Bootstrapping Matrix Quantum Mechanics

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June 2020

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arXiv: 2004.10212 [hep-th]

Matrix Quantum Mechanics

- It is clear at this point that space is emergent.
- Quantum mechanical theories of large N matrices can produce semiclassical, gravitating spacetime.
- We don't really understand how the matrix degrees of freedom re-arrange themselves into spacetime.
- In Matrix Quantum Mechanics there is no field-theoretic locality built in. All of space must emerge.
- Want to solve Matrix Quantum Mechanics theories.

Matrix Quantum Mechanics

- Simplest theory is of a single matrix. Solved by mapping eigenvalues to free fermions. Key part of two dimensional string theory in early 90's.

- Richest theories are BFSS ['96] and BMN ['02]. Many matrices. Maximally supersymmetric.

- Many theories in between. Can vary number of matrices and degree of supersymmetry.

- Theories with more than one matrix not solved.

Matrix Quantum Mechanics

- Our work is inspired by a beautiful paper of [Henry Lin arXiv:2002.08387]

- That paper solved large N matrix integrals using positivity constraints. We will use the same philosophy for large N matrix quantum mechanics.

- Strategy: Relate expectation values of simple operators to those of long ones. Positivity constraints on the long operators strongly constrain the simple ones.

- Obtain new results for two matrix quantum mechanics.

Plan

- Warm-up: The anharmonic oscillator revisited.
- One matrix quantum mechanics.
- Two matrix quantum mechanics.

The anharmonic oscillator

$$H = p^2 + x^2 + gx^4$$

Step 1. Recurrence relation between expectation values

$$\langle [H, \mathcal{O}] \rangle = 0$$ with $$\mathcal{O} = x^s\,, \quad \mathcal{O} = x^t p$$

[Commute operators, eliminate p^2 in terms of energy E]

$$4tE\langle x^{t-1}\rangle + t(t-1)(t-2)\langle x^{t-3}\rangle - 4(t+1)\langle x^{t+1}\rangle - 4g(t+2)\langle x^{t+3}\rangle = 0$$

Obtain all expectation values $<x^{2t}>$ from E and $<x^2>$.

The anharmonic oscillator

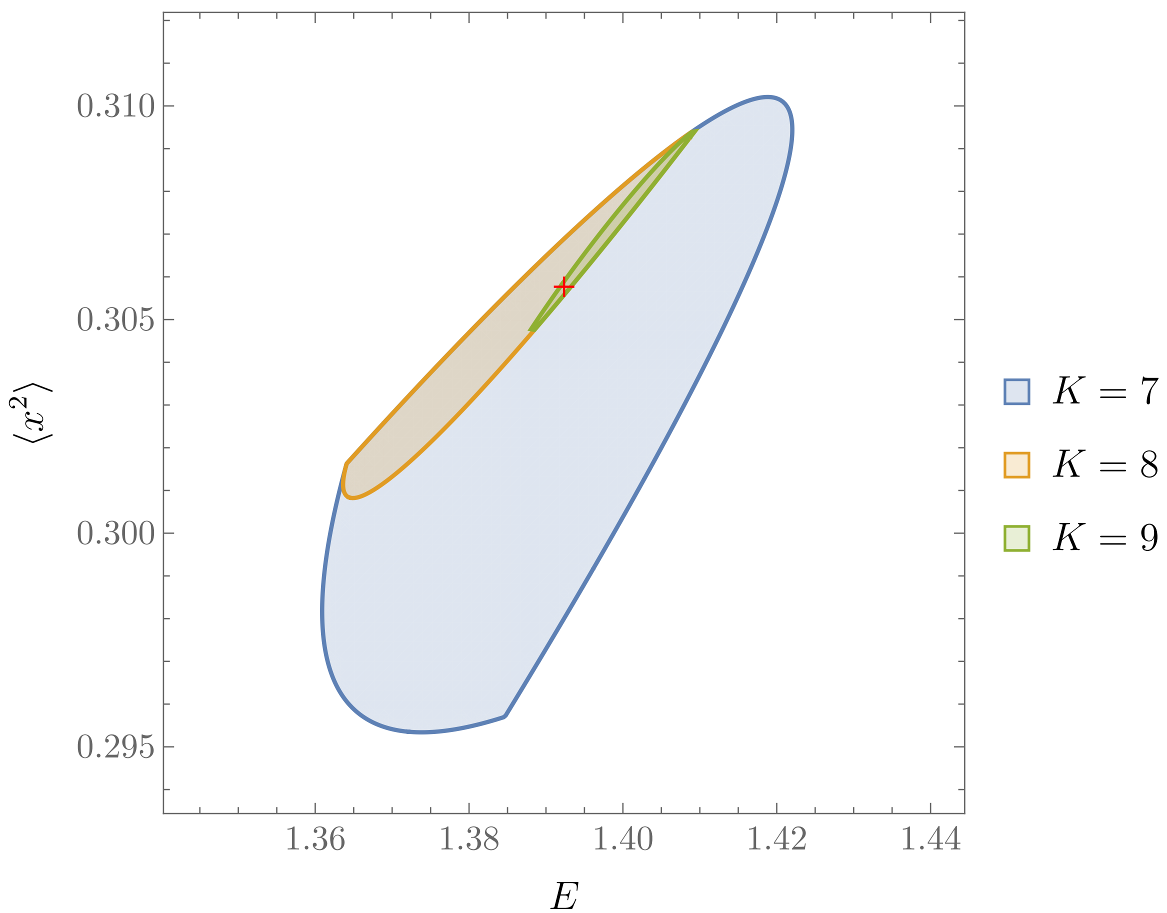
Step 2. Impose positivity constraint: [cf. Lin '20]

