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Grassmann Variable Analysis for 1D and 2D Ising Models

By

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(Lectures from 2002)

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Abstract

The fermionic formulation for low-dimensional Ising models is given in terms of purely anti-commuting Grassmann variables. The discussion includes fermionization procedures based on mirror-ordered factorization of the density matrix, free-fermion representation of the partition function and momentum-space analysis and analytic solutions. The fermionic decription of the Ising chain is considered in some detail as an introduction to the 2D Ising model on a rectangular lattice.

These notes were prepared in relation to lectures given in 2002 at the School of Theoretical Physics of the Dublin Institute for Advanced Studies.

1 Spin-variable formulation of the 1D Ising chain

The basic variable in the Ising model is the Ising spin $\sigma_m = \pm 1$ associated with a site m. The 1D chain Hamiltonian H is given by

$$-\beta H = \sum_{m=1}^{L} b_{m+1}\sigma_m\sigma_{m+1}, \qquad b_m = \frac{J_m}{kT}, \tag{1.1}$$

where T is the absolute temperature, k is Boltzmann's constant, and $\beta = 1/(kT)$. The Gibbs density matrix is

$$\begin{aligned} e^{-\beta H(\sigma)} &= \prod_{m=1}^{L} e^{b_{m+1}\sigma_m\sigma_{m+1}} && (1.2) \\ &= \prod_{m=1}^{L} (\cosh b_{m+1} + \sigma_m\sigma_{m+1}\sinh b_{m+1}) \\ &= \prod_{m=1}^{L} \cosh b_{m+1}(1 + \sigma_m\sigma_{m+1}\tanh b_{m+1}) \\ &= \left\{\prod_{m=1}^{L} \cosh b_{m+1}\right\} \sum_{m=1}^{L} (1 + t_{m+1}\sigma_m\sigma_{m+1}), && (1.3) \end{aligned}$$

where $t_m = \tanh b_m$. The partition function becomes

$$Z = \sum_{\{\sigma_m = \pm 1\}} e^{-\beta H(\sigma)} = \left\{\prod_{m=1}^{L} (2\cosh b_{m+1})\right\} Q, \tag{1.4}$$

where the reduced partition function Q is given by

$$Q = \underset{(\sigma)}{\mathrm{Sp}} \left\{\prod_{m=1}^{L} (1 + t_{m+1}\sigma_m\sigma_{m+1})\right\}, \tag{1.5}$$

in which we defined the normalization spin-averaging by

$$\underset{(\sigma)}{\mathrm{Sp}}\{\cdots\} = \prod_{m=1}^{L} \underset{\sigma_m}{\mathrm{Sp}}\{\cdots\} \text{ with } \underset{\sigma_m}{\mathrm{Sp}}(\cdots) = \frac{1}{2}\sum_{\sigma_m = \pm 1} (\cdots).$$

Possible spin-closing conditions (boundary conditions) are:

$$\sigma_{L+1}* = \sigma_1 \quad \text{(periodic)}$$
$$\sigma_{L+1} = -\sigma_1 \quad \text{(anti-periodic)}$$
$$\sigma_{L+1} = 0 \quad \text{(free)}.$$

Remark. It turns out that periodic spins correspond to aperiodic fermions and vice versa in the 1D case.

2 Fermionization for the 1D case

We want to compute the (reduced) partition function

$$Q = \operatorname*{Sp}_{(\sigma)} \left\{ \prod_{m=1}^{L} (1 + t_{m+1}\sigma_m\sigma_{m+1}) \right\}, \tag{2.1}$$

by changing the sum over spin variables into a fermionic integral. We perform this transformation in two steps as follows:

$$Q = \operatorname*{Sp}_{(\sigma)} Q(\sigma) \to \operatorname*{Sp}_{(\sigma,a)} Q(\sigma, a) \to \operatorname*{Sp}_{(a)} Q(a),$$

i.e. we start with a spin-variable densiity matrix $Q(\sigma)$, then introduce new (Grassmann) anti-commuting variables (a) by means of a factorization of local Boltzmann weights, passing to the mixed (σ, a) representation, and finally eliminate the spin variables. The result is a purely fermionic representation for the partition function Q in the form of an 'integral' over Grassmann variables. These variables may be viewed as classical fermions. For any Grassmann variable a, we set

$$\int 1\, da = 0 \text{ and } \int a\, da = 1.$$

This yields the following simple rules for Gaussian Grassmann integrals with two variables a and $\overline{a}$: For any complex variable λ, we have

$$\begin{aligned} \int d\overline{a}\, da\, e^{\lambda a\overline{a}} &= \lambda \\ \int d\overline{a}\, da\, a\, e^{\lambda a\overline{a}} &= 0 \\ \int d\overline{a}\, da\, \overline{a}\, e^{\lambda a\overline{a}} &= 0 \\ \int d\overline{a}\, da\, a\overline{a}\, e^{\lambda a\overline{a}} &= 1. \end{aligned} \tag{2.2}$$

This follows from

$$e^{\lambda a\overline{a}} = 1 + \lambda a\overline{a}.$$

(Note that, for any Grassmann variable a, $a^2 = 0$, and for two variables a and $\overline{a}$, $a\overline{a} + \overline{a}a = 0$.)

To pass to a fermionic representation, we introduce a pair of Grassmann variables a_m and $\overline{a}_m$ at each site of the lattice and factorize the local weights from $Q(\sigma)$ as follows.

$$\begin{aligned} &1 + t_{m+1}\sigma_m\sigma_{m+1} = \\ &= \int d\overline{a}_m da_m\, e^{a_m\overline{a}_m}(1 + a_m\sigma_m)(1 + t_{m+1}\sigma_{m+1}\overline{a}_m) \\ &= \underset{(a)}{\mathrm{Sp}}\{A_m\overline{A}_{m+1}\}, \end{aligned} \tag{2.3}$$

where

$$A_m = 1 + a_m\sigma_m \text{ and } \overline{A}_{m+1} = 1 + t_{m+1}\sigma_{m+1}\overline{a}_m, \tag{2.4}$$

and where $\mathrm{Sp}_{(a)}$ denotes the Grassmann integration with a Gaussian weight, i.e.

$$\underset{(a)}{\mathrm{Sp}}(\cdots) = \int d\overline{a}\, da\, e^{a\overline{a}}(\cdots).$$

An even more compact notation is to drop the integral notation altogether, and write simply

$$1 + t_{m+1}\sigma_m\sigma_{m+1} = A_m\overline{A}_{m+1}. \tag{2.5}$$

The identity (2.3) can be checked by integrating term by term the fermionic polynomial

$$
\begin{aligned}
&(1 + a_m\sigma_m)(1 + t_{m+1}\sigma_{m+1}\overline{a}_m) = \\
&\qquad = 1 + a_m\sigma_m + t_{m+1}\sigma_{m+1}\overline{a}_m + t_{m+1}a_m\overline{a}_m\sigma_m\sigma_{m+1}, \qquad (2.6)
\end{aligned}
$$

and following the above rules for Gaussian integrals. (Note that products of even numbers of Grassmann variables commute with any other element of the Grassmann algebra, while odd products in general neither commute nor anti-commute with an arbitrary element of the algebra.)

We now have to substitute the factors (2.5) into $Q(\sigma)$ and average over the spin variables. Note that, although individual factors A_m and $\overline{A}_{m+1}$ neither commute nor anti-commute with other elements, the products $A_m\overline{A}_{m+1}$ can (effectively) be commuted with each other under the integral sign of total fermionic averaging over all Grassmann variables a_m and $\overline{a}_m$. This is the case because the non-commuting fermionic terms a_m and $\overline{a}_m$ contained in $A_m\overline{A}_{m+1}$ average to zero. Note that the indices of the factors A_m and $\overline{A}_{m+1}$ have been chosen to be the same as those of the spin-variables involved. We now substitute $A_m\overline{A}_{m+1}$ into $Q(\sigma)$ and combine the factors with the same index m to be averaged over $\sigma_m = \pm 1$. Assuming free boundary conditions $\sigma_{L+1} = 0$ at this stage, we have $\overline{A}_{L+1} = 1$. The averaging then goes as follows.

$$
\begin{aligned}
Q(\sigma) &= \operatorname*{Sp}_{(a)} \left\{ \prod_{m=1}^{L} A_m\overline{A}_{m+1} \right\} \\
&= \operatorname*{Sp}_{(a)} \left\{ (A_1\overline{A}_2)(A_2\overline{A}_3)\ldots(A_L\overline{A}_{L+1}) \right\} \\
&= \operatorname*{Sp}_{(a)} \left\{ A_1(\overline{A}_2A_2)(\overline{A}_3A_3)\ldots(\overline{A}_LA_L)\overline{A}_{L+1} \right\} \\
&= \operatorname*{Sp}_{(a)} \left\{ \prod_{m=1}^{L} \overline{A}_mA_m \right\}, \qquad (2.7)
\end{aligned}
$$

where we also set $\overline{A}_1 = 1$, where formally $\overline{A}_0 = 0$ corresponds to free boundary conditions for fermion variables.

Thus, for the density matrix $Q(\sigma)$ of the 1D chain with open ends (free boundary conditions), we obtain the factorized representation

$$\begin{aligned}
Q(\sigma) &= \prod_{m=1}^{L} (1 + t_{m+1}\sigma_m\sigma_{m+1})\big|_{\sigma_{L+1}=0} \\
&= \underset{(a)}{\mathrm{Sp}} \left\{ \prod_{m=1}^{L} \overline{A}_m A_m \right\} \Bigg|_{\overline{a}_0=0} \\
&= \int \prod_{m=1}^{L} d\overline{a}_m da_m \, e^{\sum_{m=1}^{L} a_m \overline{a}_m} \prod_{m=1}^{L} \{(1 + t_m \overline{a}_{m-1}\sigma_m)(1 + a_m\sigma_m)\}.
\end{aligned} \tag{2.8}$$

Next we average over the spins $\sigma_m = \pm 1$ at each site to obtain

$$\begin{aligned}
\underset{(\sigma_m)}{\mathrm{Sp}} (\overline{A}_m A_m) &= \frac{1}{2} \sum_{\sigma_m = \pm 1} (1 + t_m \overline{a}_{m-1}\sigma_m)(1 + a_m\sigma_m) \\
&= 1 + t_m \overline{a}_{m-1} a_m \\
&= 1 - t_m a_m \overline{a}_{m-1} = e^{-t_m a_m \overline{a}_{m-1}}.
\end{aligned} \tag{2.9}$$

The partition function then becomes

$$\begin{aligned}
Q &= \underset{(\sigma)}{\mathrm{Sp}}\, Q(\sigma) = \underset{(\sigma)}{\mathrm{Sp}}\, \underset{(a)}{\mathrm{Sp}} \left\{ \prod_{m=1}^{L} \overline{A}_m A_m \right\} \\
&= \int \prod_{m=1}^{L} d\overline{a}_m da_m \, \exp \left\{ \sum_{m=1}^{L} [a_m \overline{a}_m - t_m a_m \overline{a}_{m-1}] \right\}.
\end{aligned} \tag{2.10}$$

Here $\overline{a}_0 = 0$ for free boundary conditions. This is an exact fermionic representation for the inhomogeneous 1D Ising chain with open ends (free boundary conditions).

3 The 1D Ising chain with periodic-aperiodic closing conditions in terms of (classical) fermions

In the previous section we considered the fermionization of an open chain. We now consider a chain which is closed into a ring, i.e. we assume the periodic spin-closing condition for the spin variables: $\sigma_{L+1} = \sigma_1$. The final Boltlzmann weight is then of the form $1 + t_{L+1}\sigma_L\sigma_{L+1} = 1 + t_{L+1}\sigma_L\sigma_1 = 1 + t_1\sigma_1\sigma_L$ if we set $t_1 = t_{L+1}$. In principle we could try to apply directly the same procedure as in the case of the open chain. This would lead to the representation

$$Q(\sigma) = A_1(\overline{A}_2A_2)\ldots(\overline{A}_LA_L)\overline{A}_{L+1}. \tag{3.1}$$

For free boundary conditions the last factor $\overline{A}_{L+1} = 1$, and we set $\overline{A}_1 = 1$. In the present case, $\overline{A}_{L+1} = 1 + t_{L+1}\overline{a}_L\sigma_{L+1} = 1 + t_1\overline{a}_L\sigma_1 \neq 1$. This factor may also be viewed as a factor

$$\overline{A}_1' = 1 + t_1\overline{a}_0\sigma_1\big|_{\overline{a}_0=\overline{a}_L}.$$

We then have to move this factor to the beginning of the chain from right to left. Interchanging this factor with the previous factors, the term involving $\overline{a}_L$ changes sign an odd number of times, so that we end up with

$$\overline{A}_1 = 1 - t_1\overline{a}_L\sigma_1 = 1 + t_1\overline{a}_0\sigma_1\big|_{\overline{a}_0=-\overline{a}_L}. \tag{3.2}$$

This can also be derived as follows.

$$\begin{aligned} 1 + t_{L+1}\sigma_L\sigma_{L+1} &= 1 + t_1\sigma_1\sigma_L \\ &= \int d\overline{a}_L da_L\, e^{a_L\overline{a}_L}(1 - t_1\overline{a}_L\sigma_1)(1 + a_L\sigma_L) \\ &= \int d\overline{a}_L da_L\, e^{a_L\overline{a}_L}(1 + t_1\overline{a}_0\sigma_1)(1 + a_L\sigma_L) \\ &= \underset{(a_L,\overline{a}_L)}{\mathrm{Sp}}\,\{\overline{A}_1A_L\}\big|_{\overline{a}_0=-\overline{a}_L}. \end{aligned} \tag{3.3}$$

Inserting the product of factors $A_m\overline{A}_{m+1}$, we have

$$\begin{aligned} Q(\sigma) &= (\overline{A}_1A_L)\prod_{m=1}^{L-1}(A_m\overline{A}_{m+1}) \\ &= \overline{A}_1\left(\prod_{m=1}^{L-1}A_m\overline{A}_{m+1}\right)A_L = \prod_{m=1}^{L}(\overline{A}_mA_m)\big|_{\overline{a}_0=-\overline{a}_L}, \end{aligned} \tag{3.4}$$

where fermionic averaging with diagonal Gaussian weight is assumed. This is an exact expression for the periodic Ising chain. The spin-periodic condition

$\sigma_{L+1} = \sigma_1$ becomes the fermion-antiperiodic condition $\overline{a}_0 = -\overline{a}_L$. Explicitly, we have

$$\begin{aligned} Q(\sigma) &= \prod_{m=1}^{L} (1 + t_{m+1}\sigma_m\sigma_{m+1})\Big|_{\sigma_{L=1}=\sigma_1} \\ &= \int \prod_{m=1}^{L} d\overline{a}_m da_m \, e^{\sum_{m=1}^{L} a_m\overline{a}_m} \left\{ \prod_{m=1}^{L} \overline{A}_m A_m \right\} \Bigg|_{\overline{a}_0=-\overline{a}_L} \end{aligned} \tag{3.5}$$

where

$$A_m = 1 + \sigma_m a_m \text{ and } \overline{A}_m = 1 + t_m\overline{a}_{m-1}\sigma_m.$$

Averaging (3.5) over $\sigma_m = \pm 1$ at each site yields the partition function Q. This is basically identical to the case of free boundary conditions, and gives a purely fermionic representation similar to (2.10), but now with $\overline{a}_0 = -\overline{a}_L$. This is an exact transformation for the 1D chain of length L with arbitrary inhomogeneous coupling constants $t_{m+1} = \tanh(b_{m+1})$. Similarly, the chain with anti-periodic spin condition $\sigma_{L+1} = -\sigma_1$ results in an integral with the fermion periodic condition $\overline{a}_0 = \overline{a}_L$, but otherwise identical. The universal form of the integral is thus

$$\begin{aligned} Q &= \operatorname*{Sp}_{(\sigma)} \left\{ \prod_{m=1}^{L} (1 + t_{m+1}\sigma_m\sigma_{m+1}) \right\} \\ &= \int \prod_{m=1}^{L} d\overline{a}_m da_m \, \exp\left\{ \sum_{m=1}^{L} [a_m\overline{a}_m - t_m a_m \overline{a}_{m-1}] \right\}, \end{aligned} \tag{3.6}$$

with the following correspondence between boundary conditions for spins and fermions:

$$\begin{aligned} \sigma_{L+1} = 0 \quad &\leftrightarrow \quad \overline{a}_0 = 0 \\ \sigma_{L+1} = \sigma_1 \quad &\leftrightarrow \quad \overline{a}_0 = -\overline{a}_L \\ \sigma_{L+1} = -\sigma_1 \quad &\leftrightarrow \quad \overline{a}_0 = \overline{a}_L. \end{aligned} \tag{3.7}$$

Remark. The transformation of boundary conditions from spins to fermions is somewhat more sophisticated in the 2D case, where the result is a sum of terms with different periodic and anti-periodic conditions.

