$ be an open set s.t.

$$\overline{W} \cap \gamma = \emptyset\,, \quad W \cup N = X\,.$$

We let $F : N \to [0,1]$ be a smooth function s.t.

$$F = 1 \quad \text{on} \quad N_L \cap W\,,$$
$$F = 0 \quad \text{on} \quad N_R \cap W\,.$$

Let ε be so small that $z \in W^c$ implies $\exp(\varepsilon F)(z) \in N$. Define a new metric manifold $(X^\varepsilon, G^\varepsilon_{z\bar{z}})$ by

$$X^\varepsilon|_W := X|_W$$
$$G^\varepsilon_{z\bar{z}}(z)\,|dz|^2 := G_{z\bar{z}}(\exp(\varepsilon F)(z))\,|d\exp(\varepsilon F)(z)|^2\,, \quad z \in \overline{W}\,.$$

We have

$$dG_{\mu\nu}T^{\mu\nu} = dG_{\bar{z}\bar{z}}T^{\bar{z}\bar{z}} + \text{antiholomorphic contributions} + \text{Weyl terms}\,,$$

where we disregard the antiholomorphic contributions $\sim T_{\bar{z}\bar{z}}$, and the Weyl terms are absent since by assumption $\mathcal{R} = 0$. Alternatively, we can describe the change in the metric by the map

$$|dz|^2 \mapsto |dz + \mu d\bar{z}|^2 = dzd\bar{z} + \mu d\bar{z}d\bar{z} + \ldots\,,$$

where

$$\mu = \varepsilon\partial_{\bar{z}}F + O(\varepsilon^2)\,.$$

is the Beltrami differential. Thus

$$dG_{\bar{z}\bar{z}} = 2G_{z\bar{z}}\,d\mu(z,\bar{z})\,.$$

Eq. (20) yields

$$\begin{aligned}
\frac{d\langle\varphi\rangle}{d\varepsilon}\Big|_{\varepsilon=0} &= -\frac{1}{2}\iint_X \frac{\partial G_{\mu\nu}}{\partial\varepsilon}\Big|_{\varepsilon=0}\,\langle T^{\mu\nu}\,\varphi\rangle\,dvol_2 \\
&= -\frac{i}{2}\iint_X 2G_{z\bar{z}}\,\frac{\partial\mu(z,\bar{z})}{\partial\varepsilon}\Big|_{\varepsilon=0}\,(G^{z\bar{z}})^2\langle T_{zz}\,\varphi\rangle\,G_{z\bar{z}}\,dz\wedge d\bar{z} \\
&= i\iint_N (\partial_{\bar{z}}F)\,\langle T_{zz}\,\varphi\rangle\,d\bar{z}\wedge dz\,,
\end{aligned}$$

since $(G^{z\bar{z}})^k = (G_{z\bar{z}})^{-k}$ for $k \in \mathbb{Z}$. Here

$$\langle T_{zz}\,\varphi\rangle\,dz = \iota_A(\langle T_{zz}\,\varphi\rangle\,(dz)^2)$$

is the holomorphic one-form given by the contraction of the holomorpic vector field $A = \frac{\partial}{\partial z}$ with the quadratic differential $\langle T_{zz}\,\varphi\rangle\,(dz)^2$, which is holomorphic on N. By

Stokes' Theorem,

$$
\begin{aligned}
\frac{d\langle\Phi\rangle}{d\varepsilon}|_{\varepsilon=0} &= \iint_N \partial_{\bar{z}} (F \langle T_{zz}\, \varphi\rangle)\ d\bar{z} \wedge dz \\
&= \oint_{W_R} F\, \langle T_{zz}\, \varphi\rangle\, dz + \oint_{W_L} F\, \langle T_{zz}\, \varphi\rangle\, dz \\
&= -\oint_{W_L} F\, \langle T_{zz}\, \varphi\rangle\, dz\,.
\end{aligned}
$$

Here $W_R = N_R \cap \partial W$ and $W_L = N_L \cap \partial W$ are the left and right boundary of W in N, respectively. We conclude that

$$
\frac{d\langle\varphi\rangle}{d\varepsilon}|_{\varepsilon=0} = -\oint_{W_L} \langle T_{zz}\, \varphi\rangle\, dz = -\oint_{\gamma} \langle T_{zz}\, \varphi\rangle\, dz\,,
$$

by holomorphicity on $N_L \cup \gamma$. ◻

Remark 6. *The construction is independent of F. When F approaches the discontinuous function defined by*

$$
\begin{cases} F = 1 & \text{on} \quad N_L\,, \\ F = 0 & \text{on} \quad N_R\,, \end{cases}
$$

we obtain a description of $(X^\varepsilon, G^\varepsilon_{z\bar{z}})$ by cutting along γ and pasting back after a transformation by $\exp(\varepsilon)$ *on the left.*

There is a way to check the result of Theorem 1: Let φ be a holomorphic field whose position lies in a sufficiently small open set $U \subset X$ with boundary $\partial U = \gamma$. We can use a translationally invariant metric in U and corresponding coordinates $z, \bar{z}$. Then

$$
T_{zz} = \frac{1}{2\pi} T(z)
$$

in eq. (24). For $A = \frac{d}{dw}$, we have

$$
\langle A\varphi(w)\ldots\rangle = \frac{1}{2\pi i} \oint_\gamma \langle T(z)\varphi(w)\ldots\rangle\, dz\,, \tag{27}
$$

This can be seen in two ways.

1. Eq. (27) follows from the residue theorem for the OPE of $T(z) \otimes \varphi(w)$. Indeed, the Laurent coefficient of the first order pole at $z = w$ is $N_{-1}(T,\varphi)(w) = \partial_w \varphi$, which is holomorphic.

2. Alternatively, by Theorem 1,

$$
\frac{d}{d\varepsilon}|_{\varepsilon=0} \langle\varphi(w+\varepsilon)\ldots\rangle = \frac{1}{2\pi i} \oint_\gamma \langle T(z)\varphi(w)\ldots\rangle\, dz\,.
$$

The two approaches are compatible!

