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Theory of an Inverted Pendulum with Trifilar Suspension

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ABSTRACT

The theory of small oscillations is applied to the inverted, trifilar suspension pendulum. The potential energy in a small displacement is calculated from geometrical considerations using vectorial methods. The periods of the principal modes of oscillation are found. The theory is applied to the O'LEARY Seismograph (mass $1\frac{3}{4}$ tons) at Rathfarnham Castle and gives results in close agreement with the measured values.

The pendulum discussed in the following pages was designed by the REV. WILLIAM O'LEARY, S.J., and incorporated in two seismographs he built and in a later model constructed at Rathfarnham. It is an inverted, vertical pendulum and achieves a long period by its

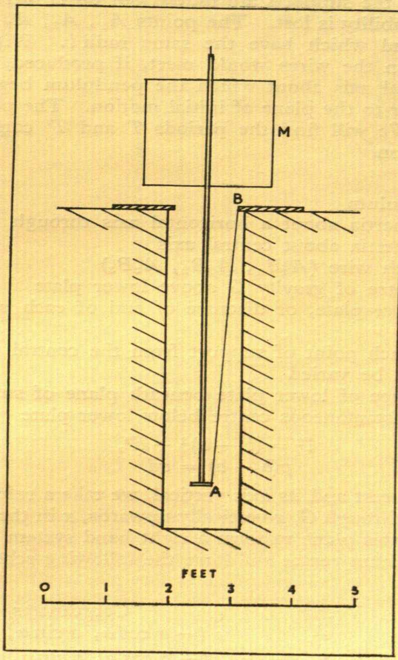

FIG. I — Diagrammatic Section of the O'LEARY Inverted Pendulum Seismograph at Rathfarnham Castle, Dublin trifilar suspension. The fundamental periods of this pendulum are calculated and compared with the existing models. Previous, unpublished solutions for the motion in one particular plane were given by REV. D. O'CONNELL, S.J., REV. P. HEELAN, S.J. and REV. R. E. INGRAM, S.J. and also by DR. A. CONWAY, PROF. A. O'RAHILLY and REV. W. O'LEARY, though of these there is no record in the observatory.

The theory, as developed here, makes three assumptions. (1) The suspension wires remain constant in length when the pendulum oscillates. (2) The stiffness of the wires is not involved in the energy terms of the system. (3) The motion of the pendulum is small.

The pendulum is a heavy cylindrical mass M fitted to a long shaft at the end of which there is a small circular plate A (*Fig.* I). Three suspension wires of equal length l are attached to symmetrical points A_1, A_2, A_3, apices of an equilateral triangle (*Fig.* II) at equal distances a from the centre of the plate A and to adjustable points of support B_1, B_2, B_3 in a horizontal plane just below the cylindrical mass. These points of support can move on lines which, if produced, would meet on the central axis and have angles of intersection equal to $2\pi/3$. But they are always kept at equal distances b from the central axis and with b greater than a. As the supports are moved out (b is increased) the period of the oscillation increases until stability is lost. The points A_1, A_2, A_3 move on spheres whose centres are B_1, B_2, B_3 and which have the same radii l. With the pendulum in its undisplaced, central position the wires would meet, if produced, in a point I below the lower plate. The horizontal axis about which the pendulum begins to rotate is through this point and perpendicular to the plane of initial motion. The pendulum may also rotate about the vertical axis. We will find the periods T and T' corresponding to these two principal modes of oscillation.

Let

M = Mass of pendulum
Mk^2 = Moment of inertia about a horizontal axis through the centre of gravity
Mk'^2 = Moment of inertia about central axis
l = Length of each wire (A_1B_1, A_2B_2, A_3B_3)
h = Height of centre of gravity G above lower plate
a = Radius of lower plate, or distance of end of each wire from central axis of pendulum
b = Distance of each point of support from the central axis of the pendulum at rest ; b can be varied
d = Vertical distance of lower plate beneath plane of support
p = Distance of instantaneous centre below lower plate

$$l^2 = (b - a)^2 + d^2,$$
$$p(b - a) = ad$$

With the pendulum at rest and its axis vertical, we take a reference frame of Cartesian co-ordinates fixed in space through G, z vertically upwards, x in the plane GB_1I and towards B_1 and y perpendicular to this plane to form a right hand system (*Fig.* II). We use $\mathbf{a}'$ as a row vector and $\mathbf{a}$ as a column vector and have the following scheme for the pendulum at rest in its central position.