4 The Fourier substitution in the 1D Ising chain (Grassmann variables)

The partition function (3.6) can be explicitly calculated in the homogeneous case by Fourier substitution for fermions, i.e. by a linear change of variables $\{a_m, \overline{a}_m\}_{m=1}^{L} \mapsto \{a_p, \overline{a}_p\}_{p=0}^{L-1}$ in the integral. The transformed variables a_p and $\overline{a}_p$ are also Grassmann variables, and the rules of transformation (appearance of Jacobian, etc.) are known. Thus we consider the integral (3.6) for Q with $t_{m+1} = t$ the same at all sites:

$$Q = \int \prod_{m=1}^{L} d\overline{a}_m da_m \, \exp\left\{ \sum_{m=1}^{L} (a_m \overline{a}_m - t a_m \overline{a}_{m-1}) \right\}, \tag{4.1}$$

where

$$t = \tanh(b) = \tanh(\beta J).$$

The Fourier substitution is simplest for periodic fermions, so we first consider this case:

$$\overline{a}_0 = +\overline{a}_L \qquad \text{(periodic fermions)}. \tag{4.2}$$

This corresponds to the anti-periodic spin condition $\sigma_{L+1} = -\sigma_1$. Assuming this condition, we make the standard Fourier substitution

$$\begin{aligned} a_m &= \frac{1}{\sqrt{L}} \sum_{p=0}^{L-1} a_p \, e^{2\pi i p m/L} \\ \overline{a}_m &= \frac{1}{\sqrt{L}} \sum_{p=0}^{L-1} \overline{a}_p \, e^{-2\pi i p m/L}, \end{aligned} \tag{4.3}$$

where the condition (4.2) is automatically satisfied. Taking into account the orthogonality properties of the Fourier eigenfunctions,

$$\frac{1}{L} \sum_{m=1}^{L} e^{2\pi i m (p \pm p')/L} = \delta(p \pm p' \bmod L) \tag{4.4}$$

the fermionic action becomes

$$
\begin{aligned}
S &= \sum_{m=1}^{L} (a_m \overline{a}_m - t a_m \overline{a}_{m-1}) \\
&= \sum_{p=0}^{L-1} \left(a_p \overline{a}_p - t a_p \overline{a}_p e^{2\pi i p/L} \right) \\
&= \sum_{p=0}^{L-1} \left(1 - t e^{2\pi i p/L} \right) a_p \overline{a}_p
\end{aligned} \tag{4.5}
$$

and the partition function becomes

$$
\begin{aligned}
Q &= \int \prod_{p=0}^{L-1} d\overline{a}_p da_p \, \exp \left\{ \sum_{p=0}^{L-1} \left(1 - t e^{2\pi i p/L} \right) a_p \overline{a}_p \right\} \\
&= \prod_{p=0}^{L-1} \int d\overline{a}_p da_p \, \exp \left\{ \left(1 - t e^{2\pi i p/L} \right) a_p \overline{a}_p \right\} \\
&= \prod_{p=0}^{L-1} \left(1 - t e^{2\pi i p/L} \right),
\end{aligned} \tag{4.6}
$$

where it is also taken into account that, in general, the Jacobian of the transformation $\{a_m, \overline{a}_m\}_{m=1}^{L} \mapsto \{a_p, \overline{a}_p\}_{p=0}^{L-1}$ will appear on the right-hand side, but this Jacobian equals 1 due to the orthogonality of the Fourier substitution (4.3). (In general, the Jacobian appears in the denominator in the right-hand side.) Writing

$$
Q_p = \int d\overline{a}_p da_p \, \exp \left\{ \left(1 - t e^{2\pi i p/L} \right) a_p \overline{a}_p \right\} = 1 - t e^{2\pi i p/L},
$$

we thus have

$$
Q = \prod_{p=0}^{L-1} Q_p = \prod_{p=0}^{L-1} \left(1 - t e^{2\pi i p/L} \right) = 1 - t^L. \tag{4.7}
$$

This is an exact expression for the partition function of a finite chain with fermion-closing condition $\overline{a}_0 = \overline{a}_L$ (that is $\sigma_{L+1} = -\sigma_1$). A comment about the reduction of the product to $1 - t^L$ is given in §6. From (4.7) one can calculate the free energy per site in the limit of an infinite chain ($L \to +\infty$). With the 2-dimensional lattice in mind, it is instructive to deal with the trigonometric product in (4.7) rather than the closed expression $1-t^L$. Thus, we have

$$-\beta f_Q = \frac{1}{L}\log Q = \frac{1}{L}\sum_{p=0}^{L-1}\log\left(1-te^{2\pi ip/L}\right). \tag{4.8}$$

In the limit $L\to+\infty$ the sum is replaced by an integral, and we have

$$\begin{aligned}
-\beta \lim_{L\to+\infty} f_Q &= \int_0^{2\pi}\frac{dp}{2\pi}\log(1-te^{ip}) \\
&= \frac{1}{2}\int_0^{2\pi}\frac{dp}{2\pi}\log(1-te^{ip})(1-te^{-ip}) \\
&= \frac{1}{2}\int_0^{2\pi}\frac{dp}{2\pi}\log[(1+t^2)-2t\cos(p)]. \qquad (4.9)
\end{aligned}$$

This expression can be compared with the corresponding solution for the 2D Ising model (Onsager, 1944) with t_1 and t_2 being the coupling parameters on the horizontal and vertical bonds, respectively,

$$\begin{aligned}
-\beta \lim_{L_1,L_2\to+\infty} f_Q^{(2D)} &= \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi}\frac{dp\,dq}{(2\pi)^2}\log\big[(1+t_1^2)(1+t_2^2) \\
&\quad -2t_1(1-t_2^2)\cos(p)-2t_2(1-t_1^2)\cos(q)\big], \qquad (4.10)
\end{aligned}$$

which reduces to (4.9) if $t_2=0$, as it should.

Remark. In fact, in (4.9), $-\beta\lim_{L\to+\infty} f_Q = 0$, as can be checked directly in the integral, for example by replacing $\log(1-te^{ip})$ by a series expansion in te^{ip}, and noting that $\int_0^{2\pi} dp\, e^{ipn} = 0$ for $n\neq 0$. Of course this holds only for $|t|<1$ which is the case since $t=\tanh(\beta J)$. As to the original Ising model, note also that the true partition function is given by $Z=(2\cosh(b))^L Q$, and the true fee energy density is given by

$$-\beta f_Z\big|_{L\to+\infty} = \log(2\cosh(\beta J)).$$

By making use of fermionic integrals like (4.1) one can also evaluate fermionic correlations $\langle a_m\overline{a}_{m'}\rangle$ and spin-spin correlations $\langle\sigma_m\sigma_{m'}\rangle$ for the 1D chain (finite or infinite). Once the latter are known, one can then sum the correlations to obtain, for instance, the magnetic susceptibility (in zero field) etc. The significance of the fermionic integral (4.1), even in the 1D case, is in fact somewhat greater than simply the free-energy calculation of (4.9), resulting in $-\beta f_Q = 0\,(L \to +\infty)$.

5 The Fourier transformation for aperiodic fermions ($\overline{a}_0 = -\overline{a}_L$)

In the previous section the standard (periodic) Fourier substitution was applied for the case of the Grassmann integral with periodic closing condition $\overline{a}_0 = \overline{a}_L$, which actually corresponds to a chain with spin-aperiodic closing condition $\sigma_{L+1} = -\sigma_1$. This may be considered as a normal chain in the form of a ring but with one 'negative bond', or kink somewhere in the chain. The partition function for this chain was found to be $Q = 1 - t^L$, which is the correct answer for this case.

Let us now consider the usual homogeneous Ising chain with periodic boundary condition $\sigma_{L+1} = \sigma_1$, and correspondingly $\overline{a}_0 = -\overline{a}_L$ in the fermionic representation.In this case we would expect that $Q = 1 + t^L$ and hence

$$Z = 2^N (\cosh(b))^N Q = 2^N [(\cosh(b)^N + \sinh(b)^N].$$

In this case we have to apply the aperiodic Fourier transformation. We have the Grassmann integral

$$Q = \int \prod_{m=1}^{L} d\overline{a}_m da_m \, \exp\left\{ \sum_{m=1}^{L} (a_m \overline{a}_m - t a_m \overline{a}_{m-1}) \right\} \Big|_{\overline{a}_0 = -\overline{a}_L}. \qquad (5.1)$$

In order to be able to satisfy the aperiodic condition

$$\overline{a} = -\overline{a}_L \qquad (5.2)$$

we apply a modified Fourier transformation with half-integer momenta $p + \frac{1}{2}$:

$$\begin{aligned} a_m &= \frac{1}{\sqrt{L}} \sum_{p=0}^{L-1} a_p \, e^{2\pi i m(p+\frac{1}{2})/L} \\ \overline{a}_m &= \frac{1}{\sqrt{L}} \sum_{p=0}^{L-1} \overline{a}_p \, e^{-2\pi i m(p+\frac{1}{2})/L}. \end{aligned} \qquad (5.3)$$

Then $a_{m+L} = -a_m$ and $\overline{a}_{m+L} = -\overline{a}_m$, and in particular the aperiodic boundary condition holds. The corresponding orthogonality relations are

$$\frac{1}{L}\sum_{m=1}^{L} e^{i\frac{2\pi m}{L}((p+\frac{1}{2})-(p'+\frac{1}{2}))} = \delta_{p,p'}. \tag{5.4}$$

The inverse transformation is given by the conjugated matrix and the Jacobian of the combined substitution (5.3) equals 1, as follows from (5.4). Otherwise, the procedure is quite similar to the periodic case, and the result is

$$\begin{aligned} Q &= \int \prod_{p=0}^{L-1} d\overline{a}_p da_p \, \exp\left\{\sum_{p=0}^{L-1}\left(1 - te^{i\frac{2\pi}{L}(p+\frac{1}{2})}\right) a_p\overline{a}_p\right\} \\ &= \prod_{p=0}^{L-1}\left(1 - te^{i\frac{2\pi}{L}(p+\frac{1}{2})}\right) = 1 + t^L. \end{aligned} \tag{5.5}$$

The physical consequences that follow from this formula are the same as those for (4.7) with $\overline{a}_0 = \overline{a}_L$, at least in the thermodynamic limit $L \to +\infty$ (the boundary effects do not play a role in the thermodynamic limit). Finally, it remains to consider the free boundary case (open chain) with $\sigma_{L+1} = 0$. In this case Fourier substitution is not appropriate. The answer is $Q = 1$, which can also be guessed of course from superposition of the periodic and aperiodic solutions:

$$Q_{\sigma_{L+1}=0} = \frac{1}{2}\left[Q_{\sigma_{L+1}=\sigma_1} + Q_{\sigma_{L+1}=-\sigma_1}\right] = \frac{1}{2}[(1-t^L) + (1+t^L)] = 1. \tag{5.6}$$

Remark. The result $Q = 1$ follows of course directly from (2.10) by integrating subsequently over $\overline{a}_1$ and a_1, then $\overline{a}_2$ and a_2, etc. The factors $\exp[-ta_m\overline{a}_{m-1}]$ then reduce to 1 at each stage.

The free-boundary case is also more difficult in the 2D Ising model, where the exact solution, for a finite lattice, is known for the torus case (spin-periodic closing of the lattice in both directions), but unknown in closed form for a lattice with free boundary conditions, or in the cylinder case.

6 Trigonometric products

In this sections we make some remarks about the trigonometric products which appeared in the 1D case, and may also be of interest in the subsequent analysis of the 2D case. First we note that

$$\prod_{p=0}^{L-1}\left(1 - te^{i\frac{2\pi p}{L}}\right) = 1 - t^L \tag{6.1}$$

follows from the expansion of the polynomial $z^L - 1$ with respect to its roots in the complex plane. Changing variables to $t \mapsto te^{i\pi/L}$ we also find

$$\prod_{p=0}^{L-1}\left(1 - te^{i\frac{2\pi}{L}(p+\frac{1}{2})}\right) = 1 + t^L. \tag{6.2}$$

These identities then evidently generalize to

$$\prod_{p=0}^{L-1}\left(A - Be^{i\frac{2\pi p}{L}}\right) = A^L - B^L \tag{6.3}$$

and

$$\prod_{p=0}^{L-1}\left(A - Be^{i\frac{2\pi}{L}(p+\frac{1}{2})}\right) = A^L + B^L. \tag{6.4}$$

Taking the logarithm of these expressions, one obtains additive identities, and in the limit $L \to +\infty$, one obtains trigonometric integral formulae:

$$\frac{1}{2\pi}\int_0^{2\pi} dp\, \log(1 - te^{\pm ip}) = \lim_{L\to+\infty}\frac{1}{L}\log(1 - t^L) = 0 \quad (|t| < 1), \tag{6.5}$$

and adding,

$$\frac{1}{2}\int_0^{2\pi}\frac{dp}{2\pi}\log(1 + t^2 - 2t\cos(p)) = 0 \quad (|t| < 1). \tag{6.6}$$

More generally,

$$\frac{1}{2}\int_0^{2\pi}\frac{dp}{2\pi}\log(a^2 + b^2 - 2ab\cos(p)) = \frac{1}{2}\log(a^2) = \log|a| \quad (|b| < |a|). \tag{6.7}$$

Reparametrizing the last equation according to

$$A = a^2 + b^2 \qquad B = 2ab$$
$$a = \frac{1}{2}[\sqrt{A+B} + \sqrt{A-B}] \qquad b = \frac{1}{2}[\sqrt{A+B} - \sqrt{A-B}], \tag{6.8}$$

it transforms into

$$\begin{aligned}\int_0^{2\pi} dp\, \log[A - B\cos(p)] &= 2\pi \log\left[\frac{1}{2}(A + \sqrt{A^2 - B^2})\right] \\ &= 2\pi \log\left(\frac{\sqrt{A+B} + \sqrt{A-B}}{2}\right)^2 .\end{aligned} \tag{6.9}$$

There are also discrete product analogues of this formula.

There are many other similar identities which can be deduced analogously, for instance,

$$2^{N-1} \prod_{p=0}^{N-1} \left[\cos(\gamma) + \cos\left(\frac{\pi}{N}(p + \frac{1}{2})\right)\right] = \cos(\gamma N) \tag{6.10}$$

and

$$2^{N-1} \prod_{p=0}^{N-1} \sin\left[\gamma + \frac{\pi}{N}(p + \frac{1}{2})\right] = \cos(\gamma N), \tag{6.11}$$

where γ is a parameter.

Still more identities can be obtained by differentiation with respect to a parameter. For example, differentiating with respect to A in (6.9), we have

$$\int_0^{2\pi} \frac{dp}{2\pi} \frac{1}{A - B\cos(p)} = \frac{1}{\sqrt{A^2 - B^2}} \tag{6.12}$$

and

$$\int_0^{2\pi} \frac{dp}{2\pi} \frac{\cos(p)}{A - B\cos(p)} = \frac{A - \sqrt{A^2 - B^2}}{B\sqrt{A^2 - B^2}}. \tag{6.13}$$

Another way of calculating integrals of this kind is integrating in the complex plane with respect to the variable $z = e^{ip}$. We may write $A - B\cos(p) = (a - be^{ip})(a - be^{-ip})$, where the correspondence $A, B \leftrightarrow a, b$ is given by (6.8). Then, assuming $|a| > |b|$, if $f(e^{ip})$ is some polynomial in $z = e^{ip}$,

$$\begin{aligned}
\int_0^{2\pi} \frac{dp}{2\pi} \frac{f(e^{ip})}{A - B\cos(p)} &= \oint \frac{dz}{2i\pi z} \frac{f(z)}{(a - bz)(a - bz^{-1})} \\
&= \oint \frac{dz}{2i\pi} \frac{f(z)}{(a - bz)(az - b)} \\
&= \oint \frac{dz}{2i\pi} \frac{f(z)}{a(a - bz)(z - \frac{b}{a})} \\
&= \frac{f(\frac{b}{a})}{a^2 - b^2} = \frac{f(\frac{b}{a})}{\sqrt{A^2 - B^2}}. \qquad (6.14)
\end{aligned}$$

The integral for $-\beta f_Q$ in the 2D Ising model (see (4.10)) is over two momenta p and q. One of these integrations can be performed explicitly by means of the above relations. The other survives, however (due to the appearance of square roots) and this integral cannot be significantly simplified. The derivatives of $-\beta f_Q^{(2D)}$ for the 2D Ising model, like the energy density $\langle \epsilon \rangle$ and the specific heat c can be expressed in terms of elliptic integrals (see e.g. K. Huang, "Statistical Mechanics", Wiley, 1987).

7 Dirac fields in the 1D Ising model

It is known[1] that the 2D Ising model may be viewed as a Majorana-Dirac field theory in 2-dimensional euclidean space. A similar Dirac structure is also visible in the 1D Ising case, using the fermionic description of the model. In the quantum-field-theoretical (QFT) interpretation, the 1D Ising model must be considered near its formal critical point $t = 1$ (zero temperature) and at low momenta, i.e. in the neighbourhood of $p = 0$ in momentum space.

[1] About the Majorana-Dirac interpretation of the 2D Ising model, see C. Drouffe & J.-M. Itzykson: "Statistical Field Theory", Vol. 1 (Cambridge Univ. Press, 1989), and V. N. Plechko, Phys. Lett. A**239** (1998), 289; and hep-th/9607053; and J. Phys. Stud. (Ukr.) **3** (1999), 312–330, issue dedicated to Prof. N. N. Bogoliubov on occasion of his 90th anniversary. For applications in the context of the disordered 2D Ising model, see also B. N. Shalaev, Phys. rep. **237** (1994), 129; as well as the above Phys. Lett A**239** (1998), 289.

We start with the exact lattice integral

$$Q = \int \prod_{m=1}^{L} d\overline{a}_m da_m \, \exp\left\{ \sum_{m=1}^{L} (a_m \overline{a}_m - t a_m \overline{a}_{m-1}) \right\}, \tag{7.1}$$

with $t = \tanh(b) = \tanh(\beta J)$. We write the action in the integral in the form

$$\begin{aligned} S_m &= a_m \overline{a}_m - t a_m \overline{a}_{m-1} \\ &= (1-t) a_m \overline{a}_m + t a_m (\overline{a}_m - \overline{a}_{m-1}), \end{aligned} \tag{7.2}$$

and define a lattice derivative ∂_m such that $\partial_m \overline{a}_m = \overline{a}_m - \overline{a}_{m-1}$, and let $\underline{m} = 1 - t$ be the Dirac mass. The total action then takes the form

$$S = \sum_m \left[\underline{m} a_m \overline{a}_m + t a_m \partial_m \overline{a}_m \right]. \tag{7.3}$$

Near the critical point $t \approx t_c = 1$ and $\underline{m} \approx 0$.

Taking the formal continuum limit $a_m, \overline{a}_m \to \psi(x), \overline{\psi}(x)$ and $\partial_m \to \partial/\partial x$, we obtain a continuum-limit counterpart to the same action:

$$S = \int dx \, (\underline{m} \psi(x) \overline{\psi}(x) + t_c \psi(x) \partial_x \overline{\psi}(x)). \tag{7.4}$$

This also assumes that we restrict ourselves to the low-momentum sector of the lattice theory $0 \leq |p| \leq K_0$, where K_0 is some cut-off of order 1. (Note that the continuum momentum corresponds to $2\pi p/L$, which is small if $p \ll L$.) The action (7.3) can be called a 1D Dirac action with corresponding 1D Dirac equation $(\underline{m} + t_c \partial_x)\overline{\psi} = 0$.

Remark. This terminology is quite conventional, being used commonly, at least in the 2D case. The true Dirac equation (for electrons) is in 3 + 1 Minkowski space and predicts electron spin and anti-particles. For the latter, one needs at least 2-dimensional space-time, whereas for spin one needs higher dimensions as well as a coupling to the electromagnetic field.