4.4 Discussion of the metric

Let X_g be the genus g hyperelliptic Riemann surface

$$X_g: \quad y^2 = p(x)\,, \quad \deg p = 2g+1\,.$$

Recall that x defines a complex coordinate on the Riemann sphere, outside the ramification points where we must change to the y coordinate. $\mathbb{P}^1_{\mathbb{C}}$ does not allow for a constant curvature metric but we shall define a metric on $\mathbb{P}^1_{\mathbb{C}}$ which is flat almost everywhere.

Suppose $n = 3$. By means of the isomorphism $\mathbb{P}^1_{\mathbb{C}} \cong \mathbb{C} \cup \{\infty\}$, we may identify the branch points of $X_{g=1}$ with points $X_1, X_2, X_3 \in \mathbb{C}$ and $X_4 = \{\infty\}$, respectively.

Let $\theta \gg 1$, but finite, such that in the flat metric of $\mathbb{C}$,

$$|X_i| < \theta\,, \quad i = 1, 2, 3\,. \tag{28}$$

We define $|X_4| := \infty$. For $\varepsilon > 0$, define a metric

$$(ds(\varepsilon))^2 = 2G_{z\bar{z}}(\varepsilon)\, dz \otimes d\bar{z} \tag{29}$$

on $\mathbb{P}^1_{\mathbb{C}}$ by

$$2G_{z\bar{z}}(\varepsilon) := \begin{cases} (1+\varepsilon\theta^2)^{-2} & \text{for} \quad |z| \le \theta\,, \\ (1+\varepsilon z\bar{z})^{-2} & \text{for} \quad |z| \ge \theta\,. \end{cases}$$

Lemma 7. *In the disc $|z| \le \theta$, the metric is flat, while in the area $|z| \ge \theta$, it is Fubini-Study of Gauss curvature $\mathcal{K} = 4\varepsilon$.*

Proof. For $\rho = 2G_{z'\bar{z}'}(\varepsilon)$ with

$$G_{z'\bar{z}'}(\varepsilon) := \frac{1}{2\varepsilon}(1 + z'\bar{z}')^{-2} \quad \text{for} \quad |z'| \ge \sqrt{\varepsilon}\theta\,,$$

we have [7]

$$\mathcal{R} = \rho^{-1}(-4\partial_z\partial_{\bar{z}}\log\rho) = \varepsilon(1+z'\bar{z}')^2(8\partial_{z'}\partial_{\bar{z}'}\log(1+z'\bar{z}')^2) = 8\varepsilon\,,$$

and $\mathcal{R} = 2\mathcal{K}$. □

Definition 1. *Let X be a genus $g = 1$ Riemann surface with conformal structure defined by the position of the ramification points $\{X_i\}_{i=1}^3$ with finite relative distance on $\mathbb{P}^1_{\mathbb{C}}$. Let $G_{z\bar{z}}(\varepsilon)$ be the metric defined by eq. (29). We define $\langle \mathbf{1}\rangle_{\{X_i\}_{i=1}^3,\varepsilon,R}$ to be the zero-point function on $(X, G_{z\bar{z}}(\varepsilon))$.*

By eq. (23) and the fact that on any surface, $\mathcal{R} = 2\mathcal{K}$,

$$T_{z\bar{z}} = \frac{c}{24\pi} G_{z\bar{z}}\mathcal{K}.1\,.$$

So we have according to eq. (19),

$$d\log\langle\mathbf{1}\rangle_{\{X_i\}_{i=1}^3,\varepsilon,\theta} = \frac{c}{48\pi}\iint_{S_R^2}(d\log G_{z\bar{z}}(\varepsilon))\,\mathcal{K}\, dvol_2\,.$$

Since $G(\varepsilon) = (G_{z\bar{z}}(\varepsilon))^2$, for $|z| > \theta$, the two-dimensional volume form is

$$dvol_2 = G_{z\bar{z}}(\varepsilon)\, dz \wedge d\bar{z} = \frac{1}{2}\frac{\pi d(r^2)}{(1+\varepsilon r^2)^2}\,.$$

Now

$$d\log\langle\mathbf{1}\rangle_{\{X_i\}_{i=1}^3,\varepsilon,R} = dI_{|z|<\theta} + dI_{|z|>\theta}\,,$$

where for $\varrho_0^2 := \varepsilon\theta^2$, the integrals yield

$$\begin{aligned} dI_{|z|<\theta} &= -\frac{c\theta^2}{12}\,d(\varepsilon)\,\frac{\varrho_0^2}{(1+\varrho_0^2)^3}\,,\\ dI_{|z|>\theta} &= -\frac{c}{12}\,(d\log\varepsilon)\int_{|\varrho|^2>\varrho_0^2}\frac{\varrho^2\,d(\varrho^2)}{(1+\varrho^2)^3} = -\frac{c}{24}\,(d\log\varepsilon)\,(1+O(\varrho_0^4))\,. \end{aligned}$$

So for $|\varrho_0| \ll 1$,

$$\langle\mathbf{1}\rangle_{\{X_i\}_{i=1}^3,\varepsilon,\theta} = \varepsilon^{-\frac{c}{24}(1+O(\varrho_0^4))}\,Z\,\exp\left(-\frac{c}{12}\,\frac{\varrho_0^4}{(1+\varrho_0^2)^3}\right)\,, \tag{30}$$

where $Z \in \mathbb{C}$ is an integration constant.

Variation of ε rescales the metric within the conformal class defined by the branch points. In the limit as $\varepsilon \searrow 0$,

$$G_{z\bar{z}} := \lim_{\varepsilon\searrow 0} G_{z\bar{z}}(\varepsilon) = \frac{1}{2} \quad \text{for} \quad |z| < \infty\,, \tag{31}$$

(and is undefined for $|z| = \infty$). Thus $\mathbb{P}^1_{\mathbb{C}}$ becomes an everywhere flat surface except for the point at infinity, which is a singularity for the metric.