Point	Vector	Coordinates
A_j	$\mathbf{a}'_j$	$(-a\cos\alpha,\ a\sin\alpha,\ -h)$
B_j	$\mathbf{b}_j$	$(-b\cos\alpha,\ b\sin\alpha,\ d-h)$

where $\alpha = \pi,\ \pi/3,\ -\pi/3$ as $j = 1, 2, 3$

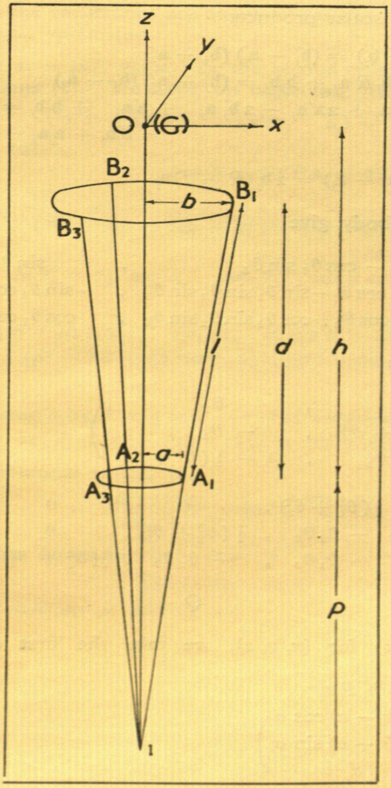

Fig. II

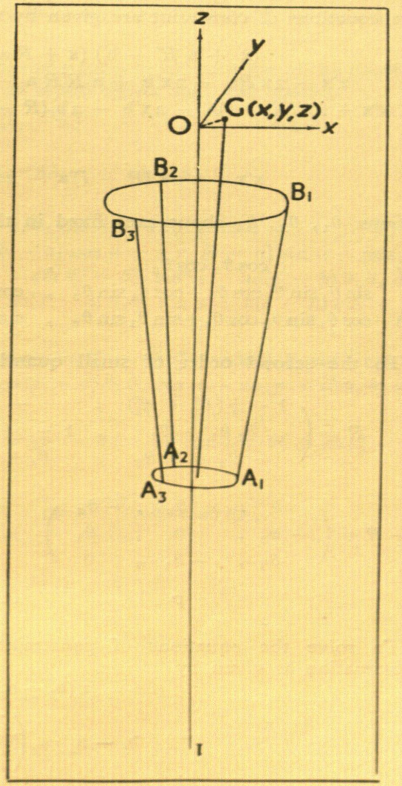

Fig. III

FIG. II — Diagram of the Pendulum in Equilibrium Position
FIG. III — Diagram of the Pendulum in Displaced Position

The most general movement of the pendulum is a translation, whereby the centre of gravity, G, moves from the origin to a new position given by the vector $\mathbf{X}'$ with coordinates (x, y, z), and a rotation about this point. Let R be the rotation matrix with $RR' = I$, the unit matrix. The points B_j are fixed. The points A move as

$$\mathbf{a}_j' \quad \rightarrow \quad \mathbf{x}' + \mathbf{a}_j' R'$$

subject to the condition that they move on spheres with centres B_j and radii equal to l.