In momentum space, the action (7.3) takes the form

$$S = \int_{|p| \leq K_0} dp \, \left[\underline{m} \psi_p \overline{\psi}_p - i p \psi_p \overline{\psi}_p \right], \tag{7.5}$$

where we put $t = t_c = 1$ in the kinetic term. Actually, one can start with $t \neq t_c$ in

$$S = \int_{|p| \leq K_0} dp \, \left[\underline{m} \psi_p \overline{\psi}_p - ipt \psi_p \overline{\psi}_p \right], \tag{7.6}$$

and rescale t to 1 by rescaling of the fields $\psi_p, \overline{\psi}_p \to \psi_p/\sqrt{t}, \overline{\psi}_p/\sqrt{t}$, which then results in a rescaled mass $\underline{m} \to \underline{m}' = \frac{1-t}{t}$. For this mass,

$$\underline{m}' = \frac{1-t}{t} = \frac{\cosh(b) - \sinh(b)}{\sinh(b)} = \frac{2e^{-b}}{e^{b} - e^{-b}} \approx 2e^{-2b},$$

where $b \to +\infty$ and $\underline{m}' \sim 2e^{-2\beta J} \to 0$ near criticality, as $\beta = 1/kT \to +\infty$.

We define the normalized averaging in the usual way:

$$\langle \cdots \rangle = \frac{\int D\, e^{S} (\cdots)}{\int D\, e^{S}}, \tag{7.7}$$

where D and S are the corresponding fermionic measure and action, respectively. The action is particularly simple in momentum space: see equation (7.5). The momentum-space fermionic correlation functions then readily follow. We have $\langle \psi_p \psi_{-p} \rangle = \langle \overline{\psi}_p \overline{\psi}_{-p} \rangle = 0$, etc. and

$$\langle \psi_p \overline{\psi}_{p'} \rangle = \delta_{p,p'} \langle \psi_p \overline{\psi}_p \rangle; \quad \langle \psi_p \overline{\psi}_p \rangle = \frac{1}{\underline{m} - ip} = \frac{\underline{m} + ip}{\underline{m}^2 + p^2}, \tag{7.8}$$

which is the 1D Dirac propagator in QFT language. In real space, we find, for large $R > 0$,

$$\langle \psi(x) \overline{\psi}(x+R) \rangle = \int \frac{dp}{2\pi} \frac{e^{-ip|R|}}{-i(p + i\underline{m})} = e^{-\underline{m}|R|}. \tag{7.9}$$

Here we assumed that the cutoff K_0 is removed to infinity, after which the integral can be performed using contour integration. The correlator (7.9) is the same as $\langle \psi(x-R) \overline{\psi}(x) \rangle$, so that (7.9) also gives the answer for $\langle \psi(x+R) \overline{\psi}(x) \rangle$ with $R = -|R| < 0$. However, note that, interestingly, the propagator $\langle \psi(x) \overline{\psi}(x+R) \rangle = 0$ for $R < 0$, and likewise $\langle \psi(x+R) \overline{\psi}(x) \rangle = 0$ for $R > 0$.

The behaviour of the fermionic correlator (7.9) can be related to the spin-spin correlator $\langle \sigma_{m+R} \sigma_x \rangle$, but we will not consider this in more detail here. Actually, in 1 dimension, the behaviour of $\langle \sigma_0 \sigma_R \rangle = \langle \sigma_m \sigma_{m+R} \rangle$ is expected to be the same as in (7.9), at least for small mass and large distance. In fact, the exact lattice result is

$$\langle \sigma_m \sigma_{m+R} \rangle = t^R = e^{-R|\log t|} \approx e^{-\underline{m}R} \tag{7.10}$$

for small mass $\underline{m} = 1 - t \approx e^{-t}$. The continuum limit approximation thus indeed reproduces the correct asymptotics for the correlations of the exact lattice theory. This is expected to be true also in the 2D and 3D cases. In the 2D case this can in fact be checked at least for fermionic correlations. A suitable continuum limit (field-theoretical) for the 3D Ising model has so far not been constructed (at least beyond a phenomenological approach based on the ϕ^4 theory).

Further insight into the continuum approximation is obtained if we consider the exact lattice expression for fermionic correlations in the thermodynamic limit, which is

$$\langle a_p \overline{a}_p \rangle = \frac{1}{1 - te^{ip}}. \tag{7.11}$$

The real-space lattice fermionic correlation follows immediately from this:

$$\langle a_m \overline{a}_{m+R} \rangle = \int_0^{2\pi} \frac{dp}{2\pi} \frac{e^{-ipR}}{1 - te^{ip}} = \begin{cases} t^R & \text{for } R \geq 1, \\ 0 & \text{for } R \leq 0. \end{cases} \tag{7.12}$$

Multi-point fermionic correlations can also be evaluated easily in terms of binary correlators by means of Wick's theorem. The fermionic correlations can in turn be used to compute the spin-spin correlations. The same approach works in the 2D case, although the correspondence between spins and fermions is more complicated. The 2D fermions are in fact superpositions of spins and the so-called disorder variables (Kadanoff, 1969). This implies that there is a non-local correspondence between spins and fermions in 2 dimensions. This feature does not appear, or can at least be circumvented in the 1D case.

8 Fermionization of the 2D Ising model on a rectangular lattice

The Hamiltonian of the 2D Ising model on a square lattice with side L is given by

$$-\beta H(\sigma) = \sum_{m=1}^{L}\sum_{n=1}^{L}\left[b^{(1)}_{m+1,n}\sigma_{m,n}\sigma_{m+1,n} + b^{(2)}_{m,n+1}\sigma_{m,n}\sigma_{m,n+1}\right], \tag{8.1}$$

with $m, n = 1, \ldots, L$ running in horizontal and vertical directions, respectively, and where $\sigma_{m,n} = \pm 1$. As before, we write the partition function in the form

$$Z = \sum_{\sigma_{m,n}=\pm 1} e^{-\beta H(\sigma)} = \left\{\prod_{m=1}^{L}\prod_{n=1}^{L} 2\cosh(b^{(1)}_{m+1,n})\cosh(b^{(2)}_{m,n+1})\right\} Q, \tag{8.2}$$

where Q is the reduced partition function

$$Q = \operatorname*{Sp}_{(\sigma)}\left\{\prod_{m,n=1}^{L}(1 + t^{(1)}_{m+1,n}\sigma_{m,n}\sigma_{m+1,n})(1 + t^{(2)}_{m,n+1}\sigma_{m,n}\sigma_{m,n+1})\right\}, \tag{8.3}$$

with

$$t^{(\alpha)}_{m,n} = \tanh(b^{(\alpha)}_{m,n}) = \tanh\left(\frac{J^{(\alpha)}_{m,n}}{kT}\right). \tag{8.4}$$

It may be noted that the prefactor in (8.2) is essentially the partition function of the disconnected bonds on the lattice, which gives an additive non-singular contribution to the free energy and the specific heat.

To evaluate the reduced partition function, we introduce a set of anti-commuting Grassmann variables $a_{m,n}, \overline{a}_{m,n}, b_{m,n}, \overline{b}_{m,n}$ and factorize the local Boltzmann weights as follows.

$$\begin{aligned}
&1 + t^{(1)}_{m+1,n}\sigma_{m,n}\sigma_{m+1,n} = \\
&= \int d\overline{a}_{m,n} da_{m,n}\, e^{a_m \overline{a}_{m,n}}(1 + a_{m,n}\sigma_{m,n})(1 + t^{(1)}_{m+1,n}\overline{a}_{m,n}\sigma_{m+1,n}) \\
&= \operatorname*{Sp}_{(a_{m,n})}\left\{A_{m,n}\overline{A}_{m+1,n}\right\}
\end{aligned} \tag{8.5}$$

and

$$\begin{aligned}
1 + t^{(2)}_{m,n+1}\sigma_{m,n}\sigma_{m,n+1} = \\
&= \int d\bar{b}_{m,n} db_{m,n}\, e^{b_m \bar{b}_{m,n}}(1+b_{m,n}\sigma_{m,n})(1+t^{(1)}_{m,n+1}\bar{b}_{m,n}\sigma_{m,n+1}) \\
&= \operatorname*{Sp}_{(b_{m,n})} \left\{ B_{m,n}\overline{B}_{m,n+1} \right\}, \qquad (8.6)
\end{aligned}$$

where $A_{m,n}$, $\overline{A}_{m,n}$, $B_{m,n}$ and $\overline{B}_{m,n}$ are given by

$$\begin{aligned}
A_{m,n} &= 1 + \sigma_{m,n} a_{m,n} \\
B_{m,n} &= 1 + \sigma_{m,n} b_{m,n} \\
\overline{A}_{m,n} &= 1 + t^{(1)}_{m,n}\sigma_{m,n}\overline{a}_{m-1,n}, \\
\overline{B}_{m,n} &= 1 + t^{(2)}_{m,n}\sigma_{m,n}\bar{b}_{m,n-1}. \qquad (8.7)
\end{aligned}$$

These four Grassmann factors correspond to the four different bonds attached to a given site m, n with spin variable $\sigma_{m,n}$. Omitting the Gaussian fermionic averaging as in the 1D case, we write in short-hand notation,

$$\begin{aligned}
1 + t^{(1)}_{m+1,n}\sigma_{m,n}\sigma_{m+1,n} &= A_{m,n}\overline{A}_{m+1,n} \\
1 + t^{(2)}_{m,n+1}\sigma_{m,n}\sigma_{m,n+1} &= B_{m,n}\overline{B}_{m,n+1}. \qquad (8.8)
\end{aligned}$$

To obtain a purely fermionic representation for Q, we have to multiply the commuting weights (8.8) over the lattice sites m, n and sum over $\sigma_{m,n} = \pm 1$ at each site. In this procedure, we want to place the four factors $A_{m,n}$, $B_{m,n}$, $\overline{A}_{m,n}$ and $\overline{B}_{m,n}$ next to each other at the moment of averaging.

Notice that separable Grassmann factors in a total set $\cdots, A_{m,n}, \cdots, B_{m',n'}, \cdots, \overline{A}_{m'',n''}, \cdots$ are in general neither commuting nor anti-commuting with each other. However, we will use the fact that double symbols like $A_{m,n}\overline{A}_{m+1,n}$ and $B_{m,n}\overline{B}_{m,n+1}$ representing the Boltzmann weights in (8.8), effectively commute with any element of the Grassmann algebra in the total Guassian Grassmann integral, because the non-commuting linear fermionic terms involved give a contribution zero after averaging. Note also that the structure of $A_{m,n}\overline{A}_{m+1,n}$ is such that we may first take the product over m with fixed n (with subsequent rearrangements like $A_{m,n}\overline{A}_{m+1,n} \to \overline{A}_{m,n}A_{m,n}$), while the structure of $B_{m,n}\overline{B}_{m,n+1}$ is such that we first have to multiply the factors with fixed m as opposed to the m-ordering needed for $A_{m,n}\overline{A}_{m+1,n}$. This problem will be resolved in the following by using a mirror-ordered arrangement of $B_{m,n}$ and $\overline{B}_{m,n}$ (see below) with respect to the index m, even though these are 'vertical' factors.

In these special ordering arrangements the following two principles will be used, illustrated here first by tutorial examples. As a first illustration, consider the linear arrangement

$$(O_0\overline{O}_1)(O_1\overline{O}_2)(O_2\overline{O}_3)(O_3\overline{O}_4) = O_0(\overline{O}_1O_1)(\overline{O}_2O_2)(\overline{O}_3O_3)\overline{O}_4,$$

where the symbols O_i and $\overline{O}_j$ are arbitrary Grassmann factors. This simple rearrangement was used in the fermionization of the 1D Ising chain. As a second illustration, consider the mirror-ordered rearrangement

$$\begin{aligned}(O_1\overline{O}_1)(O_2\overline{O}_2)(O_3\overline{O}_3) &= (O_1(O_2(O_3\overline{O}_3)\overline{O}_2)\overline{O}_1) \\ &= O_1O_2O_3 \cdot \overline{O}_3\overline{O}_2\overline{O}_1,\end{aligned}$$

where the doubled symbols $(O_i\overline{O}_i)$ are assumed to be totally commuting with any O_j or $\overline{O}_j$.

Now let n be fixed and consider a product over m of vertical weights $B_{m,n}\overline{B}_{m,n+1}$. Since these are totally commuting with all other Grassmann factors we can use the above mirror-ordering rule to write

$$\prod_{m=1}^{L}(1 + t^{(2)}_{m,n+1}\sigma_{m,n}\sigma_{m,n+1}) = \prod_{m=1}^{L} B_{m,n}\overline{B}_{m,n+1} = \prod_{m=1}^{\overleftarrow{L}} B_{m,n} \prod_{m=1}^{\overrightarrow{L}} \overline{B}_{m,n+1}. \quad (8.9)$$

This ordering of factors is already favourable for introducing horizontal weights $A_{m,n}\overline{A}_{m+1,n}$, as can be guessed from the experience with the 1D chain. In the mean time we continue by multiplying the partial products (8.9) of vertical weights, now over $n = 1, \ldots, L$, we find

$$\begin{aligned}\prod_{n=1}^{L}\prod_{m=1}^{L}(1 + t^{(2)}_{m,n+1}\sigma_{m,n}\sigma_{m,n+1}) &= \prod_{n=1}^{\overrightarrow{L}} \left\{ \prod_{m=1}^{\overleftarrow{L}} B_{m,n} \prod_{m=1}^{\overrightarrow{L}} \overline{B}_{m,n+1} \right\} \\ &= \prod_{n=1}^{\overrightarrow{L}} \left\{ \prod_{m=1}^{\overrightarrow{L}} \overline{B}_{m,n} \prod_{m=1}^{\overleftarrow{L}} B_{m,n} \right\}, \quad (8.10)\end{aligned}$$

where we have also assumed free boundary conditions for spin variables in both directions, and, for the sake of standardization, we have modified the boundary products of factors. Namely, on the right-hand side of the final product, the factor $\overrightarrow{\prod_{m=1}^{L}}\overline{B}_{m,L+1}$ does not appear since in fact $\overline{B}_{m,L+1} = 1$ because of the free boundary condition $\sigma_{m,L+1} = 0$. Moreover, we introduced an additional product of factors $\overline{B}_{m,1}$ which are of the form

$$\overline{B}_{m,1} = 1 + t_{m,1}^{(2)}\bar{b}_{m,0}\sigma_{m,1},$$

where we set $\bar{b}_{m,0} = 0$. In this way, the condition $\sigma_{m,L+1} = 0$ is transformed into $\bar{b}_{m,0} = 0$.

In (8.10) we already have a complete product of all vertical weights prepared in a suitable, mirror-ordered, factorized form. It remains to introduce the horizontal commuting weights $A_{m,n}\overline{A}_{m+1,n}$. For evident reasons, we introduce $A_{m,n}\overline{A}_{m+1,n}$ into the sub-product of $\overline{B}_{m,n}$ of (8.10) and then apply the linear ordering of the first example above. This gives

$$\begin{aligned} \left\{\prod_{m=1}^{L} A_{m,n}\overline{A}_{m+1,n}\right\}\overrightarrow{\prod_{m=1}^{L}}\overline{B}_{m,n} &= \overrightarrow{\prod_{m=1}^{L}}(\overline{B}_{m,n}A_{m,n}\overline{A}_{m+1,n}) \\ &= \overrightarrow{\prod_{m=1}^{L}}(\overline{A}_{m,n}\overline{B}_{m,n}A_{m,n}), \end{aligned} \tag{8.11}$$

where, in the final line, we again make a boundary modification, removing $\overline{A}_{L+1,n} = 1$ and adding $\overline{A}_{1,n} = 1 + t_{1,n}^{(1)}\bar{a}_{0,n}\sigma_{1,n} = 1$ with $\bar{a}_{0,n} = 0$. Substituting (8.11) into (8.10) we obtain the mirror-ordered factorized representation for the complete density matrix of the 2D Ising model on a rectangular lattice (complete product of Boltzmann factors), as follows

$$\begin{aligned} Q(\sigma) &= \prod_{m=1}^{L}\prod_{n=1}^{L}(1 + t_{m+1,n}^{(1)}\sigma_{m,n}\sigma_{m+1,n})(1 + t_{m,n+1}^{(2)}\sigma_{m,n}\sigma_{m,n+1}) \\ &= \operatorname*{Sp}_{(a,b)}\left\{\overleftarrow{\prod_{n=1}^{L}}\left[\overrightarrow{\prod_{m=1}^{L}}(\overline{A}_{m,n}\overline{B}_{m,n}A_{m,n})\overleftarrow{\prod_{m=1}^{L}}\overline{B}_{m,n}\right]\right\}, \end{aligned} \tag{8.12}$$

where we have also reinstated the previously dropped symbol Sp for the total

Gaussian averaging given by the factorized product

$$\operatorname*{Sp}_{(a,b)}(\cdots) = \int \prod_{m=1}^{L} \prod_{n=1}^{L} d\overline{a}_{m,n} da_{m,n} d\overline{b}_{m,n} db_{m,n}$$
$$\times \exp\left[\sum_{m,n=1}^{L} (a_{m,n}\overline{a}_{m,n} + b_{m,n}\overline{b}_{m,n})\right](\cdots), \qquad (8.13)$$

and a completely explicit form of (8.12) then follows by substituting the expressions (8.7 for the Grassmann factors. On the left-hand side, free boundary conditions $\sigma_{m,L+1} = \sigma_{L+1,n} = 0$ are assumed, and on the right-hand side the fermionic conditions $\overline{a}_{0,n} = \overline{b}_{m,0} = 0$ are assumed. With these conditions equation (8.12) is a mixed spin-fermion representation for the density matrix, suitable for elimination of the spin variables. This will yield a fermionic integral for the partition function $Q = \mathrm{Sp}_{(\sigma)}\, Q(\sigma)$. The averaging over $\sigma_{m,n} = \pm 1$ will be done step-by-step by averaging each product of four neighbouring factors with the same indices m, n, as follows.