Definition 2. *Let X be a genus $g = 1$ Riemann surface with conformal structure defined by the position of the ramification points $\{X_i\}_{i=1}^3$ with finite relative distance on $\mathbb{P}^1_{\mathbb{C}}$. Let $G_{z\bar{z}}$ be the metric on X defined by eq. (31). We define the zero-point function on $(X_{g=1}, G_{z\bar{z}})$ by*

$$\langle\mathbf{1}\rangle_{\{X_i\}_{i=1}^3} := \lim_{\rho_0\searrow 0}\,\varepsilon^{\frac{c}{24}(1+O(\varrho_0^4))}\langle\mathbf{1}\rangle_{\{X_i\}_{i=1}^3,\varepsilon,\theta}\,.$$

Thus $\langle\mathbf{1}\rangle_{\{X_i\}_{i=1}^3} = Z$.

Remark 8. *The reason for introducing ε and performing $\lim_{\varepsilon\searrow 0}$ is the fact that the logarithm of the Weyl factor $\mathcal{W}$ is not defined for surfaces with a singular metric and infinite volume. Let $\langle\mathbf{1}\rangle_z$ be the zero-point function on $(X_{g=1}, dzd\bar{z})$. We have*

$$d\log\frac{\langle\mathbf{1}\rangle_{\{X_i\}_{i=1}^3}}{\langle\mathbf{1}\rangle_z} = d\log\mathcal{W}\,,$$

so $\mathcal{W}$ is determined only up to a multiplicative constant. This constant is infinite for $\varepsilon = 0$.

Our method is available for any surface $X_g : y^2 = p(x)$ with $\deg p = n \geq 3$. When n is odd, the point at infinity is a non-distinguished element in the set of ramification points on X_g. We shall distribute the curvature of X_g evenly over these. Using the Gauss-Bonnet theorem, the total curvature is recovered as

$$\int_{X_g}\mathcal{K}\,dvol_2 = 2\pi\,\chi(X_g) = 4\pi(1-g) = 8\pi - 2\pi(2g+2)\,.$$

We interpret 8π as the contribution to the curvature from the $g = 0$ double covering and -2π from any branch point.

The method is now available for arbitrary genus $g \geq 1$ hyperelliptic Riemann surfaces and will in the following be checked against the case $g = 1$.

4.5 The main theorem

We now get to an algebraic description of the effect on an N-point function as the position of the ramification points of the surface is changed.

Theorem 2. *Let X_g be the hyperelliptic Riemann surface*

$$X_g : \quad y^2 = p(x) , \quad n = \deg p = 2g + 1 ,$$

with roots X_j. We equip range$(x) = \mathbb{P}^1_{\mathbb{C}}$ with the singular metric which is equal to

$$|dz|^2 \quad on\ \mathbb{P}^1_{\mathbb{C}} \setminus \{X_1 \ldots , X_n\} .$$

We define a deformation of the conformal structure by

$$\xi_j = dX_j \quad for \quad j = 1, \ldots , n .$$

Let (U_j, z) be a chart on X_g containing X_j but no field position. We have

$$d\langle\varphi \ldots\rangle = \sum_{j=1}^{n} \left(\frac{1}{2\pi i} \oint_{\gamma_j} \langle T(z)\varphi \ldots\rangle \, dz \right) \xi_j , \tag{32}$$

where γ_j is a closed path around X_j in U_j.

Proof. On the chart (U, z), we have $\frac{1}{2\pi} T(z) = T_{zz}$ in eq. (24), outside the points which project onto one of the X_j for $j = 1, \ldots , n$ on $\mathbb{P}^1_{\mathbb{C}}$. Moreover, γ does not pick up any curvature for whatever path γ we choose. Since

$$d\langle\mathbf{1}\rangle = \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial X_i} \langle\mathbf{1}\rangle ,$$

the theorem follows from Theorem 1. □

5 Application to the case $g = 1$

5.1 Algebraic approach

Let $X_{g=1}$ be the genus 1 Riemann surface

$$X_1 : \quad y^2 = p(x) , \quad \deg p = 3 ,$$

with ramification points $X_1, X_2.X_3$. Throughout this section, we shall assume that

$$\sum_{i=1}^{3} X_i = 0 . \tag{33}$$

We introduce some notation: Let $m(X_1, \xi_1, \dots, X_n, \xi_n)$ be a monomial. We denote by

$$\overline{m(X_1, \xi_1, \dots, X_n, \xi_n)}$$

the sum over all distinct monomials $m(X_{\sigma(1)}, \xi_{\sigma(1)}, \dots, X_{\sigma(n)}, \xi_{\sigma(n)})$, where σ is a permutation of $\{1, \dots, n\}$. E.g. eq. (33) reads $\overline{X_1} = 0$, and $\overline{X_1 X_2} = X_1X_2 + X_1X_3 + X_2X_3$ (for $n = 3$).

The Virasoro one-point function on X is given by [11]

$$\langle T(x)\rangle = \frac{c}{32}\frac{[p']^2}{p^2}\langle \mathbf{1}\rangle + \frac{\Theta(x)}{4p} , \tag{34}$$

where $\Theta(x) = \Theta^{[1]}(x)$ in the previous notations (the polynomial $\Theta^{[y]}$ is absent),

$$\Theta(x) = -ca_0 x\langle \mathbf{1}\rangle + \mathbf{A}_1 , \tag{35}$$

where a_0 is the leading coefficient of p, and $\mathbf{A}_1 \propto \langle \mathbf{1}\rangle$ is constant. The connected Virasoro two-point function on X_1 has been computed in [11]. We note that for $n = 3$, the state specific symmetric polynomial, previously denoted by

$$P(x_1, x_2, y_1, y_2) = P^{[1]}(x_1, x_2) , \tag{36}$$

is constant. For the one-forms $\xi_j = dX_j$ ($j = 1, 2, 3$) we introduce the matrices

$$\Xi_{3,0} = \begin{pmatrix} X_1 & X_2 & X_3 \\ 1 & 1 & 1 \\ \xi_1 & \xi_2 & \xi_3 \end{pmatrix}, \quad \Xi_{3,1} = \begin{pmatrix} X_1 & X_2 & X_3 \\ 1 & 1 & 1 \\ \xi_1 X_1 & \xi_2 X_2 & \xi_3 X_3 \end{pmatrix} ,$$

and the 3×3 Vandermonde matrix

$$V_3 := \begin{pmatrix} 1 & X_1 & X_1^2 \\ 1 & X_2 & X_2^2 \\ 1 & X_3 & X_3^2 \end{pmatrix} .$$