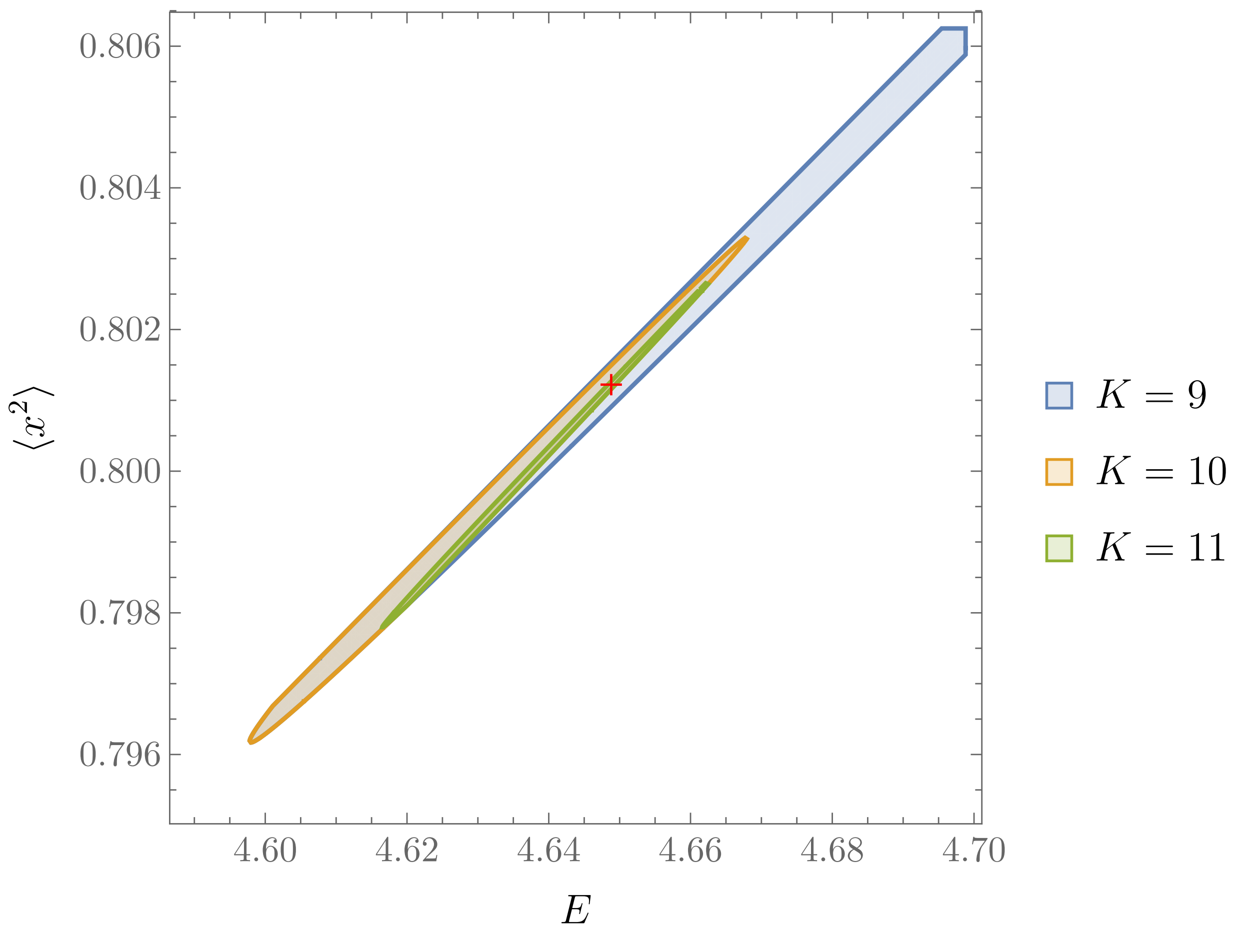
$$\langle \mathcal{O}^\dagger \mathcal{O} \rangle \geq 0\,, \quad \forall \mathcal{O} = \sum_{i=0}^{K} c_i x^i\,,$$