$$\begin{aligned}
(\sigma_{m,n})&\left\{\overline{A}_{m,n}\overline{B}_{m,n}A_{m,n}B_{m,n}\right\} = \\
&= \frac{1}{2}\sum_{\sigma_{m,n}=\pm 1} (1 + t^{(1)}_{m,n}\overline{a}_{m-1,n}\sigma_{m,n})(1 + t^{(2)}_{m,n}\overline{b}_{m,n-1}\sigma_{m,n}) \\
&\qquad \times (1 + a_{m,n}\sigma_{m,n})(1 + b_{m,n}\sigma_{m,n}) \\
&= 1 + t^{(1)}_{m,n}t^{(2)}_{m,n}\overline{a}_{m-1,n}\overline{b}_{m,n-1} + a_{m,n}b_{m,n} \\
&\qquad + (t^{(1)}_{m,n}\overline{a}_{m-1,n} + t^{(2)}_{m,n}\overline{b}_{m,n-1})(a_{m,n} + b_{m,n}) \\
&\qquad + t^{(1)}_{m,n}t^{(2)}_{m,n}\overline{a}_{m-1,n}\overline{b}_{m,n-1}a_{m,n}b_{m,n} \\
&= \exp\left[t^{(1)}_{m,n}t^{(2)}_{m,n}\overline{a}_{m-1,n}\overline{b}_{m,n-1} + a_{m,n}b_{m,n}\right. \\
&\qquad \left. + (t^{(1)}_{m,n}\overline{a}_{m-1,n} + t^{(2)}_{m,n}\overline{b}_{m,n-1})(a_{m,n} + b_{m,n})\right]. \qquad (8.14)
\end{aligned}$$

This identity can be checked by series expansion of the exponential (the series terminates at the 4-fermion order) or it may be understood in a more general framework, in view of identities like

$$(1 + \sigma_0 L_1)(1 + \sigma_0 L_2) = e^{L_1 L_2}(1 + \sigma_0(L_1 + L_2)), \qquad (8.15)$$

where $\sigma_0 = \pm 1$ and L_1 and L_2 are arbitrary forms linear in Grassmann variables. For example,

$$(1 + a_{m,n}\sigma_{m,n})(1 + b_{m,n}\sigma_{m,n}) = e^{a_{m,n}}b_{m,n}(1 + (a_{m,n} + b_{m,n})\sigma_{m,n}).$$

The complete spin-variable averaging in the representation (8.12) now has to be performed step-by-step at the junction of the two m-ordered products inside the n-product. Let n therefore be fixed. Then, at the junction of two m-products for the given n, we have a product of four neighbouring factors of the type (8.14) with $m = L$, and after averaging over $\sigma_{L,n}$ we obtain a totally commuting exponential factor (8.14), which can be removed from the product. At the junction we then have again a set of four Grassmann factors, now with $m = L - 1$. This procedure can thus be repeated for $m = L - 1, L - 2, \ldots, 2, 1$ for the given n, and then for subsequent values of n. It is important that after each local averaging over $\sigma_{m,n}$ we obtain an even polynomial in Grassmann variables, which can be moved outside the product, thus allowing the next set of four Grassmann factors with a given m, n to appear side-by-side. This makes it possible to eliminate completely the spin variables in the mixed (σ, a, b) factorized representation for $Q(\sigma)$.

Remark. If this were not the case, i.e. if a non-commuting fermionic expression would result after averaging over $\sigma_{m,n}$, then it would not be possible to perform the averaging in a sequential fashion. This is just the case for the 2D Ising model in a magnetic field. Introducing an additional weight $1+t_0\sigma_{m,n}$, where $t_0 = \tanh(\beta h)$ and h is a non-zero magnetic field, the resulting polynomial in fermions also has odd terms, which effectively terminates the elimination of spin variables at the junction. Alternatively, one can insist on averaging by transposing (commuting) the factors according to the algebraic commutation relations, but this produces a large amount of non-local terms in the action.

Eliminating all spin variables, we obtain an expression for the partition function as a product of fermionic factors (8.14) integrated with respect to the total Gaussian averaging (8.13). It reads

$$Q = \int \prod_{m=1}^{L} \prod_{n=1}^{L} d\overline{a}_{m,n} da_{m,n} d\overline{b}_{m,n} db_{m,n} \exp \left[\sum_{m,n=1}^{L} (a_{m,n}\overline{a}_{m,n} + b_{m,n}\overline{b}_{m,n}) \right]$$
$$\times \exp \left[\sum_{m,n=1}^{L} \big(t^{(1)}_{m,n} t^{(2)}_{m,n} \overline{a}_{m-1,n} \overline{b}_{m,n-1} + a_{m,n} b_{m,n} \right.$$
$$\left. + (t^{(1)}_{m,n} \overline{a}_{m-1,n} + t^{(2)}_{m,n} \overline{b}_{m,n-1})(a_{m,n} + b_{m,n}) \big) \right]. \qquad (8.16)$$

This is an exact fermionic representation for the partition function of the 2D Ising model on a rectangular lattice with free boundary conditions $\sigma_{m,L+1} = \sigma_{L+1,n} = 0$, where it assumed that $\overline{a}_{0,n} = \overline{b}_{m,0} = 0$. Note that this Gaussian fermionic integral for Q is obtained for the most general set of coupling parameters $t^{(1)}_{m,n}$ and $t^{(2)}_{m,n}$.[2] This will enable us to compute the spin-spin and fermion-fermion correlation functions by perturbation of the partition function: any correlator can typically be expressed as a ratio Q'/Q where Q' is some perturbed partition function.

In the following section we discuss general features of Gaussian fermionic integrals, and in the next section we discuss the evaluation of the analogous integral (8.16) with periodic boundary conditions in the homogeneous case by transformation to momentum space. This will result in Onsager's solution for the free energy.

[2] The inhomogeneous fermionic integral (8.16) was first obtained in V. N. Plechko, Dokl. Akad. Nauk SSSR **281** (1985) 834 (English translation: Sov. Phys. Doklady **30** (1985) 271). The mirror-ordered factorization method for obtaining a fermionic representation (and hence an exact solution) for the 2D Ising model was introduced here. The solution for a finite torus is in: V. N. Plechko, Teor. Mat. Fiz. **64** (1985) 150 (Sov. Phys.-Theor. Math. Phys. **64** (1985) 748). Other important references are: F. A. Berezin, Usp. Matem. Nauk **24**, No. 3 (1969), 3 [English translation in Russian Math. Surveys **24** (1961), 1]. This is the first paper where Grassmann variables were applied to the 2D Ising model, and in a sense, the first practical use of fermionic integrals for explicit calculations. In S. Samuel, J. Math. Phys. **21** (1980), 2806, another version of combinatorial solution of the 2D Ising model using Grassmann variables is presented. A good discussion of diverse related aspects and perspectives from a QFT point of view is in C. Itzykson, Nucl. Phys. B**210** (1982), 448. His method of introducing Grassmann variables is rather based on the transfer matrix formulation. See also Vol.1 of the book by Drouffe and Itzykson, and a short review on Grassmann variables by V. N. Plechko, hep-th 9609044.

9 Gaussian fermionic (Grassmann) integrals

The Gaussian fermionic (Grassmann variable) integral of the first kind equals the determinant of the coefficient matrix:

$$\int \prod_{i=1}^{N} d\overline{a}_i da_i \, \exp\left\{ \sum_{i,j=1}^{N} a_i A_{ij} \overline{a}_j \right\} = \det(A), \tag{9.1}$$

where $a_1, \ldots, a_N, \overline{a}_1, \ldots, \overline{a}_N$ are Grassmann variables, and the matrix $A = (A_{ij})_{i,j=1}^{N}$ is arbitrary.

The Gaussian integral of the second kind is given by the Pfaffian of the associated skew-symmetric matrix:

$$\int \overleftarrow{\prod_{i=1}^{N}} da_i \, \exp\left\{ \frac{1}{2} \sum_{i,j=1}^{N} a_i A_{ij} a_j \right\} = \mathrm{Pfaff}(A), \tag{9.2}$$

where A is an arbitrary skew-symmetric matrix, i.e. $A + A^T = 0$, or $A_{ij} + A_{ji} = 0$. In particular, $A_{ii} = 0$. Another way of writing this integral is

$$\int \overleftarrow{\prod_{i=1}^{N}} da_i \, \exp\left\{ \sum_{1 \le i < j \le N} a_i A_{ij} a_j \right\} = \mathrm{Pfaff}(A), \tag{9.3}$$

The number N of variables in these integrals must be even, otherwise the integrals are identically zero. In general, the determinantal integral of the first kind may be viewed as a particular case of the integral of the second kind. Indeed, the Pfaffian of a skew-symmetric matrix equals the square root of its determinant[3]

$$\mathrm{Pfaff}(A) = \sqrt{\det(A)}, \qquad A + A^T = 0. \tag{9.4}$$

[3]The correct form of this identity is $\det(A) = (\mathrm{Pfaff}(A))^2$ with $A + A^T = 0$. The knowledge of $\det(A)$ does not yet determine the sign of the Pfaffian. The Pfaffian is in fact defined as a certain combinatorial polynomial associated with a triangular array of elements $(A_{ij})_{1 \le i < j \le N}$. The notion of Pfaffian and the determinant identity were known in mathematics already in the 19th century. The integrals of the second kind can also be used as alternative definitions of the Pfaffian. In physics, the notion of Pfaffian was first introduced by E. Caianiello in the early 1950s in the context of the study of the general structure of perturbation series in quantum electrodynamics (see his book from 1973).

This identity is in fact easily proved using Grassmann integrals.

Now define the normalized averaging $\langle \cdots \rangle$ associated with the integral (9.1) by

$$\langle \cdots \rangle = \frac{\int \prod_{i=1}^{N} d\overline{a}_i da_i \, \exp\left\{\sum_{i,j=1}^{N} a_i A_{ij} \overline{a}_j\right\} (\cdots)}{\int \prod_{i=1}^{N} d\overline{a}_i da_i \, \exp\left\{\sum_{i,j=1}^{N} a_i A_{ij} \overline{a}_j\right\}} \tag{9.5}$$

where we assume that

$$\det(A) = \int \prod_{i=1}^{N} d\overline{a}_i da_i \, \exp\left\{\sum_{i,j=1}^{N} a_i A_{ij} \overline{a}_j\right\} \neq 0.$$

The binary fermionic correlation function can expressed in terms of the elements of the inverse matrix:

$$\langle a_i \overline{a}_j \rangle = \frac{\int \prod_{i=1}^{N} d\overline{a}_i da_i \, \exp\left\{\sum_{i,j=1}^{N} a_i A_{ij} \overline{a}_j\right\} (a_i \overline{a}_j)}{\int \prod_{i=1}^{N} d\overline{a}_i da_i \, \exp\left\{\sum_{i,j=1}^{N} a_i A_{ij} \overline{a}_j\right\}} = (A^{-1})_{ji}, \tag{9.6}$$

while $\langle a_i a_j \rangle = \langle \overline{a}_i \overline{a}_j \rangle = 0$ as can be guessed from symmetries. Also, $\langle a_i \rangle = \langle \overline{a}_{i'} \rangle = 0$ and similarly all multi-point correlations of an odd number of fermion variables equals zero.

In a similar way, if $\langle \cdots \rangle$ is the normalized averaging associated with an integral of the second kind, then

$$\langle a_i a_j \rangle = \frac{\int \prod_{i=1}^{N} da_i \, \exp\left\{\frac{1}{2}\sum_{i,j=1}^{N} a_i A_{ij} a_j\right\} (a_i a_j)}{\int \prod_{i=1}^{N} da_i \, \exp\left\{\frac{1}{2}\sum_{i,j=1}^{N} a_i A_{ij} a_j\right\}} = (A^{-1})_{ji}, \tag{9.7}$$

assuming $\mathrm{Pfaff}(A) \neq 0$. The multi-point fermionic averages can be expressed in terms of binary averages by means of Wick's theorem.[4] An example of these rules is

$$\begin{aligned} \langle A_1 A_2 A_3 A_4 \rangle &= \langle A_1 A_2 \rangle \langle A_3 A_4 \rangle - \langle A_1 A_3 \rangle \langle A_2 A_4 \rangle \\ &\quad + \langle A_1 A_4 \rangle \langle A_2 A_3 \rangle, \end{aligned} \tag{9.8}$$

[4] By Wick's theorem we mean combinatorial rules of disentangling the (fermionic) multi-point correlation functions in terms of binary correlators, which are analogous to the rules established by Wick (G. C. Wick, Phys. Rev. **80** (1950) 268) for multi-point vacuum averages for fermions in quantum electrodynamics. Similar rules hold also for (classical or quantum) bosons. That Wick's theorem holds for Gaussian Grassmann-variable integral averages might be expected, but nevertheless needs a separate proof.

where $A_1, \ldots, A_4$ are arbitrary linear forms in Grassmann variables involved in the integration, and $\langle \cdots \rangle$ is a Gaussian averaging of the first or second kind. A more general statement is

$$\langle A_1 A_2 \ldots A_{N'} \rangle = \text{Pfaff}\left((\langle A_i A_j \rangle)_{i,j=1}^{N'} \right), \tag{9.9}$$

where the meaning of the symbols is the same as in the example, and $N' \leq N$ is assumed to even (otherwise the average is identically zero). In some cases (if certain selection rules hold), the Pfaffian can be reduced to a determinant. In practical calculations, Wick's theorem is often more conveniently expressed in the form of known recursive rules, lowering the order of correlations step-by-step. The Pfaffian in (9.9) is the result of a repeated use of these rules.

In field-theoretical language, binary averages like $\langle a_i \overline{a}_j \rangle$ or $\langle a_i a_j \rangle$ in (9.7) are called Green's functions (or propagators, in suitable context). Integrals of the first kind are naturally associated with Dirac-like theories, where there are two kinds of 'charge-conjugated' fermions a_i and $\overline{a}_i$, while integrals of the second kind correspond to 'neutral' Majorana-type fermions.

10 Momentum-space analysis for the 2D Ising model (exact solution)

The fermionic integral (8.16) can be evaluated explicitly for the homogeneous lattice $t^{(1)}_{m,n} = t_1$, $t^{(2)}_{m,n} = t_2$ in the thermodynamic limit by passing to momentum space for fermions and changing the boundary conditions to periodic: $\overline{a}_{0,n} = \overline{a}_{L,n}$ and $\overline{b}_{m,0} = \overline{b}_{m,L}$. This results in the exact solution (Onsager solution) of the 2D Ising model on a rectangular lattice. The homogeneous partition function is given by

$$
\begin{aligned}
Q &= \int \prod_{m=1}^{L}\prod_{n=1}^{L} d\overline{a}_{m,n}da_{m,n}d\overline{b}_{m,n}db_{m,n}\ \exp\left[\sum_{m,n=1}^{L}(a_{m,n}\overline{a}_{m,n}+b_{m,n}\overline{b}_{m,n}\right] \\
&\quad\times\exp\left[\sum_{m,n=1}^{L}\left(t_1t_2\overline{a}_{m-1,n}\overline{b}_{m,n-1}+a_{m,n}b_{m,n}\right)\right] \\
&\quad\times\exp\left[\sum_{m,n=1}^{L}(t_1\overline{a}_{m-1,n}+t_2\overline{b}_{m,n-1})(a_{m,n}+b_{m,n})\right]. \qquad (10.1)
\end{aligned}
$$

We are interested in the free energy density in the thermodynamic limit $L \to +\infty$, and will change the boundary conditions from $\overline{a}_{0,n} = \overline{b}_{m,0} = 0$ to periodic closing conditions for fermions, i.e. $\overline{a}_{0,n} = \overline{a}_{L,n}$ and $\overline{b}_{m,0} = \overline{b}_{m,L}$. These conditions are more suitable for Fourier transformation, and a change of boundary conditions is an inessential modification in the limit $L \to +\infty$. (Note that, these fermionic periodic boundary conditions do not correspond to periodic boundary conditions for spins at finite volume!)

We now define momentum-space fermion variables $a_{p,q}, \overline{a}_{p,q}, b_{p,q}$ and $\overline{b}_{p,q}$ by

$$
\begin{aligned}
a_{m,n} &= \frac{1}{L}\sum_{p=0}^{L-1}\sum_{q=0}^{L-1} a_{p,q}\, e^{i\frac{2\pi}{L}(mp+nq)}, \\
b_{m,n} &= \frac{1}{L}\sum_{p=0}^{L-1}\sum_{q=0}^{L-1} b_{p,q}\, e^{i\frac{2\pi}{L}(mp+nq)}, \\
\overline{a}_{m,n} &= \frac{1}{L}\sum_{p=0}^{L-1}\sum_{q=0}^{L-1} \overline{a}_{p,q}\, e^{-i\frac{2\pi}{L}(mp+nq)}, \\
\overline{b}_{m,n} &= \frac{1}{L}\sum_{p=0}^{L-1}\sum_{q=0}^{L-1} \overline{b}_{p,q}\, e^{-i\frac{2\pi}{L}(mp+nq)} \qquad (10.2)
\end{aligned}
$$

Substituting (10.2) into the integral (10.1), we obtain

$$
\begin{aligned}
Q &= \int \prod_{p=0}^{L-1} \prod_{q=0}^{L-1} d\overline{a}_{p,q} da_{p,q} d\overline{b}_{p,q} db_{p,q} \exp\left[\sum_{p,q=0}^{L-1} (a_{p,q}\overline{a}_{p,q} + b_{p,q}\overline{b}_{p,q})\right] \\
&\quad \times \exp\left[\sum_{p,q=0}^{L-1} \left(a_{p,q} b_{L-p,L-q} + t_1 t_2 \overline{a}_{p,q}\overline{b}_{L-p,L-q} e^{-i\frac{2\pi}{L}(p-q)}\right)\right] \\
&\quad \times \exp\left[\sum_{p,q=0}^{L-1} (t_1 e^{i\frac{2\pi}{L}p}\overline{a}_{p,q} + t_2 e^{i\frac{2\pi}{L}q}\overline{b}_{p,q})(a_{p,q} + b_{p,q})\right]. \qquad (10.3)
\end{aligned}
$$

Here the orthogonality relations for the Fourier eigenfunctions (Fourier exponentials in (10.2) were used as well as the fact that the Jacobian of the combined transformation (10.2) equals 1 as in the 1D case. It is clear from (10.3) that the integral decouples into a product of simple low-dimensional integrals, due to the block-diagonal structure of the action in (10.3). We have to take into account that, since the variables with momenta p, q and $L-p, L-q$ 'interact', (are coupled), the true independent factor includes the variables with momenta p, q as well as $L-p, L-q$ for each given value of p, q. The corresponding terms in the integral give rise to the elementary factor

$$
\begin{aligned}
Q^{(2)}_{p,q} &= \int (d\overline{b}\, db\, d\overline{a}\, da)_{p,q} (d\overline{b}\, db\, d\overline{a}\, da)_{L-p,L-q} \exp\left\{(a_{p,q}\overline{a}_{p,q} + b_{p,q}\overline{b}_{p,q})\right. \\
&\quad + (a_{L-p,L-q}\overline{a}_{L-p,L-q} + b_{L-p,L-q}\overline{b}_{L-p,L-q}) \\
&\quad + (a_{p,q} b_{L-p,L-q} + a_{L-p,L-q} b_{p,q}) \\
&\quad + (\hat{t}_1 \hat{t}_2^* \overline{a}_{p,q}\overline{b}_{L-p,L-q} + \hat{t}_1^* \hat{t}_2 \overline{a}_{L-p,L-q}\overline{b}_{p,q}) \\
&\quad + (\hat{t}_1 \overline{a}_{p,q} + \hat{t}_2 \overline{b}_{p,q})(a_{p,q} + b_{p,q}) \\
&\quad \left. + (\hat{t}_1^* \overline{a}_{L-p,L-q} + \hat{t}_2^* \overline{b}_{L-p,L-q})(a_{L-p,L-q} + b_{L-p,L-q})\right\}, \qquad (10.4)
\end{aligned}
$$

where we have introduced abbreviations for the parameters as follows,

$$
\begin{aligned}
\hat{t}_1 &= t_1 e^{i\frac{2\pi p}{L}}, \qquad \hat{t}_1^* = t_1 e^{-i\frac{2\pi p}{L}}, \\
\hat{t}_2 &= t_2 e^{i\frac{2\pi q}{L}}, \qquad \hat{t}_2^* = t_1 e^{-i\frac{2\pi q}{L}}. \qquad (10.5)
\end{aligned}
$$

The extraction of the elementary integral factor (10.4) from (10.3) can also be interpreted as a symmetrization with respect to the permutation

$p, q \longleftrightarrow L-p, L-q$ of the p,q-sum in (10.3). We symmetrize the p, q sum in (10.3) by combining the terms with p, q and with $L-p, L-q$, and reducing the sum to a half-interval with respect to either p or q in order not to repeat the same terms twice. The complete partition function is obtained by multiplying the factors (10.4) over the reduced set of points in such a way that if the factor $Q^{(2)}_{p,q}$ is included in the product then the factor $Q^{(2)}_{L-p,L-q}$ is not, and vice versa. Alternatively, we can multiply the factors (10.4) over the complete set of points in the momentum lattice $0 \leq p, q \leq L-1$ but obtain the square partition function Q^2. This can also be seen by comparing the fermionic measures in (10.3) and (10.4).