These equations of constraint are given by the scalar products

$$(\mathbf{x}' + \mathbf{a}_j' R' - \mathbf{b}_j')(\mathbf{x} + R\,\mathbf{a}_j - \mathbf{b}_j) = (\mathbf{b}_j' - \mathbf{a}_j')(\mathbf{b}_j - \mathbf{a}_j)$$

$$\mathbf{x}'\mathbf{x} + 2\mathbf{x}'R\mathbf{a}_j - 2\mathbf{x}'\mathbf{b}_j + \mathbf{a}_j' R'R\,\mathbf{a}_j - 2\mathbf{b}_j' R\,\mathbf{a}_j + \mathbf{b}_j'\mathbf{b}_j = (\mathbf{b}_j' - \mathbf{a}_j')(\mathbf{b}_j - \mathbf{a}_j)$$

$$\mathbf{x}'\mathbf{x} + 2\mathbf{x}'(R - I)\mathbf{a}_j - 2\mathbf{x}'\mathbf{b}_j - 2\mathbf{b}_j'(R - I)\mathbf{a}_j + 2\mathbf{x}'\mathbf{a}_j - 2\mathbf{b}_j'\mathbf{a}_j + \mathbf{a}_j'\mathbf{a}_j + \mathbf{b}_j'\mathbf{b}_j = $$
$$= \mathbf{b}_j'\mathbf{b}_j + \mathbf{a}_j'\mathbf{a}_j - 2\mathbf{b}_j'\mathbf{a}_j$$

$$\mathbf{x}'\mathbf{x} + 2\mathbf{x}'(R - I)\,\mathbf{a}_j - 2\mathbf{x}'(\mathbf{b}_j - \mathbf{a}_j) = 2\mathbf{b}_j'(R - I)\,\mathbf{a}_j \tag{1}$$

Rotations θ_1, θ_2, θ_3 about axes fixed in the body give

$$R = \begin{pmatrix} \cos\theta_2\cos\theta_3 , & -\cos\theta_2\sin\theta_3 , & \sin\theta_2 \\ \sin\theta_1\sin\theta_2\cos\theta_3 + \cos\theta_1\sin\theta_3 , & \cos\theta_1\cos\theta_3 - \sin\theta_1\sin\theta_2\sin\theta_3 , & -\sin\theta_1\cos\theta_2 \\ -\cos\theta_1\sin\theta_2\cos\theta_3 + \sin\theta_1\sin\theta_3 , & \sin\theta_1\cos\theta_3 + \cos\theta_1\sin\theta_2\sin\theta_3 , & \cos\theta_1\cos\theta_2 \end{pmatrix}$$

To the second order of small quantities

$$R = \begin{pmatrix} 1 - \tfrac{1}{2}(\theta_2^2 + \theta_3^2) , & -\theta_3 , & \theta_2 \\ \theta_1\theta_2 + \theta_3 , & 1 - \tfrac{1}{2}(\theta_1^2 + \theta_3^2) , & -\theta_1 \\ -\theta_2 + \theta_1\theta_3 , & \theta_1 + \theta_2\theta_3 , & 1 - \tfrac{1}{2}(\theta_1^2 + \theta_2^2) \end{pmatrix}$$

$$I - R = \begin{pmatrix} 0 , & \theta_3 , & -\theta_2 \\ -\theta_3 , & 0 , & \theta_1 \\ \theta_2 , & -\theta_1 , & 0 \end{pmatrix} + \begin{pmatrix} \tfrac{1}{2}(\theta_2^2 + \theta_3^2) , & 0 , & 0 \\ -\theta_1\theta_2 , & \tfrac{1}{2}(\theta_1^2 + \theta_3^2) , & 0 \\ -\theta_1\theta_3 , & -\theta_2\theta_3 , & \tfrac{1}{2}(\theta_1^2 + \theta_2^2) \end{pmatrix}$$

$$= \qquad P \qquad + \qquad Q$$

To solve the equations of constraint (1) for (x, y, z), we take the first order approximation as given by

$$\mathbf{x}'(\mathbf{b}_j - \mathbf{a}_j) = \mathbf{b}_j' P\,\mathbf{a}_j \tag{2}$$