Implies that the (K+1)x(K+1) matrix $\mathcal{M}_{ij} = \langle x^{i+j} \rangle$ must be positive semidefinite.

Fix K. Scan over *E* and $\langle x^2 \rangle$. Compute the eigenvalues of *M* and thereby see if values are excluded by positivity…

The anharmonic oscillator





One matrix theory

$$H = \mathrm{tr}P^2 + \mathrm{tr}X^2 + \frac{g}{N}\mathrm{tr}X^4$$

Step 1. Relations between expectation values:

$$\langle [H, \mathcal{O}] \rangle = 0, \quad \forall \mathcal{O}.$$

eg. $\langle [H, \mathrm{tr}XP] \rangle = 0 \quad \Rightarrow \quad 2\langle \mathrm{tr}P^2 \rangle = 2\langle \mathrm{tr}X^2 \rangle + \frac{4g}{N}\langle \mathrm{tr}X^4 \rangle$

$$\langle \mathrm{tr}\, G\mathcal{O} \rangle = 0\,, \quad \forall \mathcal{O}\,.$$

$$G = i[X, P] + NI$$

eg. $\langle \mathrm{tr}\, G \rangle = 0 \quad \text{and} \quad \langle [H, \mathrm{tr}\, X^2] \rangle = 0 \quad \Rightarrow$

$$\langle \mathrm{tr}XP \rangle = -\langle \mathrm{tr}PX \rangle = \frac{iN^2}{2}$$

One matrix theory

Step 2. Take a selection of operators and write down a matrix that must be non-negative.

eg.

	I	X^2	X	P
I	$\langle \mathrm{tr}\, I \rangle$	$\langle \mathrm{tr}\, X^2 \rangle$	0	0
X^2	$\langle \mathrm{tr}\, X^2 \rangle$	$\langle \mathrm{tr}\, X^4 \rangle$	0	0
X	0	0	$\langle \mathrm{tr}\, X^2 \rangle$	$\langle \mathrm{tr}\, XP \rangle$
P	0	0	$\langle \mathrm{tr}\, PX \rangle$	$\langle \mathrm{tr}\, P^2 \rangle$

We already established relationships between some of these quantities. Positivity therefore implies constraints such as:

$$\langle \mathrm{tr}\, X^2 \rangle \left(\langle \mathrm{tr}\, X^2 \rangle + \frac{2g}{N} \langle \mathrm{tr}\, X^4 \rangle \right) \geq \frac{N^4}{4}$$

One matrix theory

Take all strings of X's and P's of length ≤ L. There are 2^L such strings. These give a matrix with 2^{2L} entries.