Remark. There may be a few self-dual modes among $Q^{(2)}_{p,q}$, i.e. such that $p = L-p$ and $q = L-q$. This depends on the parity of L. We would have to calculate these special mode integrals separately, but they do not contribute to the thermodynamic limit.

It remains to evaluate the elementary integral factor $Q^{(2)}_{p,q}$ of (10.4) in explicit form. Note that the original integral (10.1) must rather be understood as a Pfaffian integral (integral of the second kind) as discussed in the previous section, Eq. (9.2), due to terms like $a_{m,n}b_{m,n}$ and $\overline{a}_{m-1,n}\overline{b}_{m,n-1}$ present in the action. The factor $Q^{(2)}_{p,q}$ can therefore also be expected to be a Pfaffian. But in fact it can be read and calculated as the determinant of a 4×4 matrix. For this, we want to interpret the factor $Q^{(2)}_{p,q}$ as a determinantal Gaussian integral of the first kind (9.1), with $N = 4$, that is,

$$Q^{(2)}_{p,q} = \int \prod_{j=1}^{4} d\overline{a}_j da_j \, \exp\left\{ \sum_{i,j=1}^{4} A_{ij} a_i \overline{a}_j \right\}. \tag{10.6}$$

The factor (10.4) can indeed be written in this form with the following choice of conjugated fields:

$$\begin{aligned} a_1, a_2, a_3, a_4 &\longleftrightarrow a_{p,q}, b_{p,q}, \overline{a}_{L-p,L-q}, \overline{b}_{l-p,L-q} \\ \overline{a}_1, \overline{a}_2, \overline{a}_3, \overline{a}_4 &\longleftrightarrow \overline{a}_{p,q}, \overline{b}_{p,q}, a_{L-p,L-q}, b_{l-p,L-q}. \end{aligned} \tag{10.7}$$

The factor $Q^{(2)}_{p,q}$ is then given in the form (10.6) with the matrix A explicitly given by

$$A = \begin{pmatrix} 1-\hat{t}_1 & -\hat{t}_2 & 0 & 1 \\ -\hat{t}_1 & 1-\hat{t}_2 & -1 & 0 \\ 0 & \hat{t}_1^*\hat{t}_2 & -1+\hat{t}_1^* & \hat{t}_1^* \\ -\hat{t}_1\hat{t}_2^* & 0 & \hat{t}_2^* & -1+\hat{t}_2^* \end{pmatrix}, \tag{10.8}$$

where the parameters are given in (10.5). Hence, $Q^{(2)}_{p,q} = \det(A)$, which can be evaluated with a straightforward but lengthy calculation, yielding

$$\begin{aligned} Q^{(2)}_{p,q} &= (1+|\hat{t}_1|^2)(1+|\hat{t}_2|^2) - (\hat{t}_1+\hat{t}_1^*)(1-|\hat{t}_2|^2) - (\hat{t}_2+\hat{t}_2^*)(1-|\hat{t}_1|^2) \\ &= (1+t_1^2)(1+t_2^2) - 2t_1(1-t_2^2)\cos\frac{2\pi p}{L} - 2t_2(1-t_1^2)\cos\frac{2\pi q}{L}. \end{aligned} \tag{10.9}$$

The product of these factors over the whole momentum space yields the squared partition function Q^2. To get Q itself, we have to take the square root, or reduce the product to either $0 \le p < L$ and $0 \le q \le L/2$ or $0 \le p \le L/2$ and $0 \le q < L$. The 2D partition function thus appears in the form

$$Q^2 = \prod_{p,q=0}^{L-1} \left[(1+t_1^2)(1+t_2)^2 - 2t_1(1-t_2^2)\cos\frac{2\pi p}{L} - 2t_2(1-t_1^2)\cos\frac{2\pi q}{L} \right]. \tag{10.10}$$

Remark. The solution (10.10) is exact for the infinite lattice, where one can neglect boundary effects. Since we have changed the original free boundary conditions for fermions to periodic boundary conditions in order to be able to apply Fourier transformation, the expression (10.10) is in fact also the exact value of the (squared) integral (10.1) with periodic closing $\overline{a}_{0,n} = \overline{a}_{L,n}$ and $\overline{b}_{m,0} = \overline{b}_{m,L}$, even at finite L.

From (10.10) we can readily obtain the exact expression for the free energy per site in the thermodynamic limit $L \to +\infty$:

$$\begin{aligned} -\beta f_Q^{(2D)} &= \lim_{L\to+\infty} \frac{1}{L^2}\log Q \\ &= \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi} \frac{dp\,dq}{(2\pi)^2} \log\left[(1+t_1^2)(1+t_2^2) - 2t_1(1-t_2^2)\cos\frac{2\pi p}{L} \right. \\ &\qquad \left. -2t_2(1-t_1^2)\cos\frac{2\pi q}{L} \right]. \end{aligned} \tag{10.11}$$

This is essentially Onsager's expression for the free energy. Taking into account the relation between the full partition function Z and the reduced partition function Q, the actual free energy of the 2D Ising model becomes

$$\begin{aligned}
-\beta f_Z^{(2D)} &= \lim_{L\to+\infty} \frac{1}{L^2}\log Z \\
&= \log 2 + \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi} \frac{dp\,dq}{(2\pi)^2}\log\left[\cosh(2b_1)\cosh(2b_2)\right. \\
&\qquad \left. -\sinh(2b_1)\cos(p) - \sinh(2b_2)\cos(q)\right], \quad (10.12)
\end{aligned}$$

where one can also change the limits in the integrals from $0, 2\pi$ to $-\pi, \pi$ if preferred. The dominant contribution to the singular part of the free energy, and the corresponding specific heat comes from integration around the origin $p = q = 0$ in momentum space (provided we are dealing with the ferromagnetic case $b_1, b_2 > 0$). The free energy expression (10.12) is Eq. (108) in: L. Onsager, Phys. Rev. **65** (1944) 117. An interesting comment on the structure of the expression (10.12) follows just after this formula Eq. (108).

We have thus obtained the exact solution for the 2D Ising model on a rectangular lattice via Grassmann integration. This method does not use the transfer matrix, nor is it combinatorial in spirit. It is more like Dirac's method of insert unity $\sum |a\rangle\langle a| = 1$ in quantum mechanics: see P. A. M. Dirac, *The Principles of Quantum Mechanics* (Oxford Univ. Press, 1934). In our case, however, the situation is somewhat more complicated than that typically occurring in quantum-mechanical problems, as we pass from commuting spins to non-commuting Grassmann factors. The crucial point in the fermionization procedure is the mirror-ordering arrangement of Grassmann factors, which enables the complete elimination of spin variables. Technically, the most laborious part of the solution is the evaluation of the 4-th order determinant $\det(A)$ where A is given by (10.8). This problem can however be significantly simplified by eliminating part of the Grassmann variables from the basic integral for Q before going to momentum space. This effectively reduces the problem to the evaluation of a 2×2 determinant. In the next section we comment briefly on the thermodynamic properties which can be derived from the exact solution for f_Z.

11 The free energy and the specific heat singularity of the 2D Ising model

Let us start from the expression for the free energy

$$\begin{aligned} -\beta f_Q^{(2D)} &= \frac{1}{2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{dp\,dq}{(2\pi)^2}\log\left[(1+t_1^2)(1+t_2^2)-2t_1(1-t_2^2)\cos\frac{2\pi p}{L}\right. \\ &\qquad \left. -2t_2(1-t_1^2)\cos\frac{2\pi q}{L}\right]. \end{aligned} \tag{11.1}$$

The singular part of this expression is caused by the singularity of the logarithm near zero. Rewriting the argument of the logarithm as

$$\begin{aligned} Q_{p,q}^{(2)} &= (1-t_1-t_2-t_1t_2)^2+2t_1(1-t_2^2)(1-\cos(p)) \\ &\quad +2t_2(1-t_1^2)(1-\cos(q)) \\ &\approx (1-t_1-t_2-t_1t_2)^2+4t_1(1-t_2^2)\sin^2\frac{p}{2}+4t_2(1-t_1^2)\sin^2\frac{q}{2} \end{aligned} \tag{11.2}$$

we see that this can only approach zero if $p,q\to 0$, in which case

$$Q_{p,q}^{(2)} \approx \underline{m}^2 + A_1p^2 + A_2q^2 \tag{11.3}$$

where

$$\underline{m} = 1-t_1-t_2-t_1t_2, \qquad A_1 = t_1(1-t_2^2), \quad A_2 = t_2(1-t_1^2). \tag{11.4}$$

We call $\underline{m}$ the mass. The condition $\underline{m}=0$ determines the critical point:

$$1-t_1-t_2-t_1t_2 = 0 \iff T = T_c. \tag{11.5}$$

This is equivalent to the more familiar criticality condition

$$\sinh(2b_1)\sinh(2b_2) = 1. \tag{11.6}$$

We are interested in the situation where $\underline{m}$ is small (T close to T_c), where the approximation (11.3) is good. The singular part of the free energy is thus

$$(-\beta f)_{\text{sing}} = \frac{1}{2}\int\int_{|(p,q)|\le K_0}\frac{dp\,dq}{(2\pi)^2}\log[\underline{m}^2+A_1p^2+A_2q^2], \tag{11.7}$$

where K_0 is some small cut-off momentum. The parameters A_1 and A_2 can be taken exactly at T_c for our purposes.

We now make some more technical remarks. By rescaling the momenta, we may replace $\underline{m}^2+A_1p^2+A_2q^2$ by $\underline{m}^2+\sqrt{A_1A_2}(p^2+q^2)$ and then extract the factor $\sqrt{A_1A_2}$ from the kinetic term altogether, giving rise to an insignificant additive term to the free energy as well as a rescaling of the mass to $\overline{m}^2 = \underline{m}^2/\sqrt{A_1A_2}$. The result is

$$(-\beta f)_{\rm sing} = \frac{1}{8\pi^2}\int\int_{|(p,q)|\leq K_0} dp\,dq\,\log[\overline{m}^2 + p^2 + q^2], \tag{11.8}$$

and evaluating the integral,

$$(-\beta f)_{\rm sing} = \frac{1}{8\pi}\overline{m}^2\log\frac{\rm const}{\overline{m}^2} + {\rm const} \qquad \text{as } \overline{m}\to 0. \tag{11.9}$$

(The integration is best done in polar coordinates: $dp\,dq = 2\pi|p|\,d|p|$ where $p^2+q^2 = |p|^2$.) Evaluating A_1A_2 at the critical point, we have

$$(A_1A_2)_c = t_1t_2(1-t_1^2)(1-t_2^2) = 4t_1^2t_2^2\frac{1-t_1^2}{2t_1}\frac{1-t_2^2}{2t_2} = (2t_1t_2)_c^2$$

since

$$\left(\frac{1-t_1^2}{2t_1}\frac{1-t_2^2}{2t_2}\right)_c = \frac{1}{\sinh(2b_1)\sinh(2b_2)} = 1.$$

The rescaled mass is therefore

$$\overline{m} = \frac{m}{\sqrt{2(t_1t_2)_c}} = \frac{1-t_1-t_2-t_1t_2}{\sqrt{2(t_1t_2)_c}}. \tag{11.10}$$

This mass also appears in the Majorana-Dirac continuum limit formulation of the 2D Ising model.

The specific heat C is given by

$$C/k = \beta^2\frac{\partial^2(-\beta f_Z)}{\partial\beta^2}, \tag{11.11}$$

where we can use the expression (11.8) to get the critical singularity of the dimensionless specific heat C/k (k is Boltzmann's constant). Near T_c, the mass $\overline{m}$ varies linearly as a function of $T - T_c$. Differentiating (11.9) twice with respect to $\overline{m}$, we have[5]

$$(C/k)_{\rm sing}(\overline{m}) = \frac{1}{4\pi}\log\frac{\rm const}{\overline{m}^2} = \frac{1}{2\pi}|\log|\overline{m}|| + {\rm const}. \tag{11.12}$$

The specific heat of the 2D Ising model is therefore logarithmically divergent as $T \to T_c$ (since $|\overline{m}| \sim |T - T_c|$). This was first established by Onsager in 1944. Note that near T_c (or $\beta_c = 1/kT_c$) the mass is

$$\overline{m} \approx (\overline{m}'')_c(\beta - \beta_c),\ \text{where}\ (\overline{m}'')_c = \frac{\partial^2\overline{m}}{\partial\beta^2}\Big|_{T=T_c}.$$

Therefore

$$(C/k)_{\rm sing} = A_c\left|\log\frac{|T-T_c|}{T_c}\right| \to +\infty\ \text{as}\ T \to T_c, \tag{11.13}$$

where

$$A_c = \beta_c\frac{(\overline{m}'')_c}{2\pi} \quad \text{(isotropic case)}. \tag{11.14}$$

A_c is called the specific heat critical amplitude. In general, A_c depends on the anisotropy of the interaction (J_1/J_2), but in the isotropic case it is a definite dimensionless number. In the isotropic case, $t = \tanh(b)$ with $b = J/kT_c$, and the criticality condition $\overline{m} = 1 - 2t - t^2 = 0$ yields $t_c = \sqrt{2} - 1$. The mass, to linear order in $b - b_c = b_c(T_c - T)/T_c$ becomes

$$\begin{aligned}\overline{m} &= \frac{1-2t-t^2}{\sqrt{2}t_c} = \frac{-2(1+t_c)}{\sqrt{2}t_c}(t-t_c) = 2\frac{t_c - t}{t_c} \\ &= \frac{2}{t_c}(1-t_c^2)(b_c - b) = 4(b_c - b) = 4b_c\frac{T-T_c}{T_c},\end{aligned} \tag{11.15}$$

tending to $+\infty$ as $T \to T_c$. Hence $(\overline{m}')_c^2 = 16$ and the amplitude for the specific heat becomes $A_c = \frac{8}{\pi}b_c^2$. From $t_c = \tanh(b_c) = e^{-2b_c}$ which follows from $\sinh(2b_c) = 1$, we have $b_c = \frac{1}{2}\log(1+\sqrt{2})$. Thus, for the isotropic lattice,

$$\begin{aligned}(C/k)_{\rm sing} &= A_c\left|\log\left|\frac{T-T_c}{T_c}\right|\right| + B_c,\ \text{where} \\ A_c &= \frac{8}{\pi}b_c^2 \approx 0.495,\ b_c = \frac{J}{kT_c} = \frac{1}{2}\log(1+\sqrt{2}) \approx 0.441\end{aligned} \tag{11.16}$$

[5]We can choose $\overline{m}$ as a measure of $|T - T_c|$ and obtain $(C/k)_{\rm sing}(\overline{m})$ in (11.12) by differentiating twice w.r.t. $\overline{m}$. In (11.13) this expression is recalculated to give the true specific heat singularity of (11.11) in the isotropic case.

The value for A_c is exact. The next term B_c can only be determined from the complete exact expression for $(-\beta f_Z)$ (see Huang, *Statistical Mechanics* (Wiley, 1987)).

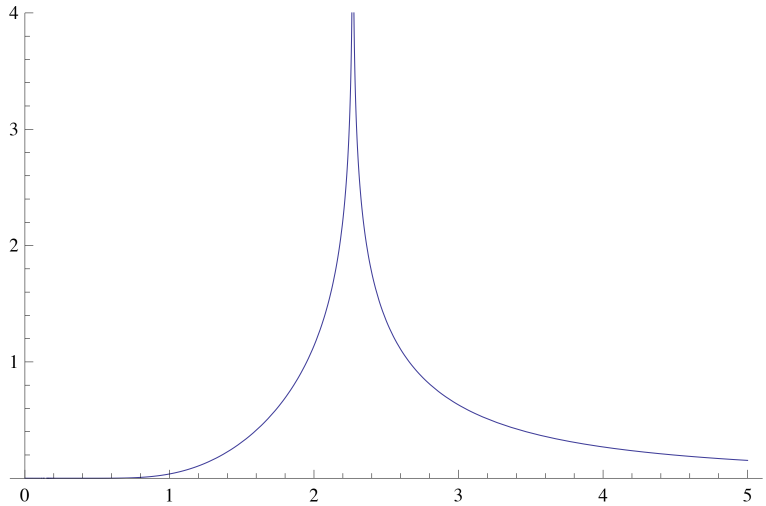

Figure 1. The specific heat of the 2D Ising model.