$$\mathbf{b}_j - \mathbf{a}_j = \begin{pmatrix} -(b - a)\cos\alpha \\ (b - a)\sin\alpha \\ d \end{pmatrix}$$

$$\mathbf{b}_j' P\,\mathbf{a}_j = -[ad + h(b - a)](\theta_1\sin\alpha + \theta_2\cos\alpha)$$

$$\begin{aligned} (b - a)\,x + dz &= [ad + h(b - a)]\,\theta_2 \\ -(b - a)\,x\cos\pi/3 + (b - a)\,y\sin\pi/3 &= -[ad + h(b - a)](\theta_1\sin\pi/3 + \theta_2\cos\pi/3) \\ -(b - a)\,x\cos\pi/3 - (b - a)\,y\sin\pi/3 &= -[ad + h(b - a)](-\theta_1\sin\pi/3 + \theta_2\cos\pi/3) \end{aligned}$$

Hence

$$x = (p + h)\,\theta_2, \quad y = -(p + h)\,\theta_1, \quad z = 0 \tag{3}$$

where

$$p = ad/(b - a)$$

Since the oscillations are about a position of equilibrium, z is obviously of the second order in small quantities.

Let $\mathbf{x}_1' = (x_1, y_1, z_1)$ be the second order approximation to the solution of the constraint equations (1). We have, on neglecting terms of order higher than the second,

$$\mathbf{x}'\mathbf{x} - 2\mathbf{x}'P\mathbf{a}_j - 2\mathbf{x}'(\mathbf{b}_j - \mathbf{a}_j) - 2\mathbf{x}_1'(\mathbf{b}_j - \mathbf{a}_j) = -2\mathbf{b}_j' P\mathbf{a}_j - 2\mathbf{b}_j' Q\mathbf{a}_j$$

using (2)

$$\mathbf{x}'\mathbf{x} - 2\mathbf{x}'P\mathbf{a}_j - 2\mathbf{x}_1'(\mathbf{b}_j - \mathbf{a}_j) = -2\mathbf{b}_j' Q\mathbf{a}_j$$

and by (3)

$$\begin{aligned}(p+h)^2(\theta_1^2+\theta_2^2) - 2(p+h)\,[h(\theta_1^2+\theta_2^2) - a\,\theta_1\theta_3\cos\alpha + a\,\theta_2\theta_3\sin\alpha\,+ \\ +\,2\,x_1(b-a)\cos\alpha - 2\,y_1(b-a)\sin\alpha - 2dz_1] = \\ = -ab(\theta_2^2+\theta_3^2)\cos^2\alpha - 2ab\,\theta_1\theta_2\sin\alpha\cos\alpha - ab(\theta_1^2+\theta_3^2)\sin^2\alpha - 2a(d-h)\,\theta_1\theta_3\cos\alpha \\ +\,2a(d-h)\,\theta_2\theta_3\sin\alpha + h(d-h)\,(\theta_1^2+\theta_2^2)\,.\end{aligned}$$

$$\begin{aligned}-2x_1(b-a)\cos\alpha + 2y_1(b-a)\sin\alpha + 2dz_1 = \\ = (p^2 - hd + ab\sin^2\alpha)\,\theta_1^2 + (p^2 - hd + ab\cos^2\alpha)\,\theta_2^2 + ab\,\theta_3^2 + 2ab\,\theta_1\theta_2\sin\alpha\cos\alpha\,+ \\ +\,2a(p+d)\,\theta_1\theta_3\cos\alpha - 2a(p+d)\,\theta_2\theta_3\sin\alpha\end{aligned}$$

Replacing α by π, $\pi/3$ and $-\pi/3$ and solving

$$2z_1 d = [p^2 - hd + ab(1 - \cos\pi/3)]\,\theta_1^2 + [p^2 - hd + ab\cos\pi/3]\,\theta_2^2 + ab\,\theta_3^2$$