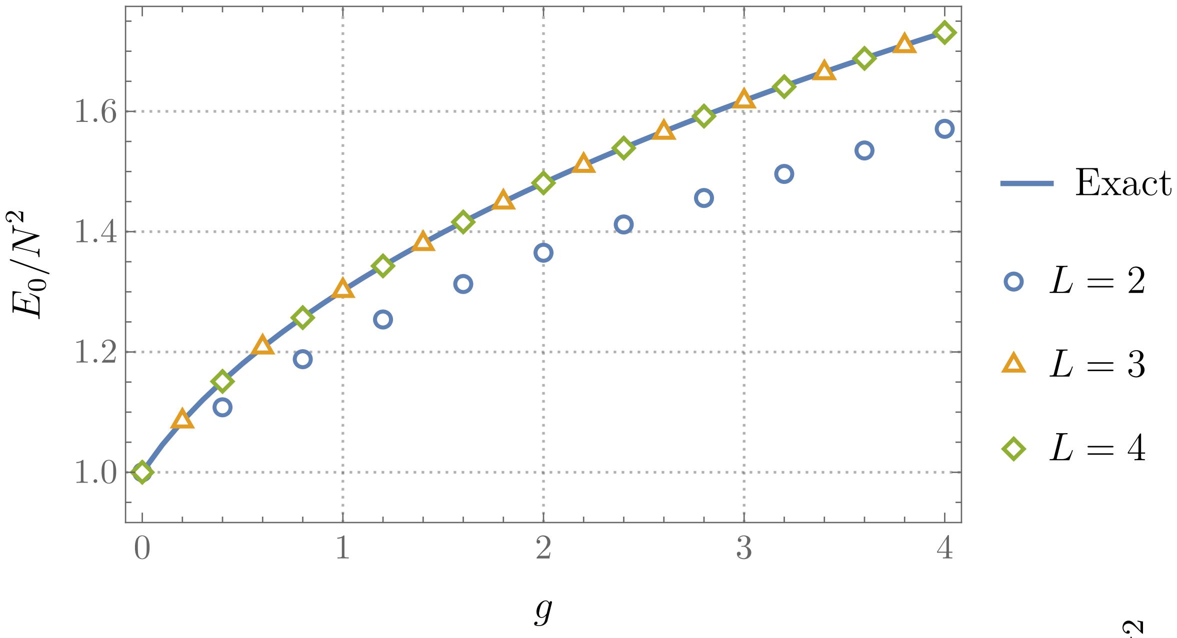
Write down all relationships of the types discussed between these strings. Furthermore consider large N cyclicity:

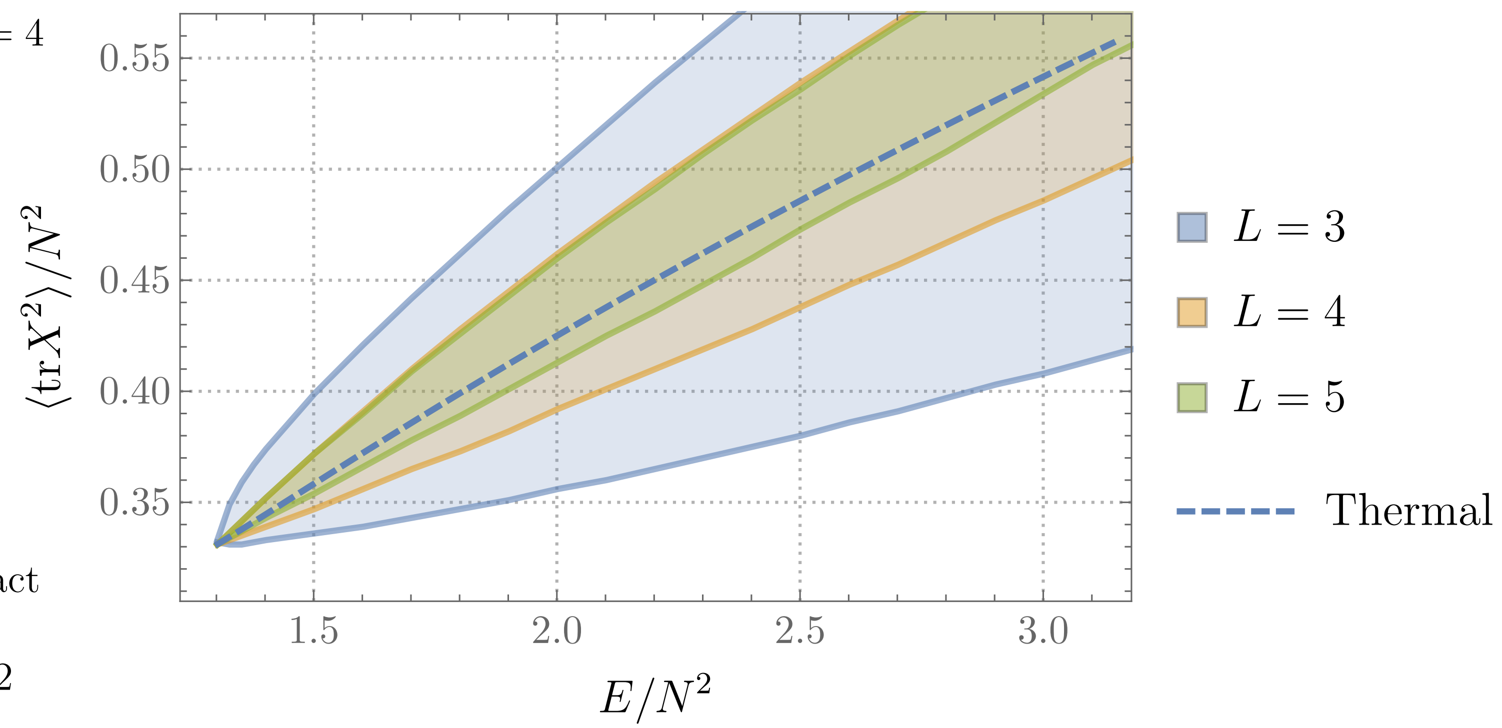
$$\langle \mathrm{tr}\, X P^3 \rangle = \langle \mathrm{tr}\, P^3 X \rangle + 2iN \langle \mathrm{tr}\, P^2 \rangle + i \langle \mathrm{tr}\, P \rangle \langle \mathrm{tr}\, P \rangle$$

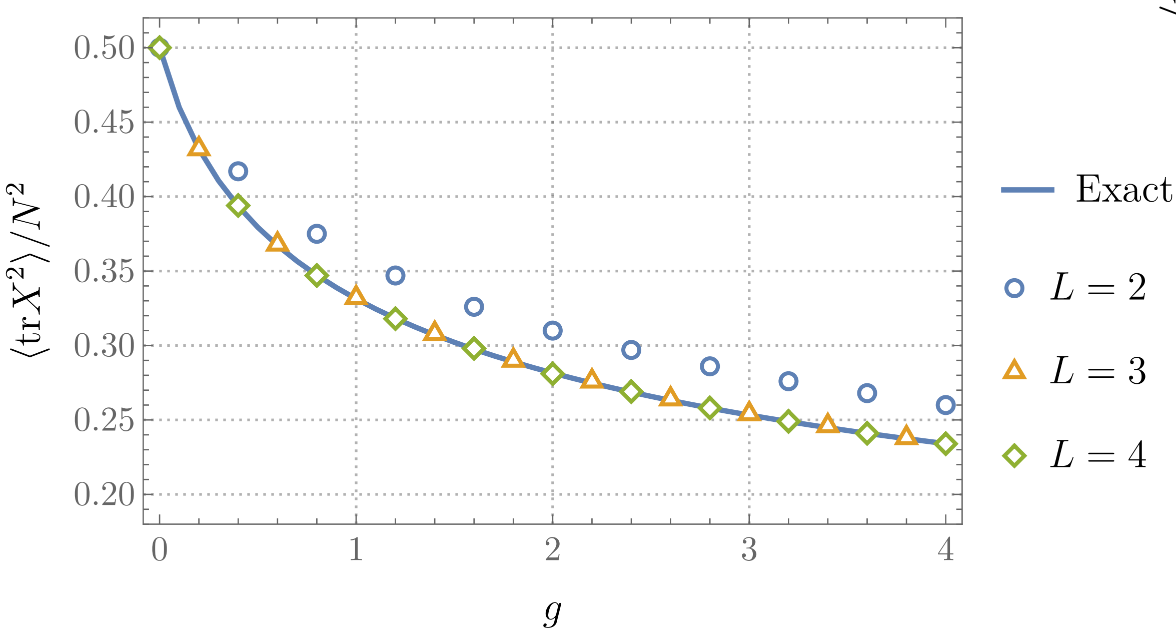
Continuum of energies allowed by the positivity constraints.