The expression (11.16) only gives the exact asymptotics for the specific heat as $T \to T_c$ (or the mass $\overline{m} \to 0$). In fact, the specific heat can be derived exactly by differentiating the free energy twice. The result can be written in terms of elliptic integrals of the first and second kind: see Huang (1987) or the original paper by Onsager (1944).[6]

Another important quantity which is known exactly for the 2D Ising model is the spontaneous magnetization at zero field. It is given by the surprisingly simple expression

$$M = \left[1 - \frac{1}{\sinh^2(2b_1)\sinh^2(2b_2)}\right]^{1/8}. \tag{11.17}$$

It implies that near T_c, $M \sim B\,|\tau|^{1/8}$, where $\tau = (T_c - T)/T_c$. Despite the relatively simple form of this expression its derivation in any of the known

[6]L. Onsager, Phys. Rev. **65** (1944) 117. and K. Huang, *Statistical Mechanics* (Wiley, New York 1963; second ed. 1987). In the book by Huang, the 2D Ising model is treated in some detail within the transfer matrix (algebraic) approach, close to Onsager's original method but in spinor-algebra formulation (Kaufman, 1949).

approaches is considerably more complicated than the evaluation of the free energy. Typically, M^2 is obtained as the limiting value of the spin-spin correlation function at infinity: $M^2 = \langle\sigma(0)\sigma(R)\rangle\big|_{R\to+\infty}$. The correlator $\langle\sigma(0)\sigma(R)\rangle$ is not a simple quantity, however, in the 2D Ising model. It is expressed as a large-size determinant of order $R \times R$ for which there is no known explicit formula. The limiting value of this determinant as $R \to +\infty$ is then obtained using the Szegö-Kac theorem, resulting in the above expression.[7] (The Szegö-Kac theorem is in fact used to compute $\log M^2$.) The formula for M^2 can also be obtained using Grassmann variables by first obtaining $\langle\sigma(0)\sigma(R)\rangle = \langle\sigma_{m,n}\sigma_{m+R,n}\rangle$ on the lattice as a large-size Toeplitz determinant, with known matrix elements and then calculating M^2 using the Szegö-Kac theorem.[8] The situation with the spontaneous magnetization in the 2D Ising model is not completely understood anyhow, and the contrast between the simple result (11.17) and its very complicated derivation remains unresolved. Other known results about the 2D Ising model include the asymptotics of the correlation functions at large distances, and asymptotics of the magnetic susceptibility above and below T_c.

---

[7]See E. W. Montroll, R. B. Potts & J. C Ward, J. Math. Phys. **4** (1963), 308, for a discussion of the spin-spin correlation and the calculation of M^2 via the Szegö-Kac theorem. See also B. M. McCoy & T. T Wu, The two-dimensional Ising model. (Harvard Univ. Press, 1973).

[8]In the Grassmann variable approach the correlator $\langle\sigma_{m,n}\sigma_{m+R,n}\rangle$ appears as a determinant of fermionic Green's functions:

$$\langle\sigma_{m,n}\sigma_{m+R,n}\rangle = \langle\prod_{r=1}^{R} A_{m,n}\overline{A}_{m+r,n}\rangle = \det\left(\langle A_{m',n}\overline{A}_{m'',n}\rangle\right).$$

12 Other representations and the correlation function with four variables $a_{m,n}, \bar{a}_{m,n}, b_{m,n}, \bar{b}_{m,n}$

The basic fermionic representation for the partition function Q was obtained in Eq. (8.16) in the form

$$\begin{aligned} Q \;=\; & \int \prod_{m,n=1}^{L} d\bar{a}_{m,n} da_{m,n} d\bar{b}_{m,n} db_{m,n} \,\exp\left[\sum_{m,n=1}^{L} (a_{m,n}\bar{a}_{m,n} + b_{m,n}\bar{b}_{m,n})\right] \\ & \times \exp\left[\sum_{m,n=1}^{L} \left(t^{(1)}_{m,n} t^{(2)}_{m,n} \bar{a}_{m-1,n}\bar{b}_{m,n-1} + a_{m,n}b_{m,n}\right)\right] \\ & \times \exp\left[\sum_{m,n=1}^{L} (t^{(1)}_{m,n}\bar{a}_{m-1,n} + t^{(2)}_{m,n}\bar{b}_{m,n-1})(a_{m,n} + b_{m,n})\right]. \qquad (12.1) \end{aligned}$$

This is a representation with four variables per site. Various modifications of this representation are also possible. For example, one can integrate out the variables $a_{m,n}$ and $b_{m,n}$ in a simple way. The resulting integral with two variables per site will be discussed later. But here we derive yet another representation with four variables per site: see (12.3). First note that for the typical Boltzmann weight in Q we can write $1 + t\sigma\sigma' = \sigma\sigma'(t + \sigma\sigma')$. Moreover, the circle product of quantities $\sigma_{m,n}\sigma_{m+1,n}$ over m or $\sigma_{m,n}\sigma_{m,n+1}$ over n equals 1. Therefore, in terms of spin variables, the partition function

$$Q = \operatorname*{Sp}_{(\sigma)} \left\{ \prod_{m,n=1}^{L} (1 + t^{(1)}_{m+1,n}\sigma_{m,n}\sigma_{m+1,n})(1 + t^{(2)}_{m,n+1}\sigma_{m,n}\sigma_{m,n+1}) \right\}$$

can also be written in the form

$$Q = \operatorname*{Sp}_{(\sigma)} \left\{ \prod_{m,n=1}^{L} (t^{(1)}_{m+1,n} + \sigma_{m,n}\sigma_{m+1,n})(t^{(2)}_{m,n+1} + \sigma_{m,n}\sigma_{m,n+1}) \right\}, \qquad (12.2)$$

provided that cyclic closing conditions for the spins are introduced. By applying the same method that led to the integral (12.1) we then obtain the following fermionic integral representation

$$
\begin{aligned}
Q &= \int \prod_{m,n=1}^{L} d\overline{a}_{m,n} da_{m,n} d\overline{b}_{m,n} db_{m,n} \, \exp\left[\sum_{m,n=1}^{L} \big(a_{m,n}b_{m,n} + \overline{a}_{m,n}\overline{b}_{m,n}\big)\right] \\
&\qquad \times \exp\left[\sum_{m,n=1}^{L} (\overline{a}_{m,n} + \overline{b}_{m,n})(a_{m,n} + b_{m,n})\right] \\
&\qquad \times \exp\left[\sum_{m,n=1}^{L} \Big(t^{(1)}_{m+1,n} a_{m,n}\overline{a}_{m+1,n} + t^{(2)}_{m,n+1} b_{m,n}\overline{b}_{m,n+1}\Big)\right]. \qquad (12.3)
\end{aligned}
$$

The starting point for deriving (12.3) is the following factorization of local weights,

$$
\begin{aligned}
& t^{(1)}_{m+1,n} + \sigma_{m,n}\sigma_{m+1,n} = \\
&\quad = \int d\overline{a}_{m+1,n} da_{m,n} e^{t^{(1)}_{m+1,n} a_{m,n}\overline{a}_{m+1,n}} (1 + \sigma_{m,n}a_{m,n})(1 + \sigma_{m+1,n}\overline{a}_{m+1,n})
\end{aligned}
\qquad (12.4)
$$

and

$$
\begin{aligned}
& t^{(2)}_{m,n+1} + \sigma_{m,n}\sigma_{m,n+1} = \\
&\quad = \int d\overline{b}_{m,n+1} db_{m,n} e^{t^{(2)}_{m,n+1} b_{m,n}\overline{b}_{m,n+1}} (1 + \sigma_{m,n}b_{m,n})(1 + \sigma_{m,n+1}\overline{b}_{m,n+1}),
\end{aligned}
\qquad (12.5)
$$

where at each site (or at adjacent bonds) we introduce variables $a_{m,n}$, $\overline{a}_{m+1,n}$, $b_{m,n}$ and $\overline{b}_{m,n+1}$. Then repeating the mirror-ordering procedure for the density matrix as in Section 8, we obtain the analogue of (8.12) with the averaging on the junction of four factors yielding

$$
\begin{aligned}
& \operatorname*{Sp}_{(\sigma_{m,n})} \; \overline{A}_{m,n}\overline{B}_{m,n}A_{m,n}B_{m,n} = \\
&\quad = \frac{1}{2} \sum_{\sigma_{m,n}=\pm 1} (1 + \overline{a}_{m,n}\sigma_{m,n}) \\
&\qquad\qquad \times (1 + \overline{b}_{m,n}\sigma_{m,n})(1 + a_{m,n}\sigma_{m,n})(1 + b_{m,n}\sigma_{m,n}) \\
&\quad = \exp\left\{\overline{a}_{m,n}\overline{b}_{m,n} + a_{m,n}b_{m,n} + (\overline{a}_{m,n} + \overline{b}_{m,n})(a_{m,n} + b_{m,n})\right\}. \qquad (12.6)
\end{aligned}
$$

The partition function integral for Q is then formed by the product of such factors plus the Gaussian terms already introduced by the factorization (12.4) and (12.5). This results in (12.3). In this representation we ignored the boundary effects. This problem may be considered in the standard way as for the integral (12.1). Notice that in (12.4) and (12.5) we introduced the variables $a_{m,n}$, $\overline{a}_{m+1,n}$, $b_{m,n}$ and $\overline{b}_{m,n+1}$, so that in the 'measure', the variables $\overline{a}_{1,n}$ and $\overline{b}_{m,1}$ are actually absent, whereas $\overline{a}_{L+1,n}$ and $\overline{b}_{m,L+1}$ do occur. We can rename the latter $\overline{a}_{1,n}$ and $\overline{b}_{m,1}$ respectively, and reorder the differentials in order that the basic 'measure' is of the standard form $\prod_{m,n=1}^{L}(d\overline{a}_{m,n}da_{m,n}d\overline{b}_{m,n}db_{m,n})$. [This might introduce an extra minus sign in front of Q.] Alternatively, the boundary terms can be evaluated properly in the action. In writing (12.3), we assumed the boundary effects to be inessential as $L \to +\infty$. For finite L, these boundary effects have to be taken into account more precisely. As compared to (12.1), the representation (12.3) has the advantage that the interaction parameters $t^{(1)}_{m,n}$ and $t^{(2)}_{m,n}$ are in a way localized in the action. On the other hand, the combinatorial analogy with spin-loops in the 2D Ising model are probably less evident in (12.3).[9]

Returning to the original observation leading to the modified representation (12.3), notice that the Boltzmann weights can also be written in the form $1+t\sigma\sigma' = t(\frac{1}{t}+\sigma\sigma')$, from which it immediately follows that the spin-variable partition function can be rewritten, by analogy with (12.3), as follows,

$$
\begin{aligned}
Q \;=\; & \left\{\prod_{m,n=1}^{L} t^{(1)}_{m+1,n}t^{(2)}_{m,n+1}\right\}\int\prod_{m,n=1}^{L} d\overline{a}_{m,n}da_{m,n}d\overline{b}_{m,n}db_{m,n} \\
& \times \exp\left[\sum_{m,n=1}^{L}\big(a_{m,n}b_{m,n}+\overline{a}_{m,n}\overline{b}_{m,n}+(\overline{a}_{m,n}+\overline{b}_{m,n})(a_{m,n}+b_{m,n})\big)\right] \\
& \times \exp\left[\sum_{m,n=1}^{L}\left(\frac{1}{t^{(1)}_{m+1,n}}a_{m,n}\overline{a}_{m+1,n}+\frac{1}{t^{(2)}_{m,n+1}}b_{m,n}\overline{b}_{m,n+1}\right)\right]. \qquad (12.7)
\end{aligned}
$$

The integral (12.7), though being in close formal analogy with (12.3), is in fact more directly related to (12.1), and can be obtained from (12.1) (this is easier to do in the homogeneous case) by transformations like

[9]The representation (12.3), which here follows by a non-combinatorial factorization argument, in its final form, can prehaps be considered closest to the representations obtained by combinatorial analysis by S. Samuel, J. Math. Phys. **21** (1980), 2806, in the homogeneous case.

$\overline{a}_{m,n}, \overline{b}_{m,n} \rightarrow \overline{a}_{m+1,n}, \overline{b}_{m,n+1}$ and $\overline{a}_{m,n}, \overline{b}_{m,n} \rightarrow \frac{1}{t_1}\overline{a}_{m,n}, \frac{1}{t_2}\overline{b}_{m,n}$. By a suitable Fourier transformation, in the homogeneous case, one can pass to momentum space in the integrals (12.3) and (12.7) and evaluate explicitly the free energy $-\beta f_Q$, which is of course the same as before. However, as a final note about (12.3) and (12.7), one may in some cases, gain an effective shift in the momenta p and q, so the free energy can appear in the form

$$\begin{aligned} -\beta f_Q^{(2D)} &= \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi}\frac{dp\,dq}{(2\pi)^2}\log\Big[(1+t_1^2)(1+t_2^2) \\ &\quad +2t_1(1-t_2^2)\cos\frac{2\pi p}{L}+2t_2(1-t_1^2)\cos\frac{2\pi q}{L}\Big], \end{aligned} \tag{12.8}$$

which is to be compared with (10.11). Both expressions are of course equivalent and the (ferromagnetic) critical point is again given by the condition $1-t_1-t_2-t_1t_2=0$, though the critical region in momentum space is now around the point (π,π) rather than $(0,0)$. (The point $(0,0)$ in (12.8) is associated with the anti-ferromagnetic transition.)

In general, the four special ferromagnetic - antiferromagnetic modes in momentum space with $(p,q)=(0,0)$, $(0,\pi)$, $(\pi,0)$ and (π,π) play an important role in the structure of the solution for finite L and the interpretation of the results for the rectangular, as well as for triangular and hexagonal lattice 2D Ising models.[10] For the rectangular lattice, these modes are

$$\begin{aligned} Q_{00} &= 1-t_1-t_2-t_1t_2 = 2\overline{\alpha}_{00}, \\ Q_{0\pi} &= 1-t_1+t_2+t_1t_2 = 2\overline{\alpha}_{0\pi}, \\ Q_{\pi 0} &= 1+t_1-t_2+t_1t_2 = 2\overline{\alpha}_{\pi 0}, \\ Q_{\pi\pi} &= 1+t_1+t_2-t_1t_2 = 2\overline{\alpha}_{\pi\pi}. \end{aligned} \tag{12.9}$$

In essence, Q_{00} is the Majorana mass in the ferromagnetic regime, while the others are masses in the anti-ferromagnetic and mixed regimes.

Another interesting feature is that the spontaneous magnetization for the

[10]See: V. N. Plechko, Physica A**152** (1988), 51. The discussion includes rectangular, triangular and hexagonal lattices. A variant of the method based on two Grassmann variables per site is applied in all three cases. The notation $2\overline{\alpha}_{00}$ etc. in (12.9) is the same as in this paper.

rectangular lattice can be written in the form

$$M = \left(\frac{-\overline{\alpha}_{00}\overline{\alpha}_{0\pi}\overline{\alpha}_{\pi 0}\overline{\alpha}_{\pi\pi}}{t_1^2 t_2^2} \right)^{1/8} \tag{12.10}$$

in the disordered phase where $T < T_c$, where the numerator in (12.10) is positive.

13 The 2D Ising model on a finite torus

The partition function of the 2D Ising model on a finite lattice wrapped around a torus (periodic boundary conditions for spins in both directions) is

$$Q = \mathop{\mathrm{Sp}}_{(\sigma)} \left\{ \prod_{m,n=1}^{L} (1 + t^{(1)}_{m+1,n}\sigma_{m,n}\sigma_{m+1,n})(1 + t^{(2)}_{m,n+1}\sigma_{m,n}\sigma_{m,n+1}) \right\}, \tag{13.1}$$

with $\sigma_{L+1,n} = \sigma_{1,n}$ and $\sigma_{m,L+1} = \sigma_{m,1}$. The Grassmann-variable representation for this expression is a superposition of four similar integrals with different periodic-aperiodic conditions for fermions,[11]

$$Q = \frac{1}{2} \left[G\big|_{--} + G\big|_{-+} + G\big|_{+-} - G\big|_{++} \right], \tag{13.2}$$

where G is the same integral as in (8.16) or (12.1), i.e.

$$\begin{aligned} G \;\; = \;\; & \int \prod_{m,n=1}^{L} d\overline{a}_{m,n} da_{m,n} d\overline{b}_{m,n} db_{m,n} \, \exp \left[\sum_{m,n=1}^{L} (a_{m,n}\overline{a}_{m,n} + b_{m,n}\overline{b}_{m,n}) \right] \\ & \times \exp \left[\sum_{m,n=1}^{L} \left(t^{(1)}_{m,n} t^{(2)}_{m,n} \overline{a}_{m-1,n} \overline{b}_{m,n-1} + a_{m,n} b_{m,n} \right) \right] \\ & \times \exp \left[\sum_{m,n=1}^{L} (t^{(1)}_{m,n}\overline{a}_{m-1,n} + t^{(2)}_{m,n}\overline{b}_{m,n-1})(a_{m,n} + b_{m,n}) \right], \end{aligned} \tag{13.3}$$

[11] See: V. N. Plechko, Teor. Mat. Fiz. **64** (1985) 150 [English transl. Sov.Phys.-Theor. Math. Phys. **64** (1985) 748]. The solution of the 2D Ising model on a torus in terms of Grassmann integrals. Also preprint JINR (1984).

and where the boundary conditions are as follows:

$$\begin{aligned}
(--) &= (\overline{a}_{0,n} = -\overline{a}_{L,n};\ \overline{b}_{m,0} = -\overline{b}_{m,L}) \\
(-+) &= (\overline{a}_{0,n} = -\overline{a}_{L,n};\ \overline{b}_{m,0} = +\overline{b}_{m,L}) \\
(+-) &= (\overline{a}_{0,n} = +\overline{a}_{L,n};\ \overline{b}_{m,0} = -\overline{b}_{m,L}) \\
(++) &= (\overline{a}_{0,n} = +\overline{a}_{L,n};\ \overline{b}_{m,0} = +\overline{b}_{m,L}). \qquad (13.4)
\end{aligned}$$

In the homogeneous case, each of these integrals can be calculated by means of Fourier transformation with the corresponding integer or half-integer momenta:

$$\begin{aligned}
(--) &\longleftrightarrow (p+\frac{1}{2}\,,\, q+\frac{1}{2}) \\
(-+) &\longleftrightarrow (p+\frac{1}{2}\,,\, q) \\
(+-) &\longleftrightarrow (p\,,\, q+\frac{1}{2}) \\
(++) &\longleftrightarrow (p\,,\, q) \qquad (13.5)
\end{aligned}$$

and one also has to take into account properly the sign of each of the integrals, which is given precisely by the exceptional modes (12.9), wherever they are present in a given integral. For instance, for the ferromagnetic case,

$$Q = \frac{1}{2}\left[|G_{--}| + |G_{-+}| + |G_{+-}| - \mathrm{sign}\left(\frac{T-T_c}{T_c}\right)|G_{++}|\right], \qquad (13.6)$$

where the integral $|G_{++}|$ is given by

$$\begin{aligned}
|G_{++}| = \prod_{p,q=0}^{L-1} \Big[&(1+t_1^2)(1+t_2^2) \\
&-2t_1(1-t_2^2)\cos\frac{2\pi p}{L} - 2t_2(1-t_1^2)\cos\frac{2\pi q}{L}\Big]^{1/2}, \qquad (13.7)
\end{aligned}$$

and where the factor $\mathrm{sign}\,(\frac{T-T_c}{T_c})$ in the last term is given by the special mode $2\overline{\alpha}_{00} = 1-t_1-t_2-t_1t_2$ with $p=q=0$: $\mathrm{sign}\,(\frac{T-T_c}{T_c}) = \mathrm{sign}\,(\overline{\alpha}_{00})$. (Notice that $|1-t_1-t_2-t_1t_2|$ is a factor in $|G_{++}|$ at $p=q=0$, as it should be.) The other three integrals are similar to (13.7) but modified according to the boundary conditions (13.5), and that they are always positive in the ferromagnetic case (where $b_1, b_2 > 0$ or $t_1, t_2 > 0$, with $b_{1,2} = J_{1,2}/kT$).