The potential energy is

$$\frac{Mg}{2d}\left\{\left(p^2 - hd + \frac{ab}{2}\right)\theta_1^2 + \left(p^2 - hd + \frac{ab}{2}\right)\theta_2^2 + ab\,\theta_3^2\right\}$$

The initial kinetic energy is

$$\tfrac{1}{2}M\,(k_1^2\dot\theta_1^2 + k_2^2\dot\theta_2^2 + k_3^2\dot\theta_3^2) + \tfrac{1}{2}M\,(p+h)^2(\dot\theta_1^2 + \dot\theta_2^2)$$

The equations of motion are

$$M\left[k_1^2 + (p+h)^2\right]\ddot\theta_1 + \frac{Mg}{d}\left(p^2 - hd + \frac{ab}{2}\right)\theta_1 = 0$$

$$M\left[k_2^2 + (p+h)^2\right]\ddot\theta_2 + \frac{Mg}{d}\left(p^2 - hd + \frac{ab}{2}\right)\theta_2 = 0$$

$$M\,k_3^2\,\ddot\theta_3 + Mg\,\frac{ab}{d}\,\theta_3 = 0$$

Since $k_1 = k_2 = k$ and $k_3 = k'$, the periods are

$$T = T_1 = T_2 = 2\pi\sqrt{\frac{d\,[k^2 + (h+p)^2]}{g\left(p^2 - hd + \dfrac{ab}{2}\right)}}$$

and

$$T' = T_3 = 2\pi\sqrt{\frac{dk'^2}{gab}}$$

T increases with b until stability is lost when $p^2 + ab/2 = hd$

T' decreases as b increases. This period is the period of a rotation about the central axis which occurs rarely.

Application of the Theory to Existing Models

Two models of the O'Leary seismograph are in existence. The larger one, with a mass of $1\frac{3}{4}$ tons, is installed at the Seismological Observatory of Rathfarnham Castle and has given daily records for the past 37 years. The smaller one, with a mass of 50 lbs., is at present in an experimental setup at the School of Cosmic Physics. It is being converted for electromagnetic pickup and photographic recording.

The large seismograph was installed in a narrow pit to reduce thermal effects but unfortunately in such a manner that it is not now possible, without dismantling, to ascertain what is the true manner of oscillation and whether assumptions 1 and 2 mentioned at the beginning are applicable. No records of details of the construction are available and it is assumed that the suspension wires consist of steel cables about a quarter of an inch in diameter which are probably strong but flexible enough to satisfy the assumptions.

For the large O'Leary Seismograph at Rathfarnham Castle

$$M = 3{,}940 \text{ lbs.}$$
$$h = 89 \text{ ins.}$$
$$k^2 = 172 \text{ ins}^2.$$
$$k'^2 = 84 \text{ ins}^2.$$
$$l = 71 \text{ ins.}$$
$$a = 3\tfrac{3}{16} \text{ ins.}$$

With $b = 5{\cdot}7$ ins., the calculated values of the periods are

$$T = 15{\cdot}7 \text{ secs and } T' = 5{\cdot}7 \text{ secs}.$$

The measured values are

$$T = 15{\cdot}6 \text{ secs and } T' = 6{\cdot}0 \text{ secs}$$

which is as good an agreement as can be expected.

It is intended that, whenever the routine observations can be suspended for a while, the seismograph will be dismantled, the various constants remeasured and the suspension wires inspected.

The case of the small seismograph is quite different. The suspension wires are of piano wire with diameters within the range 25 to 42 thousandths of an inch. For small diameter wires assumption 1 would not hold but 2 would, while for the larger diameter wires the reverse is more likely. Suitable small steel cables which might satisfy both assumptions have not been obtained as yet; the only available samples, because they are not under sufficient tension, act like small springs and in fact with them, it is not practicable to keep the pendulum upright.

The periods of this pendulum, not surprisingly, do not agree with the theory developed above and both experimental and theoretical work is being continued to solve the behaviour of this seismograph.

Seismological Observatory,
Rathfarnham Castle,
Dublin,

January 7, 1954