Lowest allowed such energy will approximate ground state. Do gradient descent of the energy within space allowed by operator and positivity constraints.

One matrix theory







Two matrix theory

$$H = \mathrm{tr}\left(P_X^2 + P_Y^2 + m^2(X^2+Y^2) - g^2[X,Y]^2\right)$$

The strategy is the same as for the one-matrix case, except that now at length L the matrix has 4^{2L} entries.

We imposed rotational invariance to reduce the number of independent variables:

$$\langle [S, \mathcal{O}] \rangle = 0\,, \quad S = \mathrm{tr}\left(X P_Y - Y P_X\right)\,.$$

Two matrix theory

The two matrix theory is not (known to be) soluble.

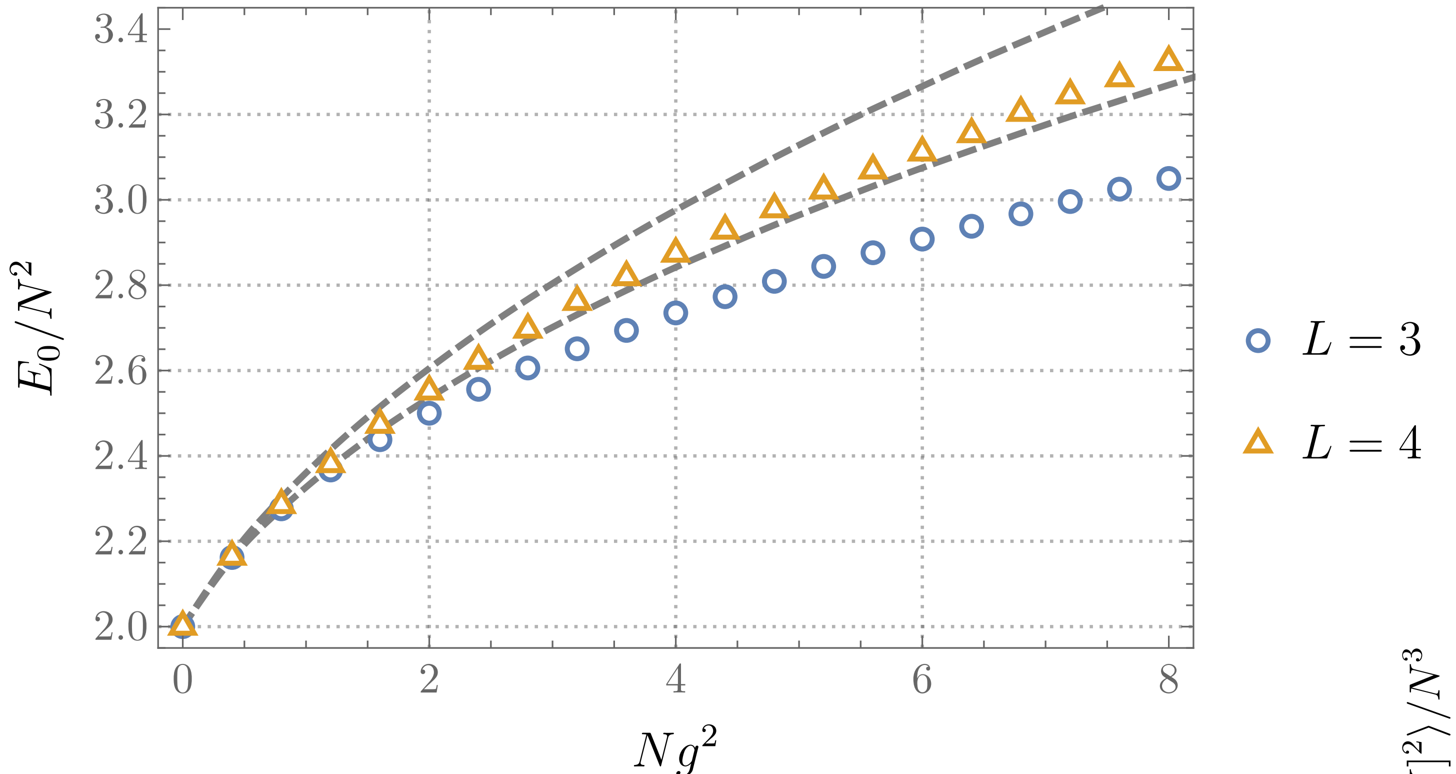
Check numerical results using a Born-Oppenheimer wavefunction. One matrix is diagonalized and the other placed in its "instantaneous" ground state:

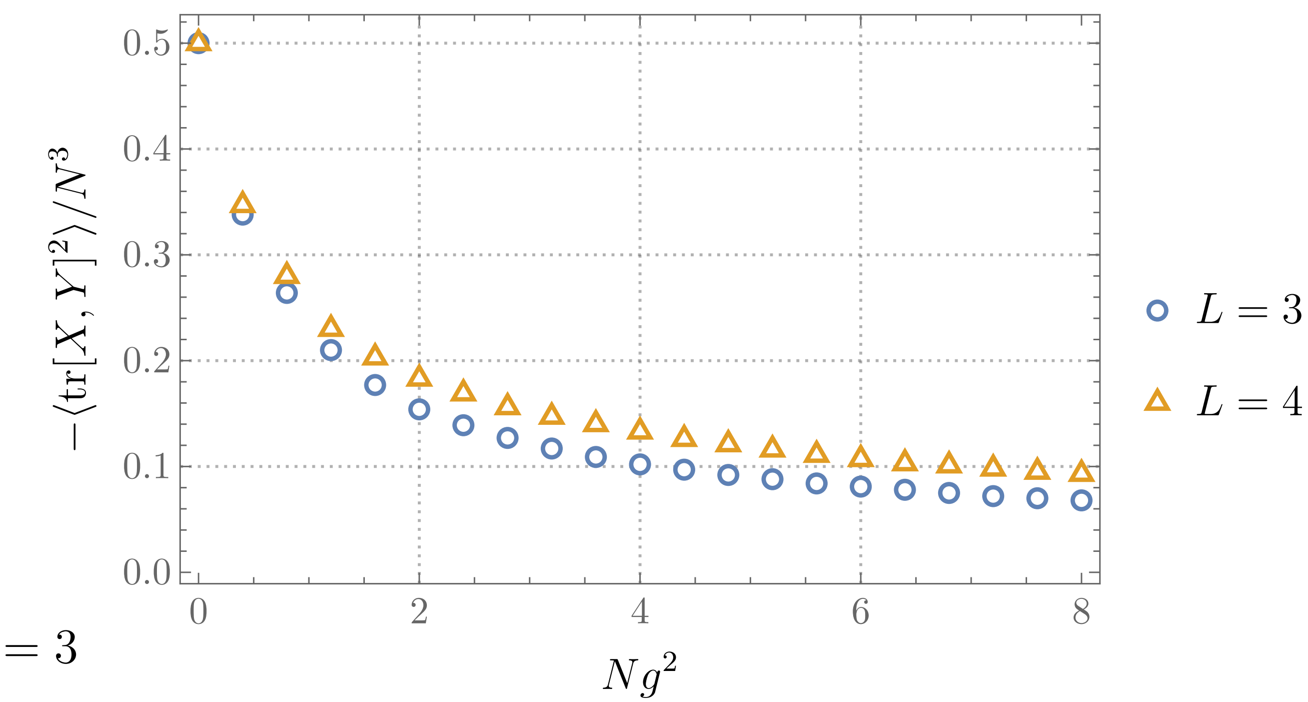
$$\Psi(X,Y) = \psi(x_i) \prod_{i,j=1}^{N} (2\omega_{ij}/\pi)^{1/4} e^{-\frac{1}{2}\omega_{ij}|y_{ij}|^2}$$