In the above, we have written the answers only for the lattice of size $L \times L$. The answers for rectangular-shaped lattices on a torus $L_1 \times L_2$ with $L_1 \neq L_2$ are the same except for the evident change of limits in sums and products from (L, L) to (L_1, L_2), and changing $\left(\frac{2\pi p}{L}, \frac{2\pi q}{L}\right) \to \left(\frac{2\pi p}{L_1}, \frac{2\pi q}{L_2}\right)$. The exceptional modes may then appear (introducing possibly a sign $\pm$ in front of $|G_{\alpha\beta}|$) in any of the integrals in (13.6) depending on the parity of (L_1, L_2) and the ferro- or anti-ferromagnetic regime. However, in the purely ferromagnetic regime, the only mode that changes sign at T_c is $2\overline{\alpha}_{00} = 1 - t_1 - t_2 - t_1 t_2$, and this mode is always located in G_{++}. So, independent of the parity of L_1, L_2, in this regime, the answer is always given by (13.6) and (13.7). Another simplification occurs if L_1 and L_2 are both even integers. Then all four special modes are present in G_{++} and Q is given by (13.6) with sign $\left(\frac{T-T_c}{T_c}\right)$ to be replaced by sign $(\overline{\alpha}_{00}\overline{\alpha}_{0\pi}\overline{\alpha}_{\pi 0}\overline{\alpha}_{\pi\pi})$. This is true in any of the ferro or anti-ferromagnetic regimes. In general, the critical modes can occur in other integrals, and one has to consider separate cases individually.

14 Fermionic correlation functions in the 2D Ising model

The 2D Ising model, having been formulated as Gaussian (free-fermion) Grassmann integrals on a lattice, the fermionic correlation functions (Green's functions) like $\langle a_{m,n}\overline{a}_{m',n'}\rangle$, $\langle a_{m,n}b_{m',n'}\rangle$, etc. can be evaluated at least in principle by standard tools. The multi-fermion correlation functions can then be decomposed in terms of binary fermionic correlations by means of Wick's theorem (see (9.9)). It then follows that any perturbations of the 2D Ising model by variation of the parameters $t^{(1)}_{m,n}$ and $t^{(2)}_{mm,n}$ in the neighbourhood of the homogeneous case $t^{(1)}_{m,n} = t_1$ and $t^{(2)}_{m,n}$ can be calculated in terms of binary correlations. The method of calculating binary fermionic correlations is to calculate first their Fourier images like $\langle a_{pq}\overline{a}_{pq}\rangle$, $\langle a_{pq}b_{-p,-q}\rangle$, etc. (since the theory is block-diagonal in momentum space). The real-space correlations then follow in an obvious way.[12] For example, we have

$$\langle a_{m,n}\overline{a}_{m',n'}\rangle = \int_0^{2\pi}\int_0^{2\pi}\frac{dp\,dq}{(2\pi)^2}\langle a_{p,q}\overline{a}_{p,q}\rangle\, e^{ip(m-m')+iq(n-n')}, \qquad (14.1)$$

where a direct calculation gives

$$\langle a_{p,q}\overline{a}_{p,q}\rangle = \frac{(1-2t_2\cos(q)+t_2^2)-t_1(1-t_2^2)e^{-ip}}{\Delta_{p,q}}, \qquad (14.2)$$

where

$$\begin{aligned}\Delta_{p,q} &= (1+t_1^2)(1+t_2^2)-2t_1(1-t_2^2)\cos(p)-2t_2(1-t_1^2)\cos(q)\\ &= A_0-A_1\cos(p)-A_2\cos(q), \qquad (14.3)\end{aligned}$$

and where, of course, we assumed that the thermodynamic limit has been performed. The correlator $\langle a_{p,q}\overline{a}_{p,q}\rangle = Q'_{p,q}/Q_{p,q}$ is the ratio of two integrals Q' and Q, where $Q_{p,q}=\Delta_{p,q}$ is known, as it is the value of the determinant $\det(A)$ given by (10.8), i.e. (10.9). $Q'_{p,q}$ is a reduced integral of this kind, in which the $a_{p,q}\overline{a}_{p,q}$-prefactor annihilates some terms in the exponential.

A systematic, more formal way to evaluate all momentum-space correlations is to compute the inverse matrix elements (or minors) of the matrix A of (10.8). Adopting the notation of (10.7), i.e.

$$\begin{aligned}a_1,a_2,a_3,a_4 &\longleftrightarrow a_{p,q},b_{p,q},\overline{a}_{L-p,L-q},\overline{b}_{L-p,L-q}\\ \overline{a}_1,\overline{a}_2,\overline{a}_3,\overline{a}_4 &\longleftrightarrow \overline{a}_{p,q},\overline{b}_{p,q},a_{L-p,L-q},b_{L-p,L-q},\end{aligned}$$

we have a Gaussian integral with 4 pairs of variables with action $S = \sum_{i,j=1}^4 a_iA_{ij}\overline{a}_j$ so that

$$\int\prod_{j=1}^4 d\overline{a}_j da_j\, e^S = \det(A) = Q_{p,q}^2,$$

and the fermionic correlations are given in terms of A^{-1} as follows,

$$\langle a_i\overline{a}_j\rangle = \frac{(-1)^{i+j}(\det(A))_{ij}}{\det(A)} = (A^{-1})_{ji}, \qquad (14.4)$$

[12] The correlation $\langle a_{m,n}\overline{a}_{m',n'}\rangle$ follows by Fourier substitution (10.2), noting that $\langle a_{p,q}\overline{a}_{p',q'}\rangle = \delta_{p,p'}\delta_{q,q'}\langle a_{p,q}\overline{a}_{p,q}\rangle$, because of the block-diagonal structure of the action in momentum space. Similarly, $\langle a_{p,q}b_{-p',-q'}\rangle = \delta_{p,p'}\delta_{q,q'}\langle a_{p,q}b_{-p,-q}\rangle$, etc.

where $\det(A))_{ij}$ is the minor of the element (ij) in A. With this notation, $\langle a_{p,q}\overline{a}_{p,q}\rangle$ becomes

$$\langle a_1\overline{a}_1\rangle = \frac{(-1)^{1+1}(\det(A))_{11}}{\det(A)}.$$

Similarly, $\langle a_1\overline{a}_2\rangle = \langle a_{p,q}\overline{b}_{p,q}\rangle$, $\langle a_1\overline{a}_3\rangle = \langle a_{p,q}a_{-p,-q}\rangle$, $\langle a_1, \overline{a}_4\rangle = \langle a_{p,q}b_{-p,-q}\rangle$, etc. In fact, the knowledge of $\langle a_1\overline{a}_1\rangle = \langle a_{p,q}\overline{a}_{p,q}\rangle$ is essentially enough for evaluating the spontaneous magnetization of the 2D Ising model. However, this needs some additional formal work since the relation of spins to fermions in the 2D Ising model is non-local. Spin-spin correlation functions like $\langle\sigma_{m,n}\sigma_{m',n'}\rangle$ for remote spins are in fact related to a string of fermionic correlations along some (arbitrary) path on the lattice connecting the points m, n and m', n'. The converse statement is also true: a fermionic correlator like $\langle a_{m,n}\overline{a}_{m',n'}\rangle$ in terms of spins is a non-local perturbation along a lattice path. The square of the spontaneous magnetization can be obtained as the limiting value of the spin-spin correlator $\langle\sigma(0)\sigma(R)\rangle = \langle\sigma_{m,n}\sigma_{m+R,n}\rangle$. This will be discussed in the next two sections.

15 The two-spin correlation function

The two-spin correlation function $\langle\sigma_{m,n}\sigma_{m',n'}\rangle$ is of interest in the theory of critical phenomena as well as because its limiting value as $R \to +\infty$, where $R^2 = (m-m')^2 + (n-n')^2$, equals the square of the magnetization.[13] In the following we will consider the correlator of two spins in the same row $\langle\sigma_{m,n}\sigma_{m+R,n}\rangle$ and obtain the limiting value using the Szegö-Kac theorem in the next section. The connection between spins and fermions is not local in general. However, a product of neighbouring spins like $\sigma_{m,n}\sigma_{m+1,n}$ can be related to $a_{m,n}\overline{a}_{m,n}$ since this product appears in the weights $1+$

[13] The magnetization in the 2D Ising model was first computed by C. N. Yang in Phys. Rev **85** (1952) 808–816. For the evaluation of $M^2 = \lim_{R\to+\infty}\langle\sigma_{m,n}\sigma_{m+R,n}\rangle$, see also E. W. Montroll, R. B. Potts & J. C. Ward, J. Math. Phys. **4** (1963) 308–322. In this paper they use dimer-like combinatorics to express $\langle\sigma_{m,n}\sigma_{m+R,n}\rangle$ as a ratio of two determinants. The result is a Toeplitz determinant of large size. They were the first to then apply the Szegö-Kac theorem to this Toeplitz determinant to evaluate M^2. Here we obtain this determinant using the fermionic representation.

$t^{(1)}_{m+1,n}\sigma_{m,n}\sigma_{m+1,n}$ and one can therefore obtain the expectation of this product by differentiating the partition function with respect to $t^{(1)}_{m,n}$. Using the fact that $\sigma^2_{m,n} = 1$, we can then write

$$\langle\sigma_{m,n}\sigma_{m+R,n}\rangle = \langle\prod_{k=1}^{R}\sigma_{m+k-1,n}\sigma_{m+k,n}\rangle. \tag{15.1}$$

Since $\sigma_{m,n}\sigma_{m+1,n}$ is related to fermion binaries by means of differentiation with respect to $t^{(1)}_{m+1,n}$, it follows that $\langle\sigma_{m,n}\sigma_{m+R,n}\rangle$ is given by a string of fermion perturbations along a lattice path. Actually, differentiation with respect to a string of parameters $t^{(1)}$ in the action of (8.16) introduces too many fermionic terms, and in practice it turns out to be easier to relate $\sigma_{m,n}\sigma_{m+1,n}$ directly to $a_{m,n}\overline{a}_{m,n}$. Indeed, at the first stages of the fermionization we have applied the factorization

$$\begin{aligned} 1 + t^{(1)}_{m+1,n}\sigma_{m,n}\sigma_{m+1,n} = \\ = \int d\overline{a}_{m,n}da_{m,n}\, e^{a_{m,n}\overline{a}_{m,n}}(1 + a_{m,n}\sigma_{m,n})(1 + t^{(1)}_{m+1,n}\overline{a}_{m,n}\sigma_{m+1,n}). \end{aligned} \tag{15.2}$$

Now, introducing the factor $a_{m,n}\overline{a}_{m,n}$ into the integral on the right-hand side, it annihilates all fermion terms, and the result will be 1. Thus we have

$$1 = \int d\overline{a}_{m,n}da_{m,n}\, e^{a_{m,n}\overline{a}_{m,n}}a_{m,n}\overline{a}_{m,n}(1 + a_{m,n}\sigma_{m,n})(1 + t^{(1)}_{m+1,n}\overline{a}_{m,n}\sigma_{m+1,n}). \tag{15.3}$$

The presence of the factor $a_{m,n}\overline{a}_{m,n}$ by itself is not an obstacle to performing fermionization in the usual way, and the result will be an integral like (8.16) but with a prefactor $a_{m,n}\overline{a}_{m,n}$. This integral corresponds to a perturbed partition function Q' in which the weight $1 + t^{(1)}_{m+1,n}\sigma_{m,n}\sigma_{m+1,n}$ is eliminated (replaced by 1). We can recover this weight in the density matrix by means of the identity

$$\frac{1}{1 + t\sigma\sigma'}(1 + t\sigma\sigma') = 1.$$

Therefore, we have a correspondence between $a_{m,n}\overline{a}_{m,n}$ and $\sigma_{m,n}\sigma_{m+1,n}$ or rather

$$\langle a_{m,n}\overline{a}_{m,n}\rangle = \langle\frac{1}{1 + t^{(1)}_{m+1,n}\sigma_{m,n}\sigma_{m+1,n}}\rangle, \tag{15.4}$$

where the average is taken over fermions on the left-hand side and over spins on the right-hand side. Using the identity

$$\frac{1}{1+t^{(1)}_{m+1,n}\sigma_{m,n}\sigma_{m+1,n}} = \frac{1-t^{(1)}_{m+1,n}\sigma_{m,n}\sigma_{m+1,n}}{1-(t^{(1)}_{m+1,n})^2} \tag{15.5}$$

we can write, in the homogeneous case,

$$\begin{aligned}\langle\sigma_{m,n}\sigma_{m+1,n}\rangle &= \frac{1}{t_1}\langle 1-(1-t_1^2)a_{m,n}\overline{a}_{m,n}\rangle \\ &= \frac{1}{t_1}\langle\exp\left[-(1-t_1^2)a_{m,n}\overline{a}_{m,n}\right]\rangle\end{aligned} \tag{15.6}$$

and therefore

$$\begin{aligned}\langle\sigma_{m,n}\sigma_{m+R,n}\rangle &= \langle\prod_{m'=m}^{m+R-1}\sigma_{m',n}\sigma_{m'+1,n}\rangle \\ &= \frac{1}{t_1^R}\langle\prod_{m'=m}^{m+R-1}\left[1-(1-t_1^2)a_{m',n}\overline{a}_{m',n}\right]\rangle \\ &= \det\left(\frac{1}{t_1}\left(\delta_{m',m''}-(1-t_1^2)\langle a_{m',n}\overline{a}_{m'',n}\rangle\right)\right).\end{aligned} \tag{15.7}$$

That the average of the exponential perturbation string is the determinant of an $R\times R$ matrix with elements $\frac{1}{t_1}\left(\delta_{m',m''}-(1-t_1^2)\langle a_{m',n}\overline{a}_{m'',n}\rangle\right)$ can also be shown directly in terms of general Gaussian integrals, while a more understandable way is to apply Wick's theorem, rewriting the exponential as a product of linear fermionic forms. For each factor, we introduce fermionic Grassmann variables $c_{m,n}, \overline{c}_{m,n}$ and write, with $\lambda_{m,n}=1-(t^{(1)}_{m+1,n})^2$,

$$\begin{aligned}1+\lambda_{m,n}a_{m,n}\overline{a}_{m,n} &= \int d\overline{c}_{m,n}dc_{m,n}\,e^{c_{m,n}\overline{c}_{m,n}} \\ &\qquad\times(c_{m,n}+a_{m,n})(\overline{c}_{m,n}+\lambda_{m,n}\overline{a}_{m,n}) \\ &= \langle L_{m,n}\overline{L}_{m,n}\rangle\end{aligned} \tag{15.8}$$

with

$$L_{m,n}=c_{m,n}+a_{m,n}, \quad \overline{L}_{m,n}=\overline{c}_{m,n}+\lambda_{m,n}\overline{a}_{m,n}. \tag{15.9}$$

Writing

$$\begin{aligned}A_{m,n} &= L_{m,n}=c_{m,n}+a_{m,n} \text{ and} \\ \overline{A}_{m,n} &= t_1^{-1}\overline{L}_{m,n}=t_1^{-1}\left(\overline{c}_{m,n}+(1-t_1^2)\,\overline{a}_{m,n}\right),\end{aligned} \tag{15.10}$$

it follows that

$$
\begin{aligned}
&\langle\prod_{k=0}^{R-1} t_1^{-1}\left(1-(1-t_1^2)a_{m+k,n}\overline{a}_{m+k,n}\right)\rangle_{(a)} = \\
&= \langle\prod_{k=0}^{R-1}(A_{m+k,n}\overline{A}_{m+k,n})\rangle_{(a,c)} \\
&= \det\left[A_{kl}\equiv\langle A_{m+k,n}\overline{A}_{m+l,n}\rangle\right] \\
&= \det\left[\frac{1}{t_1}\left(\langle c_{m+k,n}\overline{c}_{m+l,n}\rangle+(1-t_1^2)\langle a_{m+k,n}\overline{a}_{m+l,n}\rangle\right)\right] \\
&= \det\left[\frac{1}{t_1}\left(\delta_{k,l}+(1-t_1^2)\langle a_{m+k,n}\overline{a}_{m+l,n}\rangle\right)\right]
\end{aligned}
\tag{15.11}
$$

as was stated already in (15.7). In this derivation we have also made use of the facts that $\langle a_{m,n}a_{m',n}\rangle = 0$ and $\langle\overline{a}_{m,n}\overline{a}_{m'',n}\rangle = 0$. This causes the result to be a determinant rather than a Pfaffian, as it would be in more general situations. The identities $\langle a_{m,n}a_{m',n}\rangle = 0$ and $\langle\overline{a}_{m,n}\overline{a}_{m'',n}\rangle = 0$ follow by direct calculation, but this property probably only holds for the corresponding Green's functions in the case of variables on the same row: $n' = n$. For example, since

$$\langle a_{p,q}a_{-p,-q}\rangle = \langle a_{p,q}a_{L-p,L-q}\rangle = \frac{2it_1^2\sin(q)}{\Delta_{p,q}},$$

we have that

$$\langle a_{m,n}a_{m',n}\rangle = \int_0^{2\pi}\int_0^{2\pi}\frac{dp\,dq}{(2\pi)^2}\frac{2it_1^2\sin(q)}{A_0-A_1\cos(p)-A_2\cos(q)} = 0$$

by symmetry.