For later use, we note that

$$\det V_3 = \prod_{1\le i<j\le 3}(X_j - X_i)$$
$$= (X_1 - X_2)(X_2 - X_3)(X_3 - X_1) ,$$
$$\frac{\det \Xi_{3,0}}{\det V_3} = \frac{\xi_1}{(X_1 - X_2)(X_3 - X_1)} + \text{cyclic} . \tag{37}$$
$$\frac{\det \Xi_{3,1}}{\det V_3} = \frac{\xi_1 X_1}{(X_1 - X_2)(X_3 - X_1)} + \text{cyclic} . \tag{38}$$

We let

$$\Delta^{(0)} := (\det V_3)^2 \,. \tag{39}$$

It shall be convenient to work with the one-form $\omega := -3\frac{\det \Xi_{3,1}}{\det V_3}$. A simple calculation using eq. (33) shows that

$$d \det V_3 = \,-\, 3X_1(dX_1)(X_2 - X_3) + \text{cyclic} = -3 \det \Xi_{3,1} \,,$$

so that

$$\omega = \frac{1}{2} d \log \Delta^{(0)} = \frac{\xi_1 - \xi_2}{X_1 - X_2} + \text{cyclic} \,. \tag{40}$$

Lemma 9. *Let*

$$p = 4(x - X_1)(x - X_2)(x - X_3).$$

We define a deformation of the Riemann surface by

$$\xi_j = dX_j \,, \quad j = 1, 2, 3.$$

Provided that $\overline{\xi_1} = 0$, *we have*

$$\omega = \pi i \, E_2 \, d\tau - 6\frac{d\lambda}{\lambda} \,.$$

Proof. We have

$$p(x) = 4(x^3 + ax + b) \,,$$

where on the one hand,

$$a = \overline{X_1 X_2} \,, \qquad b = -X_1 X_2 X_3 \,.$$

On the other hand,

$$a = -\frac{\pi^4}{3}\lambda^4 E_4 \,, \qquad b = -\frac{2\pi^6}{27}\lambda^6 E_6 \,,$$

[14]. We will show that for a, b thus introduced, we have

$$\det(\Xi_{3,1} V_3) = 2a^2 \, da + 9b \, db \,. \tag{41}$$

Indeed, we may take $\xi \propto X$ since eq. (33) is satisfied by assumption on the ξ_j. So

$$\det \Xi_{3,1}|_{\xi = X} \det V_3 \propto - \det \begin{pmatrix} 1 & X_1 & X_1^2 \\ 1 & X_2 & X_2^2 \\ 1 & X_3 & X_3^2 \end{pmatrix}^2 = -\Delta^{(0)} \,, \tag{42}$$

where [14]

$$\Delta^{(0)} = \,-\, 4a^3 - 27b^2 = \frac{4\pi^{12}}{27}\lambda^{12}(E_4^3 - E_6^2) \,. \tag{43}$$

On the other hand, for $\xi \propto X$,

$$\begin{aligned} da &= \overline{\xi_1 X_2} \propto 2\overline{X_1 X_2} = 2a \,, \\ db &= \,-\, \overline{\xi_1 X_2 X_3} \propto -3\overline{X_1 X_2 X_3} = 3b \,. \end{aligned}$$

So eq. (41) follows from eqs (43) and (42). Now $d = d\tau\frac{\partial}{\partial\tau} + d\lambda\frac{\partial}{\partial\lambda}$. Since

$$\mathfrak{D}_4 E_4 = -\frac{E_6}{3}\,, \qquad \mathfrak{D}_6 E_6 = -\frac{E_4^2}{2}\,, \tag{44}$$

([16], Proposition 15, p. 49), where $\mathfrak{D}_{2\ell}$ is the Serre derivative defined in (7), we have by eq. (41),

$$2\,a^2\frac{d}{d\tau}a + 9\,b\,\frac{d}{d\tau}b = -\frac{i\,\pi}{3}E_2\Delta^{(0)}\,.$$

From eq. (43) follows

$$\frac{1}{2}\frac{\partial}{\partial\lambda}\log\Delta^{(0)} = \frac{6}{\lambda}\,.$$

We conclude that

$$\omega = -3\,\frac{\det\Xi_{3,1}}{\det V_3} = i\,\pi E_2\,d\tau - 6d\log\lambda\,.$$

This completes the proof. □

Lemma 10. *Under the conditions of Lemma 9, we have*

$$d\tau = -\,i\,\pi\lambda^2\,\frac{\det\Xi_{3,0}}{\det V_3}\,. \tag{45}$$

Proof. Writing

$$p(x) = 4(x^3 + ax + b)\,,$$

we have

$$\det(\Xi_{3,0}V_3) = 9b\,da - 6a\,db\,. \tag{46}$$

Indeed, if we set

$$\xi_i \propto X_i^2 - \xi_0\,, \quad \xi_0 := \frac{1}{3}\left(\sum_{i=1}^{3}X_i^2\right) = \frac{1}{3}\,\overline{X_1^2}$$

then the condition (33) continues to hold, and we have

$$\det\begin{pmatrix}\xi_1 & \xi_2 & \xi_3\\ X_1 & X_2 & X_3\\ 1 & 1 & 1\end{pmatrix} \propto \det\begin{pmatrix}X_1^2 & X_2^2 & X_3^2\\ X_1 & X_2 & X_3\\ 1 & 1 & 1\end{pmatrix} - \det\begin{pmatrix}\xi_0 & \xi_0 & \xi_0\\ X_1 & X_2 & X_3\\ 1 & 1 & 1\end{pmatrix}\,,$$