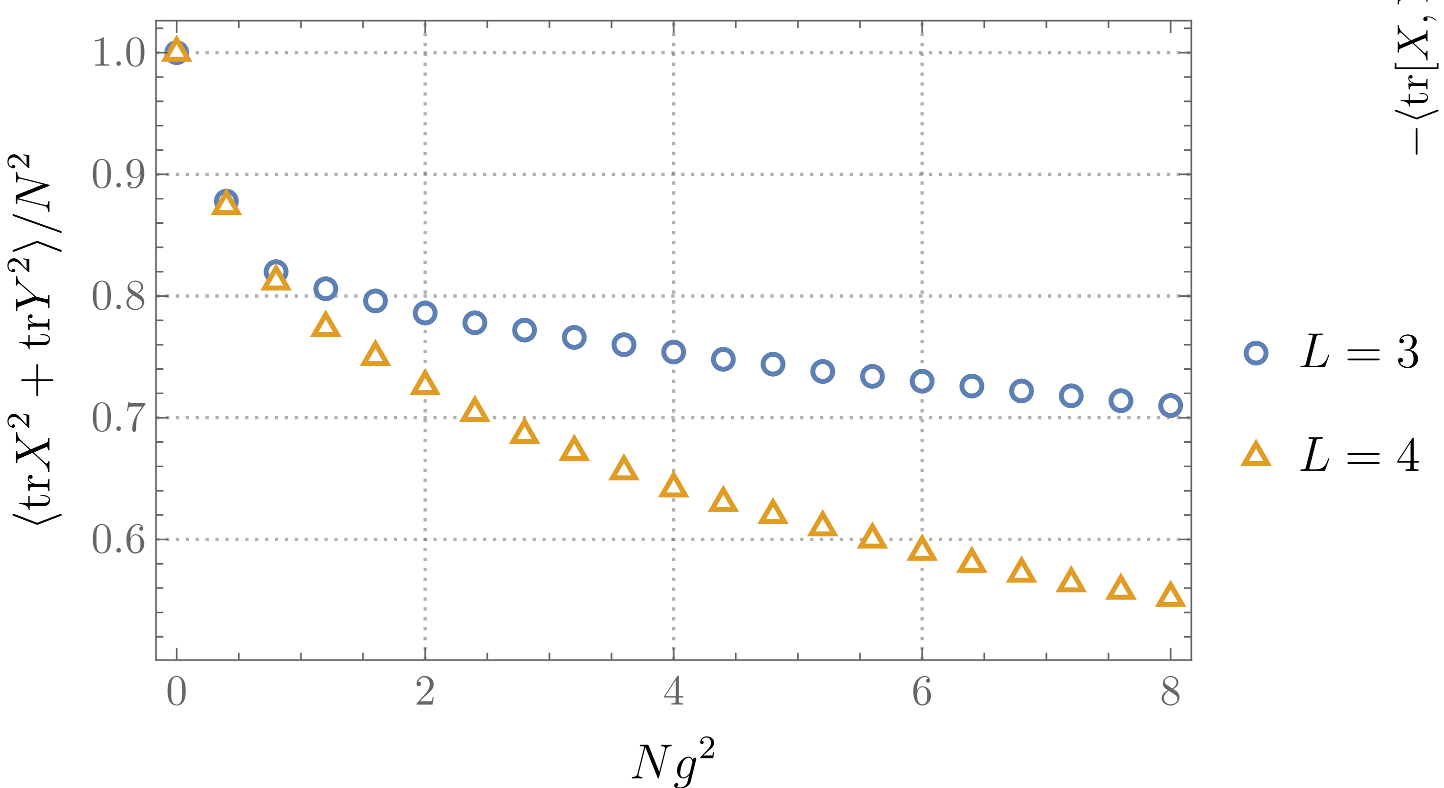
$$\omega_{ij}^2 = m^2 + g^2(x_i - x_j)^2$$

Gives rigorous upper and lower bounds on the ground state energy, using single-matrix techniques.

Two matrix theory







Final comments

- Positivity constraints give a powerful approach to matrix quantum mechanics wavefunctions.
- New results for two matrix quantum mechanics.
- More matrices and fermions doable, especially with rotational invariance. Plan: characterize the BFSS ground state!?
- Nonzero temperatures are possible. Connection to existing Monte-Carlo results? Black hole microstates?
- More fine-grained information about holographic quantum states?