We now have to properly determine the matrix elements $\langle A_{m,n}\overline{A}_{m',n}\rangle$. By equation (14.2) we have

$$
\begin{aligned}
\langle A_{m,n}\overline{A}_{m,n}\rangle &= \frac{t_1}{\delta_{\ m,m'}} - \frac{1-t_1^2}{t_1}\langle a_{m,n}\overline{a}_{m',n}\rangle \\
&= \frac{1}{t_1}\int_0^{2\pi}\int_0^{2\pi}\frac{dp\,dq}{(2\pi)^2}e^{ip(m-m')}[1-(1-t_1^2)\langle a_{p,q}\overline{a}_{p,q}\rangle],
\end{aligned}
\tag{15.12}
$$

and substituting for $\langle a_{p,q}\overline{a}_{p,q}\rangle$ from (14.2),

$$\begin{aligned}\langle A_{p,q}\overline{A}_{p,q}\rangle &= \frac{1}{t_1} - \frac{1-t_1^2}{t_1}\langle a_{p,q}\overline{a}_{p,q}\rangle \\ &= \frac{1}{t_1} - \frac{(1-t_1^2)[(1+t_2^2-2t_2\cos(q)) - t_1(1-t_2^2)e^{-ip}]}{t_1[\Delta_{p,q} = A_0 - A_1\cos(p) - A_2\cos(q)]},\end{aligned} \tag{15.13}$$

where

$$\begin{aligned}A_0 &= (1+t_1^2)(1+t_2^2), \\ A_1 &= 2t_1(1-t_2^2), \\ A_2 &= 2t_2(1-t_1^2).\end{aligned} \tag{15.14}$$

Further algebraic manipulation leads to

$$\langle A_{p,q}\overline{A}_{p,q}\rangle = \frac{-(1-t_1^2)[(1+t_2^2) - t_1(1-t_2^2)e^{-ip}] + D}{t_1\Delta_{p,q}}, \tag{15.15}$$

where $D = A_0 - A_1\cos(p)$. Note that in the numerator, the momentum variable q has disappeared[14], so that we can integrate over the variable q without problems, leading to

$$\begin{aligned}\int_0^{2\pi}\frac{dq}{2\pi}\frac{1}{\Delta_{p,q}} &= \int_0^{2\pi}\frac{1}{A_0 - A_1\cos(p) - A_2\cos(q)} \\ &= \frac{1}{\sqrt{(A_0 - A_1\cos(p))^2 - A_2^2}} \\ &= \frac{1}{\sqrt{(A_0 - A_1\cos(p) - A_2)(A_0 - A_1\cos(p) + A_2)}} \\ &= \frac{1}{\sqrt{\Delta_{p,0}\Delta_{p,\pi}}} = \frac{1}{|\alpha_{p,0}\alpha_{p,\pi}|},\end{aligned} \tag{15.16}$$

[14]The variable q is still present in

$$\langle a_{p,q}\overline{a}_{p,q}\rangle = \frac{|1-t_2e^{-iq}|^2 - t_1(1-t_2^2)e^{-ip}}{A_0 - A_1\cos(p) - A_2\cos(q)}$$

where[15]

$$\begin{aligned} |\alpha_{p,0}| &= |1 - t_1 e^{ip} - t_2 - t_1 t_2 e^{ip}|, \\ |\alpha_{p,\pi}| &= |1 - t_1 e^{ip} + t_2 + t_1 t_2 e^{ip}|. \end{aligned} \tag{15.17}$$

Remarkably, the same expressions $\alpha_{p,0}$ and $\alpha_{p,\pi}$ appear also as factors in the numerator of (15.15). Indeed,

$$\begin{aligned} &\frac{1}{t_1}\left(-(1-t_1^2)[(1+t_2^2) - t_1(1-t_2^2)e^{-ip}] + D\right) = \\ &= 2t_1(1+t_2^2) - t_1^2(1-t_2^2)e^{-ip} - (1-t_2^2)e^{ip} \\ &= -e^{ip}(1 - t_1e^{-ip} - t_2 - t_1t_2e^{-ip})(1 - t_1e^{-ip} + t_2 + t_1t_2e^{-ip}) \\ &= -e^{ip}(1 - t_2 - t_1(1+t_2)e^{-ip})(1 + t_2 - t_1(1-t_2)e^{-ip}) \\ &= -e^{ip}(1+t_2)^2\left(\frac{1-t_2}{1+t_2} - t_1e^{-ip}\right)\left(1 - t_1\frac{1-t_2}{1+t_2}e^{-ip}\right) \\ &= -e^{ip}(1+t_2)^2\left(t_2^* - t_1e^{-ip}\right)\left(1 - t_1t_2^*e^{-ip}\right) \\ &= -(1+t_2)^2(t_2^*e^{ip} - t_1)(1 - t_1t_2^*e^{-ip}) \\ &= t_1(1+t_2)^2\left(1 - \frac{t_2^*}{t_1}e^{ip}\right)(1 - t_1t_2^*e^{-ip}), \end{aligned} \tag{15.18}$$

where t_2^* is the Kramers-Wannier conjugation of t_2 given by

$$t_1^* = \frac{1-t_1}{1+t_1} = e^{-2b_1} \text{ and } t_2^* = \frac{1-t_2}{1+t_2} = e^{-2b_2}. \tag{15.19}$$

(Note that this transformation is self-dual.)

Using (15.16), we now arrive at the following representation

$$\int_0^{2\pi} \frac{dq}{2\pi}\langle A_{p,q}\overline{A}_{p,q}\rangle = \frac{(1 - t_1t_2^*e^{-ip})(1 - \frac{t_2^*}{t_1}e^{ip})}{\left|(1 - t_1t_2^*e^{-ip})(1 - \frac{t_2^*}{t_1}e^{ip})\right|}. \tag{15.20}$$

[15]Note that another representation for $\Delta_{p,q}$ is

$$\Delta_{p,q} = |1 - t_1e^{ip} - t_2e^{iq} - t_1t_2e^{i(p+q)}|^2 - 4t_1t_2\sin(p)\sin(q)$$

and in particular,

$$\Delta_{p,0} = |1 - t_1e^{ip} - t_2 - t_1t_2e^{ip}|^2$$

and

$$\Delta_{p,\pi} = |1 - t_1e^{ip} + t_2 + t_1t_2e^{ip}|^2.$$

Hence, the spin-spin correlation function $\langle\sigma_{m,n}\sigma_{m+R,n}\rangle$ is given by the determinant of a matrix with elements $\langle A_{m',n}\overline{A}_{m'',n}\rangle$ $(m, m' = 1\ldots, R)$, which are essentially fermionic Green's functions related to $\langle a_{m',n}\overline{a}_{m'',n}\rangle$:

$$\begin{aligned}
\langle A_{m,n}\overline{A}_{m',n}\rangle &= \int_0^{2\pi}\frac{dp}{2\pi}e^{ip(m-m')}\frac{(1-t_1t_2^*e^{-ip})(1-\frac{t_2^*}{t_1}e^{ip})}{\left|(1-t_1t_2^*e^{-ip})(1-\frac{t_2^*}{t_1}e^{ip})\right|}\\
&= \int_0^{2\pi}\frac{dp}{2\pi}e^{ip(m-m')}\left[\frac{(1-t_1t_2^*e^{-ip})(1-\frac{t_2^*}{t_1}e^{ip})}{(1-t_1t_2^*e^{ip})(1-\frac{t_2^*}{t_1}e^{-ip})}\right]^{1/2}\\
&= \int_0^{2\pi}\frac{dp}{2\pi}e^{-ip(m-m')}\left[\frac{(1-t_1t_2^*e^{ip})(1-\frac{t_2^*}{t_1}e^{-ip})}{(1-t_1t_2^*e^{-ip})(1-\frac{t_2^*}{t_1}e^{ip})}\right]^{1/2} \qquad (15.21)
\end{aligned}$$

The spin-spin correlator is thus

$$\langle\sigma_{m,n}\sigma_{m+R,n}\rangle = \det\left[\left(A_{m',n}\overline{A}_{m'',n}\right)_{m',m''=1}^{R}\right]. \qquad (15.22)$$

(Note that $C(m-m') = \langle A_{m,n}\overline{A}_{m',n}\rangle$ only depends on the difference $m-m'$ as a result of the translation-invariance of the model. It is therefore a Toeplitz determinant. The expression (15.21) is in agreement with the results in the literature: see Montroll, Potts and Ward (1963) and the book by McCoy and Wu (1973). In the next section we compute the square M^2 of the spontaneous magnetization as a limit of the correlator as $R \to +\infty$.

16 The spontaneous magnetization of the 2D Ising model

The result (11.17) for the spontaneous magnetization of the 2D Ising model was apparently known to Onsager, who reported it at a few conferences and seminars (for the symmetric case $t_1 = t_2$), though without proof. The first published derivation is due to C. N. Yang (1952)[16]. It is based on the algebraic transfer matrix approach and is mathematically very complicated as Yang himself in fact remarks in a comment at the end of his paper. Another

[16]Phys. Rev **85** (1952) 808–816.

(combinatorial) proof was given by Montroll, Potts and Ward[17] in which the problem is reduced to the evaluation of the limiting value of a Toeplitz determinant by means of the Szegö-Kac theorem. Here we evaluate the limiting value in the special case of horizontally separated spins: $M^2 = \lim_{R\to+\infty}\langle\sigma_{m,n}\sigma_{m+R,n}\rangle$.

We consider the matrix elements in the form of an integral as in the last line of (15.21). The magnetization is then given by

$$\begin{aligned} M^2 &= \lim_{R\to+\infty}\langle\sigma_{m,n}\sigma_{m+R,n}\rangle \\ &= \lim_{R\to+\infty}\det\left[\langle A_{m',n}\overline{A}_{m'',n}\rangle\right] \\ &= \lim_{R\to+\infty}\det[C(m'-m'')], \end{aligned} \tag{16.1}$$

where C is the Toeplitz matrix with elements $C(r)$ $(r = m' - m'')$ given by

$$\begin{aligned} C(r) &= \int_0^{2\pi}\frac{dp}{2\pi}e^{-ipr}\left[\frac{(1-\lambda_1 e^{ip})(1-\lambda_2 e^{-ip})}{(1-\lambda_1 e^{-ip})(1-\lambda_2 e^{ip})}\right]^{1/2} \\ &= \int_0^{2\pi}\frac{dp}{2\pi}e^{-ipr}C(p) \end{aligned} \tag{16.2}$$

where $\lambda_1 = t_1 t_2^*$ and $\lambda_2 = t_2^*/t_1$.

According to the Szegö-Kac theorem, we first have to evaluate the Fourier transform of $\log C(p)$, i.e.

$$g(n) = \int_0^{2\pi}\frac{dp}{2\pi}\,e^{-ipn}\log C(p). \tag{16.3}$$

Then the Szegö-Kac theorem states[18] that (with C given by (16.2))

$$M^2 = \lim_{R\to+\infty}\det[C(m'-m'')] = \exp\left\{\sum_{n=1}^{\infty} n\,g(n)\,g(-n)\right\}. \tag{16.4}$$

Now, $\log C(p)$ is given by

$$\log C(p) = \log\left[\frac{(1-\lambda_1 e^{ip})(1-\lambda_2 e^{-ip})}{(1-\lambda_1 e^{-ip})(1-\lambda_2 e^{ip})}\right]^{1/2}, \tag{16.5}$$

[17] E. W. Montroll, R. B. Potts & J. C. Ward, J. Math. Phys. **4** (1963) 308–322.

[18] As regards the application to the 2D Ising model, we we must assume that $T \le T_c$, i.e. $\lambda_2 \le 1$

where

$$\lambda_1 = t_1 t_2^* = t_1 \frac{1-t_2}{1+t_2}, \qquad \lambda_2 = \frac{t_2^*}{t_1} = \frac{1-t_2}{t_1(1+t_2)}, \tag{16.6}$$

and the elements $g(n)$ can be computed by series expansion of the logarithms and integrating term by term. For example, using

$$\log \frac{1}{1-\lambda_1 e^{ip}} = \sum_{n'=1}^{\infty} \frac{(\lambda_1 e^{ip})^{n'}}{n'},$$

we have

$$-\int_0^{2\pi} \frac{dp}{2\pi} \log(1-\lambda_1 e^{ip}) = \int_0^{2\pi} \frac{dp}{2\pi} \sum_{n'=1}^{\infty} \frac{\lambda_1^{n'}}{n'} e^{in'p} = \frac{\lambda_1^n}{n},$$

and in general,

$$g(n) = \int_0^{2\pi} \frac{dp}{2\pi} e^{-ipn} \log C(p) = \frac{\lambda_1^n - \lambda_2^n}{2n}, \tag{16.7}$$

$$g(-n) = -g(n) \tag{16.8}$$

and

$$b\, g(n)\, g(-n) = -\frac{n}{4n^2}(\lambda_1^n - \lambda_2^n)^2 = \frac{-\lambda_1^{2n} - \lambda_2^{2n} + 2\lambda_1^n \lambda_2^n}{4n}. \tag{16.9}$$

Hence,

$$\begin{aligned} \log M^2 &= \sum_{n=1}^{\infty} \frac{-\lambda_1^{2n} - \lambda_2^{2n} + 2\lambda_1^n \lambda_2^n}{4n} \\ &= \frac{1}{4}\log(1-\lambda_1^2) + \frac{1}{4}\log(1-\lambda_2^2) - \frac{1}{2}\log(1-\lambda_1\lambda_2) \\ &= \frac{1}{4}\log\left[\frac{(1-\lambda_1^2)(1-\lambda_2^2)}{(1-\lambda_1\lambda_2)^2}\right]. \end{aligned} \tag{16.10}$$

The spontaneous magnetization of the 2D Ising model is therefore

$$M = \left[\frac{(1-\lambda_1^2)(1-\lambda_2^2)}{(1-\lambda_1\lambda_2)^2}\right]^{1/8}. \tag{16.11}$$

Expressing this in terms of the variables t_1 and t_2, we have

$$M = \left[1 - \frac{(1-t_1^2)^2(1-t_2^2)^2}{(4t_1t_2)^2}\right]^{1/8}, \tag{16.12}$$

or, equivalently, in terms of $b_1 = \beta J_1$ and $b_2 = \beta J_2$,

$$M = \left[1 - \frac{1}{\sinh^2(2b_1)\,\sinh^2(2b_2)} \right]^{1/8} . \tag{16.13}$$

This holds in the ordered phase where $1 - t_1 - t_2 - t_1 t_2 < 0$ (with $t_1, t_2 > 0$). In the disordered phase, $M = 0$.

Other correlation functions can also be expressed in terms of determinants of large size (of order R, where R is the lattice distance between spins). Their structure is in general more complicated than that of the correlator considered here in detail. However, significant simplifications occur for the diagonal correlator $\langle \sigma_{m,n} \sigma_{m+R,n+R} \rangle$ at $T = T_c$. For this correlator one finds that at T_c, it behaves like $\sim 1/R^4$ (Onsager and Kaufman, 1949).

17 Concluding comments

In the above discussion, the 2-dimensional Ising model has been analysed as a theory of free fermions on a lattice. Anti-commuting (Grassmann) variables were used. For the most general, inhomogeneous set of coupling parameters $t^{(1)}_{m,n}$ and $t^{(2)}_{m,n}$ on the lattice, the partition function Q was presented as a fermionic Gaussian integral (8.16) in the case of free boundary conditions.[19] A Gaussian fermionic representation also exists for the 2D Ising model on a finite periodic lattice (lattice wrapped around a torus). In this case the spin-variable partition function is expressed as a superposition of four fermionic Gaussian integrals with different periodic-aperiodic boundary conditions for the fermions. The fermionization procedure is based on a mirror-ordered factorization of the local Boltzmann weights as discussed in §13. In subsequent sections we discussed the momentum-space analysis and the exact

[19] A fermionic representation of the 2D Ising model with two Grassmann variables per site rather than the four variables $a_{m,n}$, $b_{m,n}$, $\overline{a}_{m,n}$ and $\overline{b}_{m,n}$ in equation (8.16) first appeared in the context of a spin-polynomial interpretation of the 2D Ising model on triangular-like lattices in V. N. Plechko, Physica A**152** (1988), 51, and more recently was discussed in Phys. Lett A**239** (1998), 289 for the inhomogeneous rectangular lattice in the context of a disordered Ising model (site and bond dilution). In the latter reference, the Majorana-Dirac interpretation is also presented.

solution for the homogeneous 2D Ising model on the standard rectangular lattice, the evaluation of fermionic Green's functions and the determinantal representation for binary spin correlation functions $\langle \sigma_{m,n} \sigma_{m+R,n} \rangle$. The squared spontaneous magnetization M^2 was then obtained by means of the Szegö-Kac theorem. Other subjects of interest would be a Gaussian fermionic integral representation for Q with two Grassmann variables per site, which admits in particular the field-theoretical Majorana-Dirac interpretation of the 2D Ising model, already at the level of a lattice, and the consideration of the continuum limit of the 2D Ising model near the critical point ($T \sim T_c$) in the region of low momenta.

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