the latter determinant being zero. Thus for the present choice of ξ,

$$\det\Xi_{3,0}|_{\xi=X^2-\xi_0}\det V_3 \propto \det\begin{pmatrix}\overline{\xi_1} & \overline{\xi_1X_1} & \overline{\xi_1X_1^2}\\ \overline{X_1} & \overline{X_1^2} & \overline{X_1^3}\\ 3 & \overline{X_1} & \overline{X_1^2}\end{pmatrix} = -\Delta^{(0)}\,,$$

where $\Delta^{(0)}$ is the dicriminant (39). On the other hand, by the fact that $\overline{X_1} = 0$,

$$\begin{aligned}\xi_0 &= \frac{1}{3}\,\overline{X_1^2} = -\frac{2}{3}\,\overline{X_1X_2} = -\frac{2a}{3}\,,\\ \overline{X_1^3} &= -\,3\overline{X_1^2X_2} - 6b\,,\\ \overline{X_1^2X_2} &= \overline{X_1X_2(X_1+X_2)} = -3b\,,\end{aligned}$$

so

$$da = -\overline{\xi_1 X_1} \propto -\overline{X_1^3} + \xi_0 \overline{X_1} = -\overline{X_1^3} = -3b\,,$$

$$db = -\overline{\xi_1 X_2 X_3} \propto -\overline{X_1^2 X_2 X_3} + \xi_0 \overline{X_1 X_2} = b\overline{X_1} + \xi_0 a = \xi_0 a = -\frac{2}{3}\,a^2\,.$$

We conclude that

$$\Delta^{(0)} = -4a^3 - 27b^2 = \alpha\,a\,db + \beta\,b\,da = -\frac{2}{3}\alpha\,a^3 - 3\beta\,b^2$$

and thus $\alpha = 6$ and $\beta = 9$. Now by eqs (43) and (44),

$$9\,b\,\frac{d}{d\tau}a - 6\,a\,\frac{d}{d\tau}b = 2\pi i\,(9\,b\,\mathfrak{D}_4 a - 6\,a\,\mathfrak{D}_6 b) = \frac{i}{\pi\lambda^2}\Delta^{(0)}\,.$$

There is no dependence on λ:

$$9\,b\,\frac{d}{d\lambda}a - 6\,a\,\frac{d}{d\lambda}b = 0\,.$$

From this follows the claim. □

Theorem 3. *Under the conditions of Lemma 9, we have the closed system of linear differential equations*

$$(d + \frac{c}{24}\,\omega)\langle\boldsymbol{1}\rangle = -\frac{1}{8}\mathbf{A}_1\,\frac{\det \Xi_{3,0}}{\det V_3} \tag{47}$$

$$(d + \frac{c-8}{24}\omega)\,\mathbf{A}_1 = C\,\frac{\det \Xi_{3,0}}{\det V_3}\,.$$

Here the quotient of determinants and ω are given by eqs (37) and (40), respectively, and C is a constant. We have

$$C := -2P^{[1]} - \frac{1}{8}\langle\boldsymbol{1}\rangle^{-1}\mathbf{A}_1^2 - \frac{2ca_2}{3}\langle\boldsymbol{1}\rangle$$

with $P^{[1]}$ as in eq. (36), and $a_2 = 4\overline{X_1 X_2}$. In particular, in the $(2,5)$*-minimal model,*

$$C = \frac{11}{150}\langle\boldsymbol{1}\rangle a_2\,.$$

Note that the occurrence of a term $\sim \mathbf{A}_1^2$ in the definition of the constant C is an artefact of our presentation since $P^{[1]}$ is defined by the *connected* Virasoro two-point function.

Remark 11. *In contrast to the ODE (4) for the zero-point function $\langle\boldsymbol{1}\rangle_z$ (in the metric $|dz|^2$), the corresponding differential equation (47) for $\langle\boldsymbol{1}\rangle_x$ (in the singular metric) comes with a covariant derivative: By changing coordinates, $z \mapsto x = \wp(z)$, we find $\langle\mathbf{T}\rangle_z = \frac{\mathbf{A}_{1z}}{4}$, where $\mathbf{A}_{1z} \propto \langle\boldsymbol{1}\rangle_z$. For comparison, let $\mathbf{A}_{1x} \propto \langle\boldsymbol{1}\rangle_x$. We have*

$$d\log\frac{\langle\boldsymbol{1}\rangle_x}{\langle\boldsymbol{1}\rangle_z} = -\frac{c}{24}\omega + \frac{1}{8\pi i\,\lambda^2}\,(\mathbf{A}_{1x} - \mathbf{A}_{1z})\,d\tau\,.$$

From eq. (40) follows

$$\langle\boldsymbol{1}\rangle_x \sim \Delta^{(0)\,-\frac{c}{48}}\langle\boldsymbol{1}\rangle_z\,. \tag{48}$$

In particular, $\langle\boldsymbol{1}\rangle_x$ is not a modular function. This is due to the non-vanishing of the scalar curvature $\mathcal{R}$ in the Weyl factor $\mathcal{W}$ (cf. Remark 8)

Proof. (of the Theorem) For $j = 1, 2, 3$, let γ_j be a closed path enclosing $X_j \in \mathbb{P}^1_{\mathbb{C}}$ and no other zero of p. x does not define a coordinate close to X_j, however y does. On the ramified covering, a closed path has to wind around X_j by an angle of 4π. We shall be working with the x coordinate, and mark the double circulation along γ_j in $\mathbb{P}^1_{\mathbb{C}}$ by a symbolic $2 \times \gamma_j$ under the integral. Thus for $j = 1$ we have

$$\begin{aligned}
\frac{1}{2\pi i} \oint_{2\times\gamma_1} \langle T(x)\rangle \; dx &= 2 \lim_{x \to X_1} (x - X_1) \left\{ \langle T(x)\rangle - \frac{c}{32} \sum_{j=1}^{3} \frac{\langle \mathbf{1}\rangle}{(x - X_j)^2} \right\} \\
&= \frac{1}{8} \left(\frac{c\langle \mathbf{1}\rangle}{X_1 - X_2} + \frac{c\langle \mathbf{1}\rangle}{X_1 - X_3} + \frac{\Theta(X_1)}{(X_1 - X_2)(X_1 - X_3)} \right) \\
&= \frac{1}{8} \frac{c(-2X_1 + X_2 + X_3)\langle \mathbf{1}\rangle - A_0 X_1 - A_1}{(X_1 - X_2)(X_3 - X_1)}.
\end{aligned}$$

So

$$\begin{aligned}
d\langle \mathbf{1}\rangle = \sum_{i=1}^{3} \left(\frac{1}{2\pi i} \oint_{2\times\gamma_i} \langle T(x)\rangle \, dx \right) dX_i &= -\left(\frac{c}{4}\langle \mathbf{1}\rangle + \frac{A_0}{8} \right) \frac{\det \Xi_{3,1}}{\det V_3} - \frac{1}{8} A_1 \frac{\det \Xi_{3,0}}{\det V_3} \\
&\quad + \frac{c}{8}\langle \mathbf{1}\rangle \left(\frac{\xi_1 (X_2 + X_3)}{(X_1 - X_2)(X_3 - X_1)} + \text{cyclic} \right),
\end{aligned}$$

using eqs (37) and (38). When (33) is imposed and $A_0 = -4c\langle \mathbf{1}\rangle$ is used, we obtain the differential equation (47) for $\langle \mathbf{1}\rangle_x$. When $\langle T(x)\rangle$ is varied by changing all ramifications points X_1, X_2, X_3 simultaneously, we must require the position x not to lie on or be enclosed by any of the corresponding three curves γ_1, γ_2 and γ_3. Then we have

$$\begin{aligned}
d\langle T(x)\rangle &= \sum_{j=1}^{3} \left(\frac{1}{2\pi i} \oint_{2\times\gamma_j} \langle T(x')T(x)\rangle \, dx' \right) dX_j \\
&= \sum_{j=1}^{3} \left(\frac{\langle \mathbf{1}\rangle}{2\pi i} \oint_{2\times\gamma_j} \langle T(x')T(x)\rangle_c \, dx' \right) dX_j + \langle \mathbf{1}\rangle^{-1} \langle T(x)\rangle \, d\langle \mathbf{1}\rangle.
\end{aligned}$$

Here $\langle T(x_2)\rangle$ is given by formula (34). The connected Virasoro two-point function $\langle T(x)T(x_2)\rangle_c$ has been computed in [11]. The terms $\propto y_1 y_2$ do not contribute: As $X_j \in \mathbb{P}^1_{\mathbb{C}}$ is wound around twice along the closed curve γ_j, the square root y changes sign after one tour, so the corresponding terms cancel. Thus for $j = 1$ we have, using eq. (35) for Θ_2,

$$\begin{aligned}
&\frac{\langle \mathbf{1}\rangle}{2\pi i} \oint_{2\times\gamma_1} \langle T(x')T(x)\rangle_c \, dx' \qquad (49) \\
&= 2 \lim_{x' \to X_1} (x' - X_1) \Big\{ \frac{c}{4} \frac{\langle \mathbf{1}\rangle}{(x' - x)^4} + \frac{c}{32} \frac{p'(x')p'\langle \mathbf{1}\rangle}{(x' - x)^2 p(x')p} + \frac{1}{8} \frac{p(x')\Theta + p\Theta(x')}{(x' - x)^2 p(x')p} \\
&\qquad + \frac{P^{[1]}}{p(x')p} - \frac{a_0}{8} \frac{x'\Theta + x\Theta(x')}{p(x')p} - \frac{a_0^2 c}{8} \frac{x' x \langle \mathbf{1}\rangle}{p(x')p} \Big\} \\
&= \frac{c}{16} \frac{\langle \mathbf{1}\rangle}{(X_1 - x)^2} \frac{p'}{p} + \frac{1}{4} \frac{\Theta(X_1)}{(X_1 - x)^2 p'(X_1)} \qquad (50) \\
&\quad + \frac{2P^{[1]}}{p'(X_1)p} - \frac{a_0}{4} \frac{X_1 \mathbf{A}_1}{p'(X_1)p} - \frac{a_0}{4} \frac{x\Theta(X_1)}{p'(X_1)p}.
\end{aligned}$$

Multiplying the first term on the r.h.s. of eq. (50) by ξ_1 and adding the corresponding terms as j takes the values $2, 3$ yields

$$\frac{c}{16}\langle\mathbf{1}\rangle\frac{p'}{p}\left(\frac{\xi_1}{(x-X_1)^2}+\text{cyclic}\right)=\frac{c}{32}\langle\mathbf{1}\rangle d\left(\frac{p'}{p}\right)^2 .$$

The cyclic symmetrisation of the remaining four terms on the r.h.s. of eq. (50) gives $d\left(\frac{\Theta(x)}{4p}\right)-\frac{\Theta(x)}{4p}\,d\log\langle\mathbf{1}\rangle$. We deduce the differential equation for $\mathbf{A}_1$. Firstly,

$$d\Theta(x)=4p\,d\left(\frac{\Theta}{4p}\right)+\Theta\,\frac{dp}{p} .$$

By the above, using $p'(X_1)=-a_0(X_1-X_2)(X_3-X_1)$ with $a_0=4$,

$$\begin{aligned}4p\,d\left(\frac{\Theta}{4p}\right) = &-\frac{p}{4}\Big(\frac{1}{(x-X_1)^2}\frac{\xi_1\Theta(X_1)}{(X_1-X_2)(X_3-X_1)}+\text{cyclic}\Big)\\&+x\left(\frac{\xi_1\Theta(X_1)}{(X_1-X_2)(X_3-X_1)}+\text{cyclic}\right)\\&-2P^{[1]}\,\frac{\det\Xi_{3,0}}{\det V_3}+\mathbf{A}_1\frac{\det\Xi_{3,1}}{\det V_3}+\Theta(x)\,d\log\langle\mathbf{1}\rangle .\end{aligned}\tag{51}$$

Secondly, using partial fraction decomposition,

$$\frac{\Theta(x)}{p}=-\frac{1}{(x-X_1)}\,\frac{\Theta(X_1)}{4(X_1-X_2)(X_3-X_1)}+\text{cyclic.}$$

Solving for Θ and using that

$$\frac{dp}{p}=-\left(\frac{\xi_1}{x-X_1}+\text{cyclic}\right) ,$$

yields

$$\Theta(x)\,\frac{dp}{p}=\frac{p}{4}\left(\frac{\Theta(X_1)}{(x-X_1)(X_1-X_2)(X_3-X_1)}+\text{cyclic}\right)\sum_{j=1}^{3}\frac{\xi_j}{(x-X_j)} .\tag{52}$$

Note that three terms in the sum on the r.h.s. of eq. (52) are equal but opposite to the first term on the r.h.s. of eq. (51). Since $\overline{\xi_1}=0$, we have for the remaining sum

$$\begin{aligned}&\frac{p}{4}\left(\frac{\Theta(X_1)}{(x-X_1)(X_1-X_2)(X_3-X_1)}\sum_{j\neq 1}\frac{\xi_j}{(x-X_j)}+\text{cyclic}\right)\\&=-\left(\frac{\Theta(X_1)(\xi_2X_3+\xi_3X_2)}{(X_1-X_2)(X_3-X_1)}+\text{cyclic}\right)-x\left(\frac{\xi_1\Theta(X_1)}{(X_1-X_2)(X_3-X_1)}+\text{cyclic}\right) ,\end{aligned}$$

where the second term on the r.h.s. is equal but opposite to the one before last on the r.h.s. of eq. (51). For the first term we have (cf. Appendix A)

$$-\frac{\Theta(X_1)(\xi_2X_3+\xi_3X_2)}{(X_1-X_2)(X_3-X_1)}+\text{cyclic}=-\frac{2}{3}ca_2\langle\mathbf{1}\rangle\frac{\det\Xi_{3,0}}{\det V_3}-2\mathbf{A}_1\frac{\det\Xi_{3,1}}{\det V_3}$$

Using $\Theta(X_1)=-4cX_1\langle\mathbf{1}\rangle+\mathbf{A}_1$, we conclude that

$$d\mathbf{A}_1=-\mathbf{A}_1\frac{\det\Xi_{3,1}}{\det V_3}+(-2P^{[1]}-\frac{2ca_2}{3}\langle\mathbf{1}\rangle)\frac{\det\Xi_{3,0}}{\det V_3}+\mathbf{A}_1\,d\log\langle\mathbf{1}\rangle .$$

Plugging in eq. (47) yields the claimed formula. To determine the constant in the $(2,5)$-minimal model, we write

$$p = 4x^3 + a_1 x^2 + a_2 x + a_3 \,.$$

By Lemma 5 in [11], using $c = -\frac{22}{5}$, we find

$$P^{[1]} = -\frac{77}{400} a_1^2 \langle \mathbf{1} \rangle + \frac{2}{20} a_1 \mathbf{A}_1 + \frac{143}{100} a_2 \langle \mathbf{1} \rangle - \frac{1}{16} \langle \mathbf{1} \rangle^{-1} \mathbf{A}_1^2 \,.$$

□

The formulation in terms of determinants generalises easily to higher genus (cf. Section 5.3).

5.2 Comparison with the analytic approach, for the $(2,5)$ minimal model

We check that the system of linear differential equations obtained from Theorem 3 for the $(2,5)$ minimal model is consistent with the system discussed in Section 3.2. The relation (48) suggests the ansatz

$$\langle \mathbf{1} \rangle = \Delta^{(0)-\frac{c}{48}} f \,, \quad \mathbf{A}_1 = \Delta^{(0)-\frac{c}{48}} g \,, \tag{53}$$

for some functions f, g of τ, with $f, g \propto \langle \mathbf{1} \rangle_z$. We have [16]

$$\Delta^{(0)} = \prod_{i<j} (X_i - X_j)^2 \sim \eta^{24} = q - 24q^2 + O(q^3) \,,$$

and so close to the boundary of the moduli space where $X_1 \approx X_2$, we have

$$(X_1 - X_2) \sim q^{\frac{1}{2}} = e^{\pi i \tau} \,. \tag{54}$$

As before, we shall work with assumption (33). Since in this region only the difference $X_1 - X_2$ matters, we may w.l.o.g. suppose that

$$X_2 = \text{const.}$$

($\xi_2 = 0$). In view of (54) on the one hand, and the series expansion of the Rogers-Ramanujan partition functions $\langle \mathbf{1} \rangle_z$ on the other, we have to show that

$$f \sim (X_1 - X_2)^{-\frac{1}{30}} \,, \quad \text{or} \quad f \sim (X_1 - X_2)^{\frac{11}{30}} \,. \tag{55}$$

The ansatz (53) yields

$$d\langle \mathbf{1} \rangle = \Delta^{(0)-\frac{c}{48}} df - \frac{c}{24} \omega f \Delta^{(0)-\frac{c}{48}} \,,$$

using eq. (40), and a similar equation is obtained for $d\mathbf{A}_1$. So by Theorem 3,

$$df = -\frac{1}{8} g \frac{\det \Xi_{3,0}}{\det V_3} \,, \tag{56}$$

$$(d - \frac{1}{3}\omega) g = \frac{11}{150} a_2 f \frac{\det \Xi_{3,0}}{\det V_3} \,.$$

Since $f \sim (X_1 - X_2)^\alpha$ for some $\alpha \in \mathbb{R}$,

$$d f \sim \frac{\xi_1 \alpha}{X_1 - X_2} f \,. \tag{57}$$

On the r.h.s. of eq. (56), we have by the assumption (33),

$$\frac{\det \Xi_{3,0}}{\det V_3} = \frac{\xi_1}{(X_1 - X_2)(X_3 - X_1)} + \text{cyclic} \sim \frac{\xi_1}{(X_1 - X_2)(-3X_2)} \sim \frac{\omega}{(-3X_2)}$$

since $X_1 \approx X_2$, and we have omitted the regular terms. Eq. (56) thus yields

$$g \approx 24 X_2 \alpha f \,.$$

Now we use the differential equation for g,

$$24 X_2 \alpha (d - \frac{1}{3}\omega) f \sim \frac{11}{150} f a_2 \, \frac{\omega}{(-3X_2)}$$

which by eq. (57) and $a_2 \sim -12 X_2^2$ reduces to the quadratic equation

$$\alpha(\alpha - \frac{1}{3}) \sim \frac{11}{900}$$

and is solved by $\alpha = -\frac{1}{30}$ and $\frac{11}{30}$. This yields (55), so the check works.

5.3 Outlook: Generalisation to higher genus

When $\deg p = n$, we have [11]

$$\Theta(x, y) = \Theta^{[1]}(x) + y\Theta^{[y]}(x) \,, \quad \deg \Theta^{[1]}(x) = n - 2 \,.$$

$\Theta^{[y]}$ gives rise to poles of half integer order and so does not contribute to the contour integral. In the case $n = 5$ ($g = 2$), $\Theta^{[y]}$ is actually absent,

$$\langle T(x) \rangle = \frac{c}{32} \frac{[p']^2}{p^2} \langle \mathbf{1} \rangle + \frac{1}{4} \, \frac{A_0 x^3 + A_1 x^2 + A_3 x + A_4}{p} \,.$$

The matrices to consider are the 5×5 Vandermonde matrix V_5 and

$$\Xi_{5,k} := \begin{pmatrix} X_1^3 & X_2^3 & X_3^3 & X_4^3 & X_5^3 \\ X_1^2 & X_2^2 & X_3^2 & X_4^2 & X_5^2 \\ X_1 & X_2 & X_3^3 & X_4^3 & X_5 \\ 1 & 1 & 1 & 1 & 1 \\ \xi_1 X_1^k & \xi_2 X_2^k & \xi_3 X_3^k & \xi_4 X_4^k & \xi_5 X_5^k \end{pmatrix}, \quad k = 0, \ldots, 3.$$

The actual calculations are more extensive but straightforward.

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A Completion of the proof of Theorem 3 in Sect. 5.1

It remains to show that

$$-\frac{\Theta(X_1)(\xi_2 X_3+\xi_3 X_2)}{(X_1-X_2)(X_3-X_1)}+\text{cyclic} = -\frac{2}{3}ca_2\langle\mathbf{1}\rangle\frac{\det\Xi_{3,0}}{\det V_3}-2\mathbf{A}_1\frac{\det\Xi_{3,1}}{\det V_3}\,.$$

We have

$$\begin{aligned}\xi_2X_3+\xi_3X_2 &= (\xi_2+\xi_3)(X_2+X_3)-(\xi_2X_2+\xi_3X_3)\\ &= \xi_1X_1-(\xi_2X_2+\xi_3X_3)\\ &= 2\xi_1X_1-\sum_{i=1}^{3}\xi_iX_i\,.\end{aligned}$$

It follows that

$$-\frac{\Theta(X_1)(\xi_2 X_3+\xi_3 X_2)}{(X_1-X_2)(X_3-X_1)}+\text{cyclic} = \frac{8c\langle\mathbf{1}\rangle\,\xi_1X_1^2-2\mathbf{A}_1\xi_1X_1}{(X_1-X_2)(X_3-X_1)}+\text{cyclic}\,,$$

since $\sum_i\xi_iX_i$ is symmetric and both

$$\frac{1}{(X_1-X_2)(X_3-X_1)}+\text{cyclic}=0 \qquad (58)$$

$$\frac{X_1}{(X_1-X_2)(X_3-X_1)}+\text{cyclic}=0\,. \qquad (59)$$

Now

$$X_1^2 = -X_1(X_2+X_3) = -\frac{a_2}{4}+X_2X_3\,, \qquad (60)$$

we claim that

$$\frac{\xi_1X_2X_3}{(X_1-X_2)(X_3-X_1)}+\text{cyclic}=\frac{a_2}{6}\frac{\det\Xi_{3,0}}{\det V_3}\,. \qquad (61)$$

Indeed, since $\xi_1X_2X_3+\xi_2X_3X_1+\xi_3X_1X_2=\overline{\xi_1X_2X_3}$, we have by eq. (58),

$$\frac{\xi_1X_2X_3}{(X_1-X_2)(X_3-X_1)}+\text{cyclic} = -\frac{\xi_2X_3X_1+\xi_3X_1X_2}{(X_1-X_2)(X_3-X_1)}+\text{cyclic}\,.$$

Since $\overline{\xi_1}=0$, we have

$$\begin{aligned}-\frac{\xi_2X_3X_1+\xi_3X_1X_2}{(X_1-X_2)(X_3-X_1)}+\text{cyclic} &= \left(\frac{\xi_1(X_3X_1+X_1X_2)}{(X_1-X_2)(X_3-X_1)}+\text{cyclic}\right)+\left(\frac{(\xi_3X_3+\xi_2X_2)X_1}{(X_1-X_2)(X_3-X_1)}+\text{cyclic}\right)\\ &= \frac{a_2}{4}\frac{\det\Xi_{3,0}}{\det V_3}-\left(\frac{\xi_1X_2X_3}{(X_1-X_2)(X_3-X_1)}+\text{cyclic}\right)\\ &\quad-\left(\frac{\xi_1X_1^2}{(X_1-X_2)(X_3-X_1)}+\text{cyclic}\right)\,,\end{aligned}$$

using symmetry of $\sum_i\xi_iX_i$ and eq. (59) again. From eq. (60) follows eq. (61), and the proof of Theorem 3 is complete